WEAK AND STRONG CONVERGENCE TO FIXED POINTS OF NONEXPANSIVE MAPPINGS

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ABSTRACT. In this article, we discuss weak and strong convergence to fixed points of nonexpansive mappings in a Hilbert space or a Banach space. We first deal with strong convergence of approximants to fixed points of nonself-mappings in a Banach space. Next, we establish nonlinear ergodic theorems for families of nonexpansive mappings in a Hilbert space or a Banach space. Further, by the methods in the nonlinear ergodic theory, we discuss weak and strong convergence of Ishikawa iterates. Finally, we extend the strong convergence theorem obtained by Wittmann in a Hilbert space to that in a Banach space.

1. INTRODUCTION

Let C be a nonempty closed convex subset of a real Banach space E and let T be a nonexpansive mappings of C into itself such that the set $F(T)$ of fixed points of T is nonempty. In 1967, Browder [3] proved the following celebrated strong convergence theorem for nonexpansive mappings in the framework of a Hilbert space: Let x be an element of C and for each t with $0 < t < 1$, let x_t be an element of C satisfying

$$x_t = tTx_t + (1-t)x.$$

Then $\{x_t\}$ converges strongly to the element of $F(T)$ which is nearest to x in $F(T)$ as $t \to 1$. This result was extended to Banach spaces by Reich [10] and Takahashi and Ueda [28]. Marino and Trombetta [12], and Xu and Yin [31] also considered to extend Browder's theorem to the case when T is of C into E. On the other hand, Baillon [1] establised the first nonlinear ergodic theorem for nonexpansive mappings in 1975: Let C be a closed convex subset of a Hilbert space and let T be a nonexpansive mapping of C into itself. If $F(T)$ is nonempty, then for each $x \in C$, the Cesàro means

$$S_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to some $y \in F(T)$. In this case, putting $y = Px$ for each $x \in C$, P is a nonexpansive retraction of C onto $F(T)$ such that $PT^n = T^n P = P$ for all $n = 0, 1, 2, \dots$ and $Px \in \overline{co}\{T^n x : n = 0, 1, 2, \dots\}$ for each $x \in C$, where $\overline{co}A$ is the closure of the convex hull of A. By the methods in the ergodic theory, Tan and Xu [29] discussed the following iteration scheme which was introduced by Ishikawa [7]: $x_1 \in C$ and

$$x_{n+1} = \alpha_n T[\beta_n T x_n + (1 - \beta_n) x_n] + (1 - \alpha_n) x_n, \quad n = 1, 2, 3, \dots ,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1]$. Recently, in the framework of a Hilbert space, Wittmann [30] studied the convergence of the iterated sequence which is defined by $y_0 = x$ and

$$y_{n+1} = a_n x + (1 - a_n) T y_n, \quad n = 0, 1, 2, \dots ,$$

where $\{\alpha_n\}$ is a real sequence in $[0, 1]$.

In this article, we first extend Xu and Yin's results to Banach spaces by using the inwardness condition and sunny nonexpansive retractions; see Section 3. In Section 4, we give a nonlinear ergodic theorem for nonlinear semigroups without convexity in a Hilbert space. This has many applications. Further we deal with a nonlinear ergodic theorem for nonlinear semigroups in a Banach space. Finally, we extend Wittmann's result [30] to Banach spaces. This result answers affirmatively a problem posed by Reich [17]; see Section 6.

2. Preliminaries

Let E be a Banach space and let C be a nonempty closed convex subset of E. Then a mapping T of C into E is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for every $x, y \in C$. Let T be a mapping of C into E. Then we denote by $F(T)$ the set of fixed points of T and by $R(T)$ the range of T. let D be a subset of C and let P be a mapping of C into D. Then P is said to be *sunny* if

$$P(Px + t(x - Px)) = Px$$

whenever $Px + t(x - Px) \in C$ for $x \in C$ and $t \geq 0$. A mapping P of C into C is said to be a *retraction* if $P^2 = P$. If a mapping P of C into C is a retraction, then $Pz = z$ for every $z \in R(P)$.

Let E be a Banach space. Then, for every ε with $0 \leq \varepsilon \leq 2$, the *modulus* $\delta(\varepsilon)$ *of convexity* of E is defined by

$$\delta(\varepsilon) = \inf \left\{ 1 - \|\frac{x+y}{2}\| : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \varepsilon \right\}.$$

A Banach space E is said to be *uniformly convex* if $\delta(\varepsilon) > 0$ for every $\varepsilon > 0$. E is also said to be *strictily convex* if $\left\| \frac{x+y}{2} \right\| < 1$ for $x, y \in E$ with $\|x\| \leq 1$, $\|y\| \leq 1$ and $x \neq y$. A uniformly convex Banach space is strictly convex. A closed convex subset C of a Banach space E is said to have *normal structure* if for each closed convex bounded subset K of C which contains at least two points, there exists an element of K which is not a diametral point of K. It is well known that a closed convex subset of a uniformly convex Banach space has normal structure and a compact convex subset of a Banach space has normal structure. We also know the following theorem which was proved by Kirk [9].

Theorem 2.1 ([9]). *Let E be a reflexive Banach space and let C be a nonempty bounded closed convex subset of E which has normal structure. Let T be a nonexpansive mapping of C into itself. Then $F(T)$ is nonempty.*

Let E be a Banach space and let E^* be its dual, that is, the space of all continuous linear functionals x^* on E. For every $x \in E$ and $x^* \in E^*$, $< x, x^* >$ means the value of x^* at x. With each $x \in E$, we associate the set

$$J(x) = \{x^* \in E^* :< x, x^* >= \|x\|^2 = \|x^*\|^2\}.$$

Using the Hahn-Banach theorem, it is immediately clear that $J(x) \neq \phi$ for any $x \in E$. Then the multivalued operator $J : E \to E^*$ is called the *duality mapping* of E. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the norm of E is said to be *Gâteaux differentiable* (and E is said to be *smooth*) if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for all x and y in U. It is said to be *Fréchet differentiable* if for each x in U, this limit is attained uniformly for y in U. It is also said to be *uniformly Gâteaux differentiable* if for each $y \in U$, this limit is attained uniformly for x in U. It is also said to be *uniformly Fréchet differentiable* (and E is said to be *uniformly smooth*) if the limit is attained uniformly for x, y in U. It is well known that if E is smooth, then the duality mapping J is single-valued. And also we known that if E has a Fréchet differentiable norm, then J is norm to norm continuous.

3. Generalizations of Browder's Theorem

The following interesting convergence theorem of approximated sequences for nonexpansive mappings was established by Browder [3].

Theorem 3.1 ([3]). *Let C be a nonempty closed convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself such that $F(T)$ is nonempty. Let u be an element of C and define, for each $t \in (0,1)$, a mapping T_t of C into itself by*

$$T_t x = tTx + (1-t)u \quad \textit{for all} \ \ x \in C.$$

Then there exists a unique element x_t of C such that $T_t x_t = x_t$. Further $\{x_t\}$ converges strongly to the element of $F(T)$ which is nearest to u in $F(T)$ as $t \to 1$.

This theorem was extended to Banach spaces by Reich [16] and Takahashi and Ueda [28]. On the other hand, for a given $u \in C$ and $t \in (0,1)$, Marino and Trombetta [12] considered contractions S_t and U_t of C into itself by

$$S_t x = tPTx + (1-t)u$$

and

$$U_t x = P(tTx + (1-t)u)$$

for all $x \in C$, where P is the metric projection of C onto $F(T)$, and tried to extend Browder's theorem to nonself-mappings. Recently, Xu and Yin [31] obtained the following interesting results. Before stating them, we give difinitions. Let C be a nonempty convex subset of a Banach space E. Then for $x \in C$, we define the *inward set* $I_C(x)$ as follows:

$$I_C(x) = \{y \in E : y = x + a(z-x) \quad \text{for some} \quad z \in C \text{ and } a \geq 0\}.$$

A mapping $T : C \to E$ is said to be *inward* if $Tx \in I_C(x)$ for all $x \in C$. T is also said to be *weakly inward* if for each $x \in C$, Tx belongs to the closure of $I_C(x)$. The following are Xu and Yin's results.

Theorem 3.2 ([31]). *Let C be a nonempty closed convex subset of a Hilbert space H and let $T : C \to H$ be a nonexpansive nonself-mapping satisfying the weak inwardness condition. Fix $u \in C$ and for each $t \in (0,1)$, let x_t be an element of C satisfying*

$$x_t = tPTx_t + (1-t)u,$$

where P is the metric projection of H onto C. Then $F(T) \neq \phi$ if and only if $\{x_t\}$ remains bounded as $t \to 1$. In this case, $\{x_t\}$ converges strongly as $t \to 1$ to a fixed point of T.

Theorem 3.3 ([31]). *Let C be a nonempty closed convex subset of a Hilbert space H and let $T : C \to H$ be a nonexpansive nonself-mapping satisfying the weak inwardness condition. Fix $u \in C$ and for each $t \in (0,1)$, let x_t be an element of C satisfying*

$$x_t = P(tTx_t + (1-t)u),$$

where P is the metric projection of H onto C. Then $F(T) \neq \phi$ if and only if $\{x_t\}$ remains bounded as $t \to 1$. In this case, $\{x_t\}$ converges strongly as $t \to 1$ to a fixed point of T.

The following theorem is crucial in the proofs of Theorems 3.5 and 3.6.

Theorem 3.4 ([25]). *Let E be a reflexive Banach space with a uniformly Gâteaux differentiable norm. Let C be a nonempty closed convex subset of E which has normal structure, and let $T : C \to E$ be a nonexpansive nonself-mapping. Suppose that for some $u \in C$ and each $t \in (0,1)$, the contraction T_t defined by*

$$T_t x = tTx + (1-t)u \quad \textit{for all} \quad x \in C$$

has a (unique) fixed point $x_t \in C$. Then T has a fixed point if and only if $\{x_t\}$ remains bounded as $t \to 1$. In this case, $\{x_t\}$ converges strongly as $t \to 1$ to a fixed point of T.

Using Theorem 3.4, we obtain the following two theorems which extend Xu and Yin's results [31] to Banach spaces.

Theorem 3.5 ([25]). *Let E be a reflexive Banach space with a uniformly Gâteaux differentiable norm. Let C be a nonempty closed convex subset of E which has normal structure, and let $T : C \to E$ be a nonexpansive nonself-mapping satisfying the weak inwardness condition. Suppose that C is a sunny nonexpansive retract of E and for some $u \in C$ and each $t \in (0,1)$, $x_t \in C$ is a (unique) fixed point of the contraction S_t defined by*

$$S_t x = tPTx + (1-t)u \quad \text{for all } x \in C,$$

where P is a sunny nonexpansive retraction of E onto C. Then T has a fixed point if and only if $\{x_t\}$ remains bounded as $t \to 1$. In this case, $\{x_t\}$ converges strongly as $t \to 1$ to a fixed point of T.

Theorem 3.6 ([25]). *Let E be a reflexive Banach space with a uniformly Gâteaux differentiable norm. Let C be a nonempty closed convex subset of E which has normal structure, and let $T : C \to E$ be a nonexpansive nonself-mapping satisfying the weak inwardness condition. Suppose that C is a sunny nonexpansive retract of E and for some $u \in C$ and each $t \in (0,1)$, $y_t \in C$ is a (unique) fixed point of the contraction U_t defined by*

$$U_t x = P(tTx + (1-t)u) \quad \text{for all } x \in C,$$

where P is a sunny nonexpansive retraction of E onto C. Then T has a fixed point if and only if $\{y_t\}$ remains bounded as $t \to 1$. In this case, $\{y_t\}$ converges strongly as $t \to 1$ to a fixed point of T.

4. Nonliner Ergodic Theorems

The first nonlinear ergodic theorem for nonexpansive mappings was established in 1975 by Baillon [1].

Theorem 4.1 ([1]). *Let C be a closed convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself. If $F(T)$ is nonempty, then for each $x \in C$, the Cesàro means*

$$S_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converges weakly to some $y \in F(T)$.

We first extend this theorem to a nonlinear semigroup of nonexpansive mappings in a Hilbert space. Let S be a semitopological semigroup, i.e., a semigroup with Hausdorff topology such that for each $s \in S$, the mappings $t \mapsto ts$ and $t \mapsto st$ of S into itself are continuous. Let $B(S)$ be the Banach space of all bounded real valued functions on S with supremum norm and let X be a subspace of $B(S)$ containing constants. Then, an element μ of X^* (the dual space of X) is called a *mean* on X if $\|\mu\| = \mu(1) = 1$. For each $s \in S$ and $f \in B(S)$, we define elements $\ell_s f$ and $r_s f$ of $B(S)$ given by

$$(\ell_s f)(t) = f(st) \quad \text{and} \quad (r_s f) = f(ts)$$

for all $t \in S$. Let $C(S)$ be the Banach space of all bounded continuous real valued functions on S and let $RUC(S)$ be the space of all bounded right uniformly continuous functions on S, i.e., all $f \in C(S)$ such that the mapping $s \mapsto r_s f$ is continuous. Then $RUC(S)$ is a closed subalgebra of $C(S)$ containing constants and invariant under ℓ_s and r_s, $s \in S$; see [5, 13] for more details. Let $\{\mu_\alpha : \alpha \in A\}$ be a net of means on $RUC(S)$. Then $\{\mu_\alpha \in A\}$ is said to be *asymptotically invariant* if for each $f \in RUC(S)$ and $s \in S$,

$$\mu_\alpha(f) - \mu_\alpha(\ell_s f) \to \quad \text{and} \mu_\alpha(f) - \mu_\alpha(r_s f) \to 0.$$

Let us give an example of asymptotically invariant nets. Let $S = \{0, 1, 2, \dots\}$. Then for $f = (x_0, x_1, \dots) \in B(S)$ and $n \in N$, the real valued function μ_n defined by

$$\mu_n(f) = \frac{1}{n} \sum_{k=0}^{n-1} x_k$$

is a mean. Further since for $f = (x_0, x_1, \dots) \in B(S)$ and $m \in N$

$$\begin{aligned} |\mu_n(f) - \mu_n(r_m f)| &= \left| \frac{1}{n} \sum_{k=0}^{n-1} x_k - \frac{1}{n} \sum_{k=0}^{n-1} x_{k+m} \right| \\ &\leq \frac{1}{n} \cdot 2m\|f\| \to 0, \end{aligned}$$

as $n \to \infty$, $\{\mu_n\}$ is an asymptotically invariant net of means.

Let S be a semitopological semigroup and let C be a nonempty subset of a Banach space E. Then a family $\mathcal{S} = \{T_s : s \in S\}$ of mappings of C into itself is called a *Lipschitzian semigroup* on C if it satisfies the following:

(i) $T_{st}x = T_s T_t x$ for all $s, t \in S$ and $x \in C$;

(ii) for each $x \in C$, the mapping $s \mapsto T_s x$ is continuous;

(iii) for each $s \in S$, T_s is a Lipschitzian mapping of C into itself, i.e., there is $k_s \geq 0$ such that

$$\|T_s x - T_s y\| \leq k_s \|x - y\|$$

for all $x, y \in C$.

A Lipschizian semigroup $\mathcal{S} = \{T_s : s \in S\}$ on C is said to be *nonexpansive* if $k_s = 1$ for every $s \in S$. For a Lipschizian semigroup $\mathcal{S} = \{T_s : s \in S\}$ on C, we denote by $F(\mathcal{S})$ the set of common fixed points of $T_s, s \in S$. If C is a nonempty subset of a

Hilbert space H and $\mathcal{S} = \{T_s : s \in S\}$ is a nonexpansive semigroup on C such that $\{T_s x : s \in S\}$ is bounded for some $x \in C$, then we know that for each $u \in C$ and $v \in H$, the functions $f(t) = \|T_t u - v\|^2$ and $g(t) = < T_t u, v >$ are in $RUC(S)$. Let μ be a mean on $RUC(S)$. Then since for each $x \in C$ and $y \in H$, the real valued function $t \mapsto < T_t x, y >$ is in $RUC(S)$, we can define the value $\mu_t < T_t x, y >$ of μ at this function. By linearity of μ and of the inner product, this is linear in y; moreover, since

$$|\mu_t < T_t x, y > | \leq \|\mu\| \cdot \sup_t | < T_t x, y > | \leq (\sup_t \|T_t x\|) \cdot \|y\|,$$

it is continuous in y. So, by the Riesz theorem, there exists an $x_0 \in H$ such that

$$\mu_t < T_t x, y > = < x_0, y >$$

for every $y \in H$. We write such an x_0 by $T_\mu x$; see [21, 23] for more details.

Now we can state a nonlinear ergodic theorem for noncommutative semigroups of nonexpansive mappings in a Hilbert space.

Theorem 4.2 ([24]). *Let C be a nonempty subset of a Hilbert space H and let S be a semitopological semigroup such that $RUC(S)$ has an invariant mean. Let $\mathcal{S} = \{T_t : t \in S\}$ be a nonexpansive semigroup on C such that $\{T_t x : t \in S\}$ is bounded and $\bigcap_{s \in S} \overline{co}\{T_{st} x : t \in S\} \subset C$ for some $x \in C$. Then, $F(\mathcal{S}) \neq \phi$. Further, for an asymptotically invariant net $\{\mu_\alpha : \alpha \in A\}$ of means on $RUC(S)$, the net $\{T_{\mu_\alpha} x : \alpha \in A\}$ converges weakly to an element $x_0 \in F(\mathcal{S})$.*

Using Theorem 4.2, we can prove the following theorem.

Theorem 4.3. *Let C be a closed convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself. If $F(T)$ is nonempty, then for each $x \in C$,*

$$S_r(x) = (1 - r) \sum_{k=0}^{\infty} r^k T^k x,$$

as $r \uparrow 1$, converges weakly to an element $y \in F(T)$.

Let $S = [0, \infty)$ and let $\mathcal{S} = \{S(t) : t \in [0, \infty)\}$ be a nonexpansive semigroup on C. Then using Theorem 4.2, we also have the following nonlinear ergodic theorem.

Theorem 4.4. *Let C be a closed convex subset of a Hilbert space H and let $\mathcal{S} = \{S(t) : t \in [0, \infty)\}$ be a nonexpansive semigroup on C. If $F(\mathcal{S})$ is nonempty, then for each $x \in C$,*

$$S_\lambda(x) = \frac{1}{\lambda} \int_0^\lambda S(t) x dt,$$

as $\lambda \to \infty$, converges weakly to an element $y \in F(\mathcal{S})$.

In this section, we finally state a nonlinear ergodic theorem for nonexpansive semigroups in a Banach space. Before stating it, we give a definition. A net $\{\mu_\alpha\}$ of continuous linear functionals on $RUC(S)$ is called *strongly regular* if it satisfies the following conditions:

(i) $\sup_\alpha \|\mu_\alpha\| < +\infty$;
(ii) $\lim_\alpha \mu_\alpha(1) = 1$;
(iii) $\lim_\alpha \|\mu_\alpha - r_s^*\mu_\alpha\| = 0$ for every $s \in S$.

Theorem 4.5 ([6]). *Let S be a commutative semitopological semigroup and let E be a unifromly convex Banach space with a Fréchet differentiable norm. Let C be a nonempty closed convex subset of E and let $\mathcal{S} = \{T_t : t \in S\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S})$ is nonempty. Then there exists a unique nonexpansive retraction P of C onto $F(\mathcal{S})$ such that $PT_t = T_tP = P$ for every $t \in S$ and $Px \in \overline{co}\{T_tx : t \in S\}$ for every $x \in C$. Further, if $\{\mu_\alpha\}$ is a strongly regular net of continuous linear functionals on $RUC(S)$, then for each $x \in C$, $T_{\mu_\alpha}T_tx$ converges weakly to Px uniformly in $t \in S$.*

We do not know whether Theorem 4.5 would hold in the case when S is noncommutative.

5. Ishikawa Iterates

Let C be a nonempty closed convex subset of a real Banach space E and let T be a nonexpansive mapping of C into itself. Then we consider the following iteration scheme: $x_1 \in C$ and

$$x_{n+1} = \alpha_n T[\beta_n T x_n + (1-\beta_n)x_n] + (1-\alpha_n)x_n, \quad n = 1, 2, \dots, \tag{1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0,1]$. Such an iteration scheme was introduced by Ishikawa [7]; see also Mann [11] in the case of $\beta_n = 0$. For each integer $n \in N$ and $\alpha_n, \beta_n \in [0,1]$, define a mapping T_n of C into itself as follows:

$$T_n x = \alpha_n T[\beta_n T x + (1-\beta_n)x] + (1-\alpha_n)x.$$

Then, $T_n : C \to C$ is also nonexpansive; see [29]. Further we have the following: If $0 < \alpha_n \le 1$ and $0 \le \beta_n < 1$, then $F(T_n) = F(T)$. In fact, if $z \in F(T)$, it is obvious that $z \in F(T_n)$. Conversely, if $z \in F(T_n)$, we have $z = \alpha_n T[\beta_n T z + (1-\beta_n)z] + (1-\alpha_n)z$ and hence $z = T[\beta_n T z + (1-\beta_n)z]$. Suppose $Tz \ne z$. Then putting $u = \beta_n T z + (1-\beta_n)z$, we have

$$\|Tz - z\| = \|Tz - Tu\| \le \|z - u\|.$$

This is a contradiction. Therefore $z \in F(T)$. It also follows that the iterates $\{x_n\}$ defined by (1) can be written as

$$x_{n+1} = T_n T_{n-1} \cdots T_1 x_1. \tag{2}$$

Motivated by (2), we obtain the following lemma by using [4] and [8]; see also [10, 27].

Lemma 5.1. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E with a Fréchet differentiable norm and let $\{T_1, T_2, T_3, \dots\}$ be a sequence of nonexpansive mappings of C into C such that $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. Let $x \in C$ and $S_n = T_n T_{n-1} \cdots T_1$ for all $n \geq 1$. Then, the set $\bigcap_{n=1}^{\infty} \overline{co}\{S_m x : m \geq n\} \cap U$ consists of at most one point, where $U = \bigcap_{n=1}^{\infty} F(T_n)$.*

To discuss the weak convergence of iterates $\{x_n\}$, we also need the following lemma which is proved by using Kirk's fixed point theorem [9] and Shu's result [19].

Lemma 5.2. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and let $T : C \to C$ be a nonexpansive mapping. Suppose $x_1 \in C$, and $\{x_n\}$ is given by $x_{n+1} = \alpha_n T[\beta_n T x_n + (1 - \beta_n)x_n] + (1 - \alpha_n)x_n$ for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are chosen so that $\alpha_n \in [a, b]$ and $\beta_n \in [0, b]$ or $\alpha_n \in [a, 1]$ and $\beta_n \in [a, b]$ for some a, b with $0 < a \leq b < 1$. Then $F(T)$ is nonempty if and only if $\{x_n\}$ is bounded and $\{x_n - Tx_n\}$ converges strongly to zero as $n \to \infty$.*

Now we can state the following weak convergence theorem. Before stating it, we give a definition. A Banach space E is said to satisfy *Opial's condition* [14] if $x_n \rightharpoonup x$ and $x \neq y$ imply

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{x \to \infty} \|x_n - y\|,$$

where $\rightharpoonup$ denotes the weak convergence.

Theorem 5.3 ([26]). *Let E be a uniformly convex Banach space E which satisfies Opial's condition or whose norm is Fréchet differentiable, and let C be a nonempty closed convex subset of E, and let $T : C \to C$ be a nonexpansive mapping with a fixed point. Suppose $x_1 \in C$, and $\{x_n\}$ is given by $x_{n+1} = \alpha_n T[\beta_n T x_n + (1-\beta_n)x_n] + (1 - \alpha_n)x_n$ for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are chosen so that $\alpha_n \in [a, b]$ and $\beta_n \in [0, b]$ or $\alpha_n \in [a, 1]$ and $\beta_n \in [a, b]$ for some a, b with $0 < a \leq b < 1$. Then $\{x_n\}$ converges weakly to a fixed point of T.*

In this section, we finally prove a strong convergence theorem which is connected with Rhoades [18].

Theorem 5.4 ([26]). *Let E be a strictly convex Banach space and let C be a nonempty closed convex subset of E. Suppose $T : C \to C$ is a nonexpansive mapping such that $T(C)$ is contained in a compact subset of C. Suppose $x_1 \in C$, and $\{x_n\}$ is given by $x_{n+1} = \alpha_n T[\beta_n T x_n + (1 - \beta_n)x_n] + (1 - \alpha_n)x_n$ for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are chosen so that $\alpha_n \in [a, b]$ and $\beta_n \in [0, b]$ or $\alpha_n \in [a, 1]$ and $\beta_n \in [a, b]$ for some a, b with $0 < a \leq b < 1$. Then $\{x_n\}$ converges strongly to a fixed point of T.*

6. A Generalization of Wittmann's Theorem

Let C be a closed convex subset of a Banach space E and let T be a nonexpansive mapping from C into C. We deal with the iterative process: $x_0 = x \in C$ and

$$x_{n+1} = \alpha_{n+1}x + (1-\alpha_{n+1})Tx_n, \quad n = 0, 1, 2, \dots, \tag{3}$$

where $0 \le \alpha_n \le 1$ and $\alpha_n \to 0$. Concerning this process, Reich [17] posed the following problem:

Problem *Let E be a Banach space. Is there a sequence $\{\alpha_n\}$ such that whenever a weakly compact convex subset C of E possessed the fixed point property for nonexpansive mappings, then the sequence $\{x_n\}$ defined by (3) converges to a fixed point of T for all x in C and all nonexpansive $T : C \to C$?*

Though Reich [17] showed an affirmative answer in the case when E is uniformly smooth and $\alpha_n = n^{-a}$ with $0 < a < 1$, the problem has been generally open. Recently, Wittmann [30] solved the problem in the case when E is a Hilbert space.

Theorem 6.1 ([30]). *Let C be a closed convex subset of a Hilbert space and let T be a nonexpansive mapping from C into itself such that $F(T)$ is nonempty. Let x be an element of C and let $\{\alpha_n\}$ be a real sequence which satisfies*

$$0 \le \alpha_n \le 1, \quad \lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$$

Then the sequence $\{x_n\}$ defined by $x_0 = x$ and

$$x_{n+1} = \alpha_{n+1}x + (1-\alpha_{n+1})Tx_n, \quad n = 0, 1, 2, \dots$$

converges strongly to the element of $F(T)$ which is nearest to x in $F(T)$.

We extended Wittmann's result to Banach spaces. The difficulty of the proof depends on that the duality mapping is not weakly continuous. In a Hilbert space, the duality mapping is the identity mapping and hence it is weakly continuous.

Theorem 6.2 ([20]). *Let E be a Banach space whose norm is uniformly Gâteaux differentiable and let C be a closed convex subset of E. Let T be a nonexpansive mapping from C into C such that $F(T)$ is nonempty. Let $\{\alpha_n\}$ be a sequence which satisfies*

$$0 \le \alpha_n \le 1, \quad \lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$$

Let $x \in C$ and let $\{x_n\}$ be the sequence defined by $x_0 = x$ and

$$x_{n+1} = \alpha_{n+1}x + (1-\alpha_{n+1})Tx_n, \quad n = 0, 1, 2, \dots .$$

Assume that $\{z_t\}$ converges strongly to $z \in F(T)$ as $t \downarrow 0$, where for $t \in (0,1)$, z_t is a unique element of C which satisfies $z_t = tx + (1-t)Tz_t$. Then $\{x_n\}$ converges strongly to z.

To prove Theorem 6.2, we need the following two propositions. Before stating them, we give a definition. Let μ be a mean on ℓ^∞, i.e., a continuous linear functional on ℓ^∞ satisfying $\|\mu\| = 1 = \mu(1)$. Then μ is called a *Banach limit* if $\mu_n(a_n) = \mu_n(a_{n+1})$ for all $(a_0, a_1, a_2, \dots) \in \ell^\infty$, where $\mu_n(a_n)$ means the value of μ at $(a_0, a_1, a_2, \dots) \in \ell^\infty$.

Proposition 6.3. *Let a be a real number and let $(a_0, a_1, \dots) \in \ell^\infty$. Then $\mu_n(a_n) \le a$ for all Banach limits μ if and only if for each $\varepsilon > 0$, there exists a positive integer p_0 such that*

$$\frac{a_n + a_{n+1} + \cdots + a_{n+p-1}}{p} < a + \varepsilon \quad \text{for all } \ p \ge p_0 \ \text{ and } n \in N.$$

Proposition 6.4. *Let a be a real number and let $(a_0, a_1, \dots) \in \ell^\infty$ such that $\mu_n(a_n) \le a$ for all Banach limits μ and $\overline{\lim_{n\to\infty}}(a_{n+1} - a_n) \le 0$. Then $\overline{\lim_{n\to\infty}} a_n \le a$.*

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