Monotonicity of entropy for some real quadratic rational maps

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Abstract

We consider the one parameter family of real quadratic rational maps $f_c(x) = \frac{1}{x^2} + c$ (where c is a real parameter) as the family of endomorphisms on the circle $\hat{\mathbf{R}}$ which is the compactification of the real line $\mathbf{R}$ at the point at infinity. This family consists of a part of the boundary of "unimodal" region in the parameter space of real quadratic rational maps. We show the monotonicity of the topological entropy for this family. Our basic tool is the kneading theory and we reduce our claim to the monotonicity of kneading sequences for this family. We also use some techniques from the theory of complex dynamics; Thurston's theorem on the rigidity of post critically finite rational maps is essential in our argument.

1 Introduction

We can consider a rational function with real coefficient as a map from the circle $\hat{\mathbf{R}}$ the compactification of the real line $\mathbf{R}$ by adding the point at infinity, to itself. In this paper, we treat the family of real quadratic rational maps

$$f_c(x) = \frac{1}{x^2} + c$$

where c is a real parameter. Especially, we consider the parameter dependence of the topological entropy $h(f_c)$ of the map f_c. Our main result is

Theorem 1.1 *The topological entropy $h(f_c)$ is monotonely decreasing with respect to the parameter $c \in \mathbf{R}$. In other words $c_1 < c_2$ means that $h(c_1) < h(c_2)$.*

We should remark that the similar property holds for the family of real quadratic polynomials $q_c(x) = x^2 + c$ ($c \in \mathbf{R}$) ([M-T],[dM-vS]). These two families are some special classes in the set of real quadratic rational maps. In fact Milnor shows that the parameter space of real quadratic rational maps, i.e. the set of $PGL_2(\mathbf{R})$-conjugacy classes of these maps can be naturally identified with the two dimensional real affine plane $\mathbf{R}^2$ ([M]). In this space the family $\{q_c\}$ consists of maps which has a fixed critical point. On the other hand, the family $\{f_c\}$ consists of maps one of whose critical value is also a critical point. Therefore a map of these families has some restrictions on the critical orbit of it. Moreover these families are precisely the boundary of the unimodal region in the parameter space consisting of maps which can be considered as unimodal maps ([F-N]). It seems interesting to check whether the monotonicity of the topological entropy also holds on this unimodal region.

The idea on the main theorem 1.1 is as follows; by using the kneading theory, we reduce our claim to the monotonicity of the kneading sequence $k(f_c)$ of the map f_c. We show this by three steps. In the first step, we check that in the both side of the parameter line, the kneading sequence is constant and attains the maximum and the minimum. The second step is so called "intermediate value theorem" which tells us the condition for a sequence to be realized as a kneading sequence on the given closed interval on the parameter line, The remain to show is the rigidity property for the post critically finite maps which means that if f_{c_1} and f_{c_2} satisfy that $f_{c_1}^n(0) = 0$ and $f_{c_2}^n(0) = 0$ for some $n \in \mathbf{N}$ and their kneading sequences coincide, then $c_1 = c_2$. We show this by using techniques from the theory of complex dynamics. We show that considering f_{c_1} and f_{c_2} as rational maps on Riemann sphere $\hat{\mathbf{C}}$, they are equivalent in the sense of Thurston. Then Theorem of Thurston ([D-H]) shows that f_{c_1} and f_{c_2} are $PSL_2(\mathbf{C})$-conjugate, hence $c_1 = c_2$. This is the third step and after that we can easily prove our claim.

In Section 2, we show that a map of our family can also be considered as a map on a interval. Reviewing the definition of the topological entropy in Section 3, in Section 4 we reduce our claim to the monotonicity of the kneading sequence $k(f_c)$. And we prove this in Section

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2 The family of real quadratic rational maps

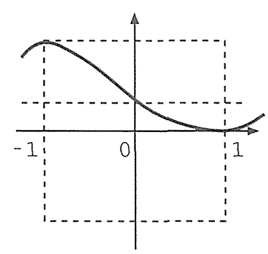
Let $\mathbf{R}$ be the real line and $\hat{\mathbf{R}}$ be the compactification of $\mathbf{R}$ be the point ∞ at infinity. Then $\hat{\mathbf{R}}$ is a circle. We are interested in analyzing the dynamics of one parameter family of real quadratic rational maps

$$f_c(x) = \frac{1}{x^2} + c \ \ (c \in \mathbf{R})$$

as the circle maps from $\hat{\mathbf{R}}$ to $\hat{\mathbf{R}}$. In the following we consider the dynamics of iterations of f_c, $\{f_c^n\}_{n \in \mathbf{N}}$. First we remark that the critical set Ω_c of f_c is $\{0, \infty\}$ and f_c satisfies $f_c(0) = \infty$ and $f_c(\infty) = f_c^2(0) = c$. Because we are mainly interested in the topological dynamics of f_c, the dynamics of f_c and one of $A \circ f_c \circ A^{-1}$ where A is a homeomorphism of $\hat{\mathbf{R}}$, are assumed to be the same. Therefore by taking the real linear fractional transformation A_c which send $\{0, c, \infty\}$ to $\{-1, 0, 1\}$, we conjugate f_c by A_c as follows

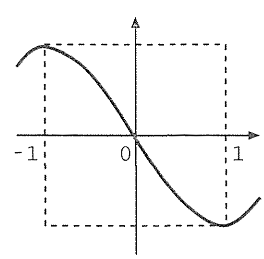
1. $c > 0$
   Put $A_c(x) = \frac{x-c}{x+c}$. Then

$$\begin{aligned} g_c(x) &:= A_c \circ f_c \circ A_c^{-1}(x) \\ &= \frac{(x-1)^2}{(x-1)^2 + 4c^3(x+1)^2}. \end{aligned}$$



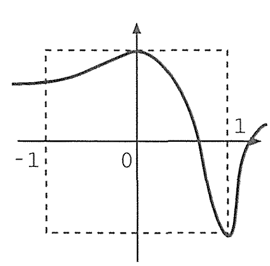
2. $c = 0$
   Put $A_0(x) = \frac{x-1}{x+1}$. Then

$$\begin{aligned} g_c(x) &:= A_c \circ f_c \circ A_c^{-1}(x) \\ &= \frac{-2x}{(x+1)^2}. \end{aligned}$$



3. $c < 0$
   Put $A_c(x) = \frac{x}{x-2c}$. Then

$$\begin{aligned} g_c(x) &:= A_c \circ f_c \circ A_c^{-1}(x) \\ &= \frac{(x-1)^2 + 4c^3x^2}{(x-1)^2 - 4c^3x^2}. \end{aligned}$$



Then each case of the graph of g_c shows that $g_c(\hat{\mathbf{R}})$ is a closed interval and g_c preserves this interval $g_c(\hat{\mathbf{R}})$ i.e., g_c can also be considered as a map on the interval $g_c(\hat{\mathbf{R}})$. Above arguments show that without the circle endomorphism $f_c : \hat{\mathbf{R}} \to \hat{\mathbf{R}}$, we can also considered the interval map $f_c : f_c(\hat{\mathbf{R}}) \to f_c(\hat{\mathbf{R}})$.

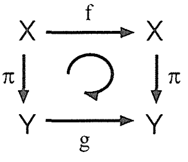
3 The topological entropy

First following [dM-vS], we review the definition and basic properties of the topological entropy. Let (X, d) be a compact metric space X with metric d and $f : X \to X$ be a continuous map. A subset $E \subset X$ is *(n, ϵ)-separated* if any distinct points $x, y \in E$, there is an integer j such that $0 \leq j < n$ and $d(f^j(x), f^j(y)) > \epsilon$. That is, any two points of E must leave at least ϵ-distance from each other until n times. If $K \subset X$ is a compact subset, we denote by $s_n(\epsilon, K, f)$ the smallest cardinality of any subset E of K which is (n, ϵ)-separated. Then the number

$$h(f, K) := \lim_{\epsilon \to 0}(\limsup_{n \to \infty} \frac{1}{n} \log s_n(\epsilon, K, f))$$

is well defined and called *the topological entropy of f with respect to K*. The number $h(f) = h(f, X)$ is called *the topological entropy of f*. We remark that $h(f, K) = 0$ if K is a finite set. Next theorem is a fundamental result of the topological entropy.

Theorem 3.1 (Bowen)
Let (X, d) and (Y, d') be compact metric spaces, $f : X \to X$, $g : Y \to Y$ be continuous maps. If $\pi : X \to Y$ is continuous and surjective such that $\pi \circ f = g \circ \pi$ then



$$h(g) \leq h(f) \leq h(g) + \sup_{y \in Y} h(f, \pi^{-1}(y)).$$

In particular this theorem shows that the topological entropy is a topological invariant, does not depend on the choice of the metric.

Corollary 3.1 *Let X be a compact space, $f : X \to X$ be continuous and finite to one. If there exists $n \geq 1$ such that $f(f^n(X)) \subseteq f^n(X)$, then $h(f) = h(f, f^n(X))$.*

Therefore by using the results of Section 2, we can conclude that the circle map $f_c : \hat{\mathbf{R}} \to \hat{\mathbf{R}}$ and the interval map $f_c : f_c(\hat{\mathbf{R}}) \to f_c(\hat{\mathbf{R}})$ have the same topological entropy.

Next we define piecewise monotone maps. Let I be the closed interval $[0,1]$ and $f: I \to I$ be a continuous map. We call f be *piecewise monotone* if there are a finite number of turning points $c_1 < c_2 < \cdots < c_l$ and f is monotone in each one of the intervals $I_0 = [0, c_1), I_1 = (c_1, c_2), \cdots, I_l = (c_l, 1]$. If f is piecewise monotone, then so are f^n for all $n \geq 1$. Let $l(f^n)$ be the number of maximal subintervals of I in which f^n is monotone. We call $l(f^n)$ *the lap number of* f^n. Next theorem shows the relation between the lap number $l(f^n)$ and the topological entropy $h(f)$.

Theorem 3.2 (Misiurewicz-Slenk)
If f is a piecewise monotone map, then

$$h(f) = \lim_{n\to\infty} \frac{1}{n} \log l(f^n).$$

Corollary 3.2 *If f and g are piecewise monotone and satisfy $l(f^n) \leq l(g^n)$ for all $n \geq 1$, then $h(f) \leq h(g)$.*

From the results of Section 2, the map f_c is piecewise monotone, and $l(f^n) = 1$ for $c \geq 0$, hence $h(f) = 0$. On the other hand $l(f_c) = 2$ for $c < 0$. In general, a piecewise monotone map f is called *unimodal* if $l(f) = 2$. In the following, putting $a = 4c^3$ for $c < 0$, we consider the one parameter family of unimodal maps $g_a = A_c \circ f_c \circ A_c^{-1}$

$$g_a = \frac{(x-1)^2 + ax^2}{(x-1)^2 - ax^2}.$$

We will show that if $a_1 < a_2$, then $l(g_{a_1}^n) < l(g_{a_2}^n)$ for any $n \geq 1$. Then by Corollary 3.2, we can conclude our claim of the monotonicity of the topological entropy for the family $\{f_c\}$. To show this monotonicity of the lap numbers $l(g_a^n)$, we prepare the combinatorial tool, the kneading theory in the next section.

4 The kneading theory for unimodal maps

First we need some definitions and notations. Let I be the closed interval $[0,1]$ and f be a unimodal map. We denote its turning point by $c \in I$ and assume that f is monotonely increasing on $[0, c)$ and monotonely decreasing on $(c, 1]$. Let us denote by $\mathbf{S}$ the symbol space $\mathbf{S} = \{L, C, R\}$ and Σ be the space of infinite sequences $A : \mathrm{N} \to \Sigma$, $A = (a_0, a_1, \cdots, a_n, \cdots)$

where $a_i = A(i)$. We also define the shift transformation $\sigma : \Sigma \to \Sigma$ by $\sigma(a_0, a_1, a_2, \cdots) = (a_1, a_2, \cdots)$. For $A = (a_0, a_1, \cdots, a_n, \cdots)$, we define $A|_n$ by the finite sequence $(a_0, a_1, \cdots, a_{n-1})$. We introduce an order structure in the space Σ. First we assume that $L < C < R$. Finite sequence of symbols L and R is called *even* if it contains even number of R. For $A = (a_0.a_1, \cdots)$ and $B = (b_0, b_1, \cdots)$, we say that $A \prec B$ if there exists $n \in \mathrm{N}$ such that $a_1 = b_i$ for $i < n$ and $a_n < b_n$ if $A|_n$ is even, and $a_n > b_n$ if $A|_n$ is not even. We call $A \in \Sigma$ *maximal* if $\sigma^n(A) \preceq A$ for all $n \in \mathrm{N}$.

Let the map $I_f : I \to \Sigma$ be defined by $I_f(x) = (i_0(x), i_1(x), \cdots, i_n(x), \cdots)$ where $i_n(x) = L$ if $f^n(x) < 0$, $i_n(x) = C$ if $f^n(x) = 0$ and $i_n(x) = R$ if $f^n(x) > 0$. The sequence $I_f(x)$ is called *the itinerary of* x. The map I_f relates the dynamics of f with the dynamics of the shift transformation: $\sigma I_f(x) = I_f(f(x))$. We can also show that $I_f(x) \prec I_f(x')$ means $x < x'$, and $x < x'$ means $I_f(x) \preceq I_f(x')$. Especially we call $I_f(f(c))$ *the kneading sequence of* f and denote it by $k(f)$. Because the function f attains the maximal value at the turning point $c \in I$, $k(f)$ is a maximal sequence.

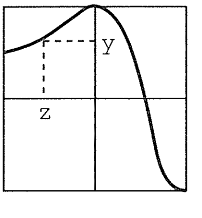
Now we go back to our original problem. For the unimodal map g_a $(a < 0)$, let $h(a)$ and $k(a)$ be the topological entropy and the kneading sequence of g_a respectively.

Proposition 4.1 *If $k(a_1) \geq k(a_2)$ for $a_1 < a_2$, then $h(a_1) \geq h(a_2)$.*

(Proof.) By Corollary 3.2, it is enough to show that under the assumption that $k(a_1) \geq k(a_2)$, $l(f^n_{a_1}) \geq l(f^n_{a_2})$ for all $n \in \mathrm{N}$. In fact $l(f^n)$ is equal to the number of finite sequences which is equal to $I_f(x)|_n$ for some point $x \in I$. Therefore it is enough for us to show that under the assumption that $k(a_1) \geq k(a_2)$ if there exist finite sequence A of L and R of length n and some point $x \in I$ such that $I_{g_{a_2}}(x)|_n = A$, then there exists $z \in I$ such that $I_{g_{a_1}}(z)|_n = A$. We prove this assertion by induction on n. It is trivial for the case of $n = 1$. We assume that it holds for $n = k$ and there exists $x \in I$ such that $I_{g_{a_2}}(x)|_{k+1} = A$. We separate our argument for the cases $A = LB$ and $A = RB$ where B is a finite sequence of L and R of length k.

1.

The case that $A = LB$.
The equation $I_{g_{a_2}}(x)|_{k+1} = A = RB$ shows that $-1 \leq x < 0$. Hence $g_{a_2}(-1) \leq g_{a_2}(x) < g_{a_2}(0)$. It means that $I_{g_{a_2}}(g_{a_2}(-1)) \leq I_{g_{a_2}}(g_{a_2}(x)) \leq I_{g_{a_2}}(g_{a_2}(0))$. In other words, $\sigma^2 k(a_2) \leq B \cdots \leq k(a_2)$.



The assumption $k(a_2) \leq k(a_1)$ and $k(a)|_2 = RL$ shows that $\sigma^2 k(a_1) \leq \sigma^2 k(a_2)$.

Therefore $I_{g_{a_1}}(g_{a_1}(-1)) \leq B \cdots \leq I_{g_{a_1}}(g_{a_1}(0))$. By the induction hypothesis, there exists $y \in I$ such that $I_{g_{a_1}}(y)|_k = B$ and from the above inequality we can assume that $g_{a_1}(-1) \leq y$.

Since g_{a_1} is continuous, by the intermediate value theorem, there exists $z < 0$ such that $y = g_{a_1}(z)$, which means that $I_{g_{a_1}}(z)|_{k+1} = LB = A$.

2. The case that $A = RB$.
   By the induction hypothesis, there exists $y \in I$ such that $I_{g_{a_1}}(y) = B$. We can assume that $y \in [-1, 1)$. Then there exists $z > 0$ such that $y = g_{a_1}(z)$ which means that $I_{g_{a_1}}(z)|_{k+1} = RB = A$.

Hence by the induction hypothesis, we can prove our claim.

From the above result, to show the monotonicity of the lap number $l(g_a^n)$, it is enough to show the monotonicity of the kneading sequence $k(a)$ which will be proved in the next section.

5 Monotonicity of kneading sequences

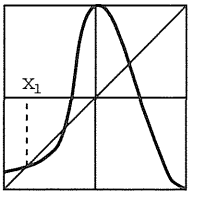
In this section we prove that the kneading sequence $k(a)$ is monotone for the family $\{g_a\}$. For this purpose, we prepare the following lemmas.

Lemma 5.1 "End points attain the extremal kneading sequences" *There exist $\mu_0 < \mu_1$ satisfying the following conditions*

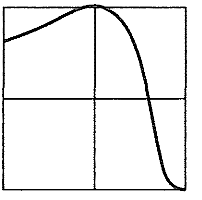
1. *For any $a < 0$ with $a \leq \mu_0$, $k(a) = RL^\infty$.*
2. *For any $a < 0$ with $\mu_1 \leq a$, $k(a) = (RL)^\infty$.*
3. *For any $a < 0$, $RL^\infty \geq k(a) \geq (RL)^\infty$.*

(Proof.) The fact that $g_a(0) = 1$ and $g_a^2(0) = g_a(1) = -1$ for all $a < 0$ means that $k(a)|_2 = RL$.

1. It is checked by direct calculation that there exists $\mu_0 < 0$ such that for any $a \leq \mu_0$ the graph of g_a and the diagonal intersects at three points. Hence if we write the smallest x-coordinate of these intersection points by x_1, $g_a^n(-1) < 0$ for any $n \in \mathrm{N}$ and $g_a^n(-1)$ goes to x_1 as n goes to infinity. This means that for any $a \leq \mu_0$, $k(a) = RLL... = RL^\infty$.



2. $g_a^3(0) = g_a(-1) = \frac{4+a}{4-a}$. $g_a^4(0) = \frac{4a+(4+a)^2}{4a-(4+a)^2}$. Hence if we put $\mu_1 = -6 + 2\sqrt{5}$, then $g_a^3(0) > 0$ and $g_a^4(0) < 0$ for any $a \geq \mu_1$. Because g_a is monotonely increasing on $[-1, 0]$ and monotonely decreasing on $[0, 1]$, $g_a^{2n+1}(0) > 0$ and $g_a^{2n+2}(0) < 0$ for all $n \in \mathrm{N}$. Therefore for $a \geq \mu_1$, $k(a) = RLRL... = (RL)^\infty$.



3. By using the definition of the order of kneading sequences and the fact that $k(a)|_2 = RL$, RL^∞ is the maximum. On the other hand, the proof of 2 shows that if $k(a)|_4 = RLRL$, then $k(a) = (RL)^\infty$. This means that $(RL)^\infty$ is the minimum.

Lemma 5.2 "Intermediate value theorem"
Let $A \in \Sigma$ be a maximal sequence and assumed that $A \neq (BL)^\infty$ and $A \neq (BR)^\infty$ for any finite sequence B of L and R. If there exist $a_1 < a_2$ such that $k(a_1) > A > k(a_2)$, then there exists $b \in (a_1, a_2)$ such that $k(b) = A$.

(Proof.) Put

$$M_A := \{a \in [a_1, a_2] | k(a) > A\}$$
$$P_A := \{a \in [a_1, a_2] | k(a) < A\}.$$

Then $a_1 \in M_A$ and $a_2 \in P_A$. Because $[a_1, a_2]$ is connected, if both of M_A and P_A are open, then there exists $b \in (a_1, a_2)$ such that $k(b) = A$. In the following we show the openness of M_A. For any element $d \in M_A$, we will show that we can take an open neighborhood U of d in $[a_1, a_2]$ which is contained in M_A. Since $k(d) = d_1, d_2, \cdots > A = a_1, a_2, \cdots$, there exists the smallest $i \in \mathrm{N}$ with $d_i \neq a_i$. We separate our arguments for the cases when $d_i = C$ and $d_i \neq C$.

1. The case that $d_i = C$.
   We can take U as $U := \{a \in [a_1, a_2] | k(a)|_i = k(d)|_i\}$.

2. The case that $d_i \neq C$.
   In this case, $k(d)$ can be written as $k(d) = (DC)^\infty$ where D is a finite sequence of R and L. Then by Lemma 11.5 of [M-T], there exists an open neighborhood of d in $[a_1, a_2]$ such that for any $a \in U$, $k(a)$ can be written as $(DL)^\infty$, $(DC)^\infty$ or $(DR)^\infty$. We remark that this claim requires the smoothness of the map g_a. Moreover there are no maximal sequences between $(DL)^\infty$ and $(DC)^\infty$, and $(DC)^\infty$ and $(DR)^\infty$. The idea of a proof of this claim is the following; if D is even, then $(DL)^\infty < (DC)^\infty < (DR)^\infty$. If there exists a maximal sequence B with $(DL)^\infty < B < (DC)^\infty$, then $B|_i = DL$. Because DL is also even and $(DL)^\infty < B$, we can conclude that $\sigma^i(B) > B$ which contradicts to the maximality of B. If there exists a maximal sequence B with $(DC)^\infty < B < (DR)^\infty$, then $B|_i = DR$. Because DR is not even, we conclude that $\sigma^i(B) > (DR)^\infty > B$ which also contradicts to the maximality of B. Above arguments show our claim for the case that D is even. It is also the same for the case that$D$ is not even.

By using the same arguments, we can also prove the openness of P_A.

Lemma 5.3 "Combinatorial rigidity for post critically finite maps" *Let $a_1 < a_2$ satisfy the following conditions*

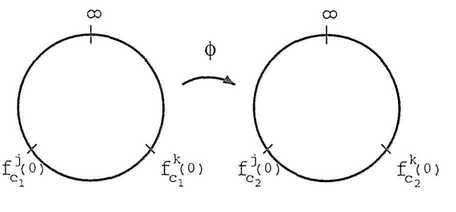
1. *There exists the smallest $n \in \mathrm{N}$ satisfying $g^n_{a_1}(0) = g^n_{a_2}(0) = 0$.*

2. *$k(a_1) = k(a_2)$.*

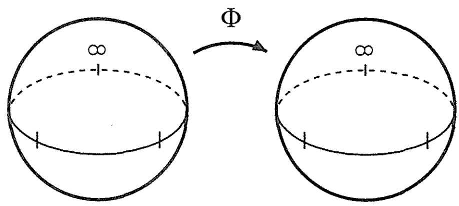
Then a_1 equals to a_2.

(Proof.)

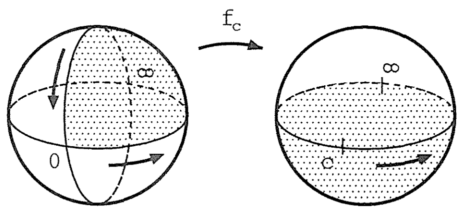
We go back to the notion of f_c. For $i = 1, 2$, we define $c_i < 0$ by $a_i = 4c_i^3$ for $i = 1, 2$. Then the condition (1) gives $f^n_{c_1}(0) = f^n_{c_2}(0) = 0$. The condition (2) shows that $f^j_{c_1}(0) < f^k_{c_1}(0)$ if and only if $f^j_{c_2}(0) < f^k_{c_2}(0)$.



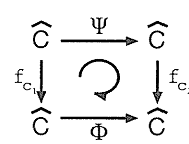
Therefore there exists an orientation preserving homeomorphism $\phi : \hat{\mathbf{R}} \to \hat{\mathbf{R}}$ which satisfies $\phi(f_{c_1}^j(0)) = f_{c_2}^j(0)$ for all $j \in \mathbf{N}$. Moreover there exists an orientation preserving homeomorphism $\Phi : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ whose restriction to $\hat{\mathbf{R}}$ is ϕ and preserves the upper and lower half planes $\mathbf{H}^+, \mathbf{H}^-$.



Next we define the orientation preserving homeomorphism $\Psi : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$. Considering f_{c_1} and f_{c_2} as rational maps from $\hat{\mathbf{C}}$ to $\hat{\mathbf{C}}$, they send the first and third quadrants to the lower half plane $\mathbf{H}^-$ and the second and fourth quadrants to the upper half plane $\mathbf{H}^+$. Therefore if z is a point of the i-th quadrant, then we can choose a unique point of $f_{c_2}^{-1} \circ \Phi \circ f_{c_1}(z)$ which is contained in the i-th quadrant. We denote this point by $\Psi(z)$. Then the map Ψ satisfies the following conditions



1. $\Psi(f_{c_1}^j(0)) = \Phi(f_{c_1}^j(0)) = f_{c_2}^j(0)$.
2. $\Phi \circ f_{c_1} = f_{c_2} \circ \Psi$.
3. Φ and Ψ are isotopic relative to the post critical set of f_{c_1}, $P_{f_{c_1}} := \{f_{c_1}(0), f_{c_1}^2(0), \cdots, f_{c_1}^n(0) = 0\}$.



In other words, this means that as rational maps on $\hat{\mathbf{C}}$, f_{c_1} and f_{c_2} are equivalent in the sence of Thurston ([D-H]). Then f_{c_1} and f_{c_2} are $PSL_2(\mathbf{C})$-conjugate by the theorem of Thurston ([D-H] Theorem 1), and in our case we conclude that $c_1 = c_2$,

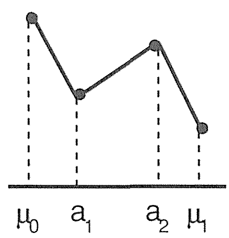
hence $a_1 = a_2$.

Now we can prove our main theorem.

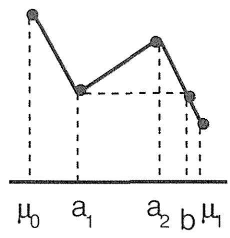
Theorem 5.1 *The kneading sequence $k(a)$ is monotonely decreasing i.e., $k(a_1) \geq k(a_2)$ for $a_1 < a_2$.*

(Proof.) By Lemma 5.1, we can restrict our attention to the closed interval $[\mu_0, \mu_1]$. We assume that there exist $a_1 < a_2$ in this interval such that $k(a_1) < k(a_2)$.

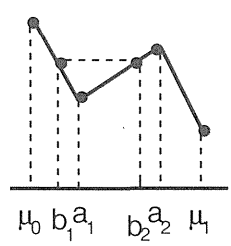
1. The case that $k(a_1) = (AC)^\infty$ where A is a finite sequence of L and R.
   The assumption that $k(a_2) > k(a_1) > k(\mu_1)$ and Lemma 5.2 show that there exists $b \in (a_2, \mu_1)$ such that $k(b) = k(a_1) = (AC)^\infty$. Then by Lemma 5.3, $a_1 = b$ which is a contradiction.



2. The case that $k(a_1)$ is a infinite sequence of L and R.
   The assumption that $k(a_1) < k(a_2)$ shows that there exists $b_2 \in (a_1, a_2]$ such that $k(b_2) = (BC)^\infty$ where B is a finite sequence of L and R. Then $k(\mu_0) > k(b_2) > k(a_1)$ and by Lemma 5.2, there exists $b_1 \in (\mu_0, a_1)$ such that $k(b_1) = k(b_2) = (BC)^\infty$. Therefore by Lemma 5.3, $b_1 = b_2$ which is a contradiction.



Above arguments show that $k(a)$ is monotonely decreasing.



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