Cores of Transferable Utility Games with Infinite Players

Hidetoshi Komiya
Department of Business and Commerce
Keio University
Hiyoshi 4-1-1, Kouhoku-ku, Yokohama 223
Japan

Abstract

We shall consider the cores of transferable utility games in this paper. The concept of core is one of the solutions in the game theory, and nonemptiness of the core ensures the stability of the grand coalition of games. A condition of the nonemptiness of cores of transferable utility games with finite players is known and it is related to balanced families of the coalitions. The purpose of this paper is to generalize the known result for games with finite players and obtain a condition for nonemptiness of cores of games with infinite players which is analogous to the corresponding condition to games of finite players. Moreover, we apply the obtained result to market games and prove that market games has nonempty cores when the production functions of agents or players are concave functions even if the number of the agents is infinite.

1 Introduction

We express *transferable utility games* mathematically as follows: Let a finite $P = \{1, \dots, p\}$ and each element of P denotes players and each nonempty subset of P denotes a coalition in the game. We denote by $\mathcal{P}$ the family of all nonempty subset of P, or the family of all coalitions. Under this setting, we define a transferable utility game with the players P by the function v of $\mathcal{P} \cup \{\emptyset\}$ to the real numbers R such that $v(\emptyset) = 0$. We call the number $v(S)$ the *value* of the coalition of S. The value of a coalition represents the maximal value obtained by the formation of the coalition of S.

We have a kind of solutions for transferable utility games which guarantees the grand coalition, that is, the coalition of all the players. We call it the *core* of the game. The mathematical definition of the core of a transferable utility game v is the set of all elements $x = (x^1, \ldots, x^p)$ of the Euclidean space R^m such that

$$\sum_{j \in S} x^j \geq v(S), \qquad S \in \mathcal{P};$$

$$\sum_{j \in P} x^j = v(P).$$

If there is an element x in the core of the game, then the allocation x is feasible by the second equation above and the first inequalities above denotes that the allocation gives the best allocation to every coalition in P. Therefore, The nonemptiness of cores of games assures the stability of the games and leads the game to the grand coalition.

From the definition above, investing whether cores are empty or not is the fundamental problem of game theory. The problem is solved in the games with finite players and a necessary and sufficient condition for the nonemptiness of cores is obtained in the relation to the concept of balancedness of the family of coalitions.

Generally speaking, given a finite set P, a family $\{S_i\}$ of the subsets of P is said to be *balanced* if there are corresponding nonnegative numbers δ_i such that the inequality

$$\sum_i \delta_i \chi_{S_i} = \chi_P$$

holds, where χ_T denotes the characteristic vector of the subset T of P, that is, it is an element of R^p defined by $\chi_T^j = 1$ if $j \in T$ and $\chi_T^j = 0$ if $j \notin T$. The nonnegative numbers δ_i are called the *balancing weights* for the balanced family $\{S_i\}$. The following is the fundamental theorem in the theory of cores of transferable utility games with finite number of players and we can, for example, find its proof in [3]:

Theorem 1.1 *The transferable utility game v has the nonempty core if and only if we have the inequality*

$$\sum_i \delta_i v(S_i) \leq v(P)$$

for any balanced family $\{S_i\}$ in $\mathcal{P}$ and the corresponding balancing weights δ_i.

We shall generalize the result above to the case that the transferable utility game has infinite number of players in the next section.

2 Definition of Games with Infinite Players

We consider countably infinite number of players and denote the individual player by a natural number. Let N be the set of all players or natural numbers $1, 2, 3, \dots$. We restrict the coalitions to ones consisting of finite number of players in this paper. Let $\mathcal{F}$ be the family of all coalitions or finite subsets of N. Next, we define a transferable utility game on this coalition structure. A transferable utility game is a real-valued function v on $\mathcal{F}$ with the properties

$$\begin{aligned} &v(\emptyset) = 0; \\ &v(S) \geq 0, \qquad S \in \mathcal{F}. \end{aligned}$$

The definition of the transferable utility game with infinite players above seems to be restrictive comparing with that with finite players by virtue of the second inequalities. However, it has no restriction. The second inequalities can be obtained if a sufficiently large common number is added to the value $v(S)$ for every $S \in \mathcal{F}$ in case the game is of finite players.

Next we proceed to the definition of balanced families in the coalition structure of infinite players. A subfamily $\{S_i\}$ of $\mathcal{F}$ is said to be a *balanced* family if

1. for each $j \in N$, the number of S_i's which contain j is finite.

2. there are nonnegative numbers δ_i such that

$$\sum_i \delta_i \chi_{S_i} = \chi_N. \tag{1}$$

The sum of the equation (1) means $\sum_i \delta_i \chi_{S_i}^j = \chi_N^j = 1$ for all $j \in N$, and is a finite sum for all $j \in N$ by the definition of balanced families mentioned above. Balanced families contain infinite number of sets because the sets in the family are finite sets.

Next we define the core for a transferable utility game with infinite players. It is same as that of finite players formally. The core of a transferable utility game v with infinite players is the set of all elements $x = (x^1, x^2, \dots)$ of ℓ_1 such that

$$\begin{aligned} &\sum_{i \in S} x^i \geq v(S), \qquad S \in \mathcal{F}. \\ &\sum_{i \in N} x^i = v(N). \end{aligned}$$

We shall prove the fundamental theorem for the existence of the core of a transferable utility game with infinite players in the next section.

3 Nonemptiness of Cores

We consider a transferable utility game v with infinite players and simply say 'a game' instead of a transferable utility game with infinite players in the sequel.

We give a few remarks before stating our main theorem and its proof. A game v is said to be *0-1 normalized* if $v(\{j\}) = 0$ for all $j \in N$ and $v(N) = 1$. We can derive a 0-1 normalized game v' from a game v as follows if the game v has the property $v(N) > \sum_{j\in N} v(\{j\})$. Define a game v' by

$$v'(S) = \frac{v(S) - \sum_{i\in S} v(\{i\})}{v(N) - \sum_{i\in N} v(\{i\})}, \quad S \in \mathcal{F}$$

from the original game v. It is easily seen that the game v' is 0-1 normalized. If we can find an element x' of the core of the game v', we can define an element $x \in \ell_1$ by

$$x^j = (v(N) - \sum_{i=1}^{\infty} v(\{i\}))x'^j + v(\{j\}).$$

On the other hand, if the inequality

$$\sum_i \delta_i v(S_i) \leq v(N)$$

holds for each balanced family $\{S_i\}$ with the balanced weights δ_i in the original game, then we can show that the similar inequality holds for the 0-1 normalized game v' as follows:

$$\begin{aligned}
\sum_i \delta_i v'(S_i) &= \frac{\sum_i \delta_i v(S_i) - \sum_i \delta_i \sum_{j\in S_i} v(\{j\})}{v(N) - \sum_{j\in N} v(\{j\})} \\
&\leq \frac{v(N) - \sum_{j\in N} \sum_{S_i \ni j} \delta_i v(\{j\})}{v(N) - \sum_{j\in N} v(\{j\})} \\
&= 1 = v'(N)
\end{aligned}$$

Now we show the main theorem of this section.

Theorem 3.1 *A game has a nonempty core if and only if*

$$\sum_i \delta_i v(S_i) \leq v(N) \tag{2}$$

holds for any balanced family $\{S_i\}$ with balanced weights δ_i.

Proof We can show that if the core of the game v is nonempty, then the inequalities (2) hold for any balanced family in Theorem 3.1 as follows: Let $\overline{x}$ be an element in the core of the game and let $\{S_i\}$ be a balanced family with the balancing weights δ_i. Then we have

$$\begin{aligned}\sum_i \delta_i v(S_i) \leq & \sum_i \delta_i \sum_{j \in S_i} \overline{x}^j \\ = & \sum_{j \in N} \sum_{S_i \ni j} \delta_i \overline{x}^j \\ = & \sum_{j \in N} \overline{x}^j \\ = & v(N)\end{aligned}$$

Hence we only need to show that the inequalities (2) for balanced families are a sufficient condition for the nonemptiness of the core. Moreover, the inequality $\sum_{j \in N} v(\{j\}) > v(N)$ never true if we assume the inequalities (2) for balanced family $\{S_i\}$ because the family $\{\,\{j\} \mid j \in N\,\}$is balanced with balancing weights 1 for all $\{j\}$. If $\sum_{j \in N} v(\{j\}) = v(N)$, then define an element x of $\ell_1$by $x^j = v(\{j\})$. For any $S \in \mathcal{F}$, consider a balanced family $\{S\} \cup \{\,\{j\} \mid j \notin S\,\}$, then we have

$$v(S) + \sum_{j \notin S} v(\{j\}) \leq v(N) = \sum_{j \in N} v(\{j\}).$$

Hence we have

$$v(S) \leq \sum_{j \in S} v(\{j\}),$$

and x belongs to the core of v. Therefore we can assume that $\sum_{j \in N} v(\{j\}) < v(N)$ without loss of generality. Now we only need to prove the following lemma in order to prove Theorem 3.1 according to the remarks mentioned just before Theorem 3.1. □

Lemma 3.1 *A 0-1 normalized game has a nonempty core if*

$$\sum_i \delta_i v(S_i) \leq 1$$

holds for any balanced family $\{S_i\}$ with balanced weights δ_i.

Proof Note that the Banach space ℓ_1 is the dual space of the Banach space c_0, and endow ℓ_1 with the weak-star topology. We always regard ℓ_1 as the linear topological space endowed with the weak-star topology in the sequel. Note that the subset $X = \{ x \in \ell_1 \mid x \geq 0, \sum_{j \in N} x^j \leq 1 \}$ of ℓ_1 is compact. Define a real-valued weak-star continuous affine functions f_S on X by

$$f_S(x) = \sum_{i \in S} x^i - v(S)$$

for each $S \in \mathcal{F}$ with $|S| \geq 2$, where $|S|$ denotes the cardinality of S. Now we shall show that, for any finite elements S_i of $\mathcal{F}$ and for any corresponding finite nonnegative numbers λ_i, there is an element x of X such that

$$\sum_i \lambda_i f_{S_i}(x) \geq 0.$$

For any S_i, its characteristic vector χ_{S_i} is obviously an element of c_0 and we have

$$\sum_i \lambda_i f_{S_i}(x) = \langle \sum_i \lambda_i \chi_{S_i}, x \rangle - \sum_i \lambda_i v(S_i)$$

for any $x \in X$. The bracket $\langle \cdot, \cdot \rangle$ denotes the dual pair between c_0 and ℓ_1. Set $\alpha = \max_j \sum_i \lambda_i \chi^j_{S_i}$, and note that $0 < \alpha$. Then we have

$$\sum \frac{\lambda_i}{\alpha} \chi_{S_i} \leq \chi_N.$$

Set $\mu_j = 1 - \sum_{S_i \ni j} \lambda_i / \alpha$ for each $j \in N$ and consider the family $\mathcal{B} = \{ S_i, \{j\} \mid j \in N \}$. Then $\mathcal{B}$ is a balanced family with balancing weights λ_i/α and μ_j. Therefore we have

$$\sum_i \frac{\lambda_i}{\alpha} v(S_i) + \sum_j \mu_j v(\{j\}) \leq 1$$

by hypothesis. Since we consider 0-1 normalized game, the inequality above becomes a simpler form

$$\sum_i \frac{\lambda_i}{\alpha} v(S_i) \leq 1,$$

or

$$\sum_i \lambda_i v(S_i) \leq \alpha.$$

Take the index j_0 where the maximum of the definition of α is attained, and choose an element x of X such that $x^{j_0} = 1$ and $x^j = 0$ for $j \neq j_0$. Then we have $\alpha = \langle \sum_i \lambda_i \chi_{S_i}, x \rangle$ and hence

$$\sum_i \lambda_i f_{S_i}(x) \geq 0.$$

Therefore, there is an element x of X such that

$$f_S(x) \geq 0$$

for all $S \in \mathcal{F}$ with $|S| \geq 2$ by virtue of the minimax theorem (cf. [2]). Since $v(\{j\}) = 0$ for all $j \in N$ and x belongs to the set X, we have $x^j \geq v(\{j\})$ for any $j \in N$. Therefore we have $\sum_{j \in S} x^j \geq v(S)$ for all $S \in \mathcal{F}$. Set $\beta = \sum_{j \in N} x^j$. If $\beta = 1$, then x belongs to the core of the game v. On the other hand, if $\beta < 1$, then set $\overline{x}^1 = x^1 + 1 - \beta$ and $\overline{x}^j = x^j$ for $j = 2, 3, \ldots$. Then, $\overline{x}$ belongs to the core of the game. □

4 Market Games

We shall apply the fundamental theorem of transferable utility games proved in the previous section to the theory of market games. Market games with finitely many players are found in [1] and our purpose of this section is to generalize the finite market games to those with infinitely many players. We shall consider an economic model consisting of infinitely many producers who produce one kind of consumption goods from finitely many kinds of production goods. We denote each producer by a natural number, and hence N denotes the set of all producers. There are m kinds of production goods, say $1, 2, \ldots, m$. Each producer j has an endowment $e_j \in R^m$, where e_j^l denotes the amount of goods l given to the producer j. We assume that the endowment e_j is nonnegative in R^m and the series $\sum_{j \in N} e_j$ converges. We denote by u_j the production function of the producer j. We assume that the domain of u_j is the nonnegative orthant of R^m and u_j is nonnegative-valued. We also denote by $\mathcal{F}$ the family of all finite subsets of N like Section 2 and 3. The sets in the family $\mathcal{F}$ represent coalitions of finitely many producers in this model.

We define the value $v(S)$ for a coalition $S \in \mathcal{F}$ as follows:

$$v(S) = \sup\{ \sum_{j \in S} u_j(x_j) \mid x_j \geq 0, \sum_{j \in S} x_j = \sum_{j \in S} e_j \}.$$

Similarly define the value $v(N)$ for the grand coalition N by

$$v(N) = \sup\{ \sum_{j \in N} u_j(x_j) \mid x_j \geq 0, \sum_{j \in N} x_j = \sum_{j \in N} e_j \}.$$

We assume that $v(N) < \infty$. Then, it is easily seen that the function v is a transferable utility game on the coalition structure $\mathcal{F}$ of N. We call this game a *market game* because it is defined be virtue of a market model in economics.

Theorem 4.1 *If all the production functions u_j in the market game are concave, then the core of the market game is nonempty.*

Proof We shall show that the inequality $\sum_i \delta_i v(S_i) \leq v(N)$ holds for any balanced family $\{S_i\} \subset \mathcal{F}$ with balancing weights δ_i. Fix $\varepsilon > 0$. For each S_i, take elements $x_{ij} \geq 0$ $(j \in S_i)$ in ℓ_1 such that

$$v(S_i) < \sum_{j \in S} u_j(x_{ij}) + \varepsilon/2^i$$

and

$$\sum_{j \in S_i} x_{ij} = \sum_{j \in S_i} e_j.$$

For each $j \in N$, define an element x_j in ℓ_1 by

$$x_j = \sum_{S_i \ni j} \delta_i x_{ij}.$$

Then, these x_j's are redistribution of the total endowment as shown in the following equation:

$$\begin{aligned}
\sum_{j \in N} x_j &= \sum_{j \in N} \sum_{S_i \in j} \delta_i x_{ij} = \sum_{i} \sum_{j \in S_i} \delta_i x_{ij} \\
&= \sum_{i} \delta_i \sum_{j \in S_i} x_{ij} = \sum_{i} \delta_i \sum_{j \in S_i} e_j \\
&= \sum_{i} \sum_{j \in S_i} \delta_i e_j = \sum_{j \in N} \sum_{S_i \ni j} \delta_i e_j \\
&= \sum_{j \in N} e_j \sum_{S_i \ni j} \delta_i = \sum_{j \in N} e_j.
\end{aligned}$$

Therefore we have the following inequalities.

$$
\begin{aligned}
v(N) &\geq \sum_{j\in N} u_j(x_j) = \sum_{j\in N} u_j(\sum_{S_i\ni j} \delta_i x_{ij}) \\
&\geq \sum_{j\in N}\sum_{S_i\ni j} \delta_i u_j(x_{ij}) = \sum_{i} \delta_i \sum_{j\in S_i} u_j(x_{ij}) \\
&> \sum_{i} \delta_i (v(S_i) - \varepsilon/2^i) = \sum_{i} \delta_i v(S_i) - \varepsilon.
\end{aligned}
$$

Since $\varepsilon > 0$ is arbitrary, we have

$$
v(N) \geq \sum_{i} \delta_i v(S_i).
$$

Therefore, the market game v has the nonempty core by virtue of Theorem 3.1. □

References

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