STABILITY ANALYSIS IN ORDER-PRESERVING SYSTEMS IN THE PRESENCE OF SYMMETRY
(対称性の入った順序保存力学系における安定性解析)

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ABSTRACT. Given an equation with certain symmetry such as symmetry with respect to rotation, translation, it is important, from the point of view of applications, to study whether or not its solutions inherit the same type of symmetry. In this note, we restrict our attention to solutions that are 'stable' in a certain sense and consider this problem. To be more precise, in the order-preserving dynamical system having a symmetry property corresponding the action of some group, we discuss the symmetry or monotonicity property of stable equilibrium points. As applications of our theory, we prove the rotational symmetry of stable equilibrium solutions and the monotonicity of stable travelling wave solutions for nonlinear diffusion equations, and the instability of stationary solutions for an evolution equation of surfaces.

1. Introduction

This note is a summary of my recent work [10] with Professor Hiroshi Matano (University of Tokyo).

Many mathematical models in physics, biology and other fields possess some kind of symmetry, such as symmetry with respect to reflection, rotation, translation, dilation, gauge transformation, and so on. Given an equation with certain symmetry, it is important, from the point of view of applications, to study whether or not its solutions inherit the same type of symmetry. As is well-known, the answer is generally negative unless we impose additional conditions on the equation or on the solutions. We will henceforth restrict our attention to solutions that are 'stable' in a certain sense and discuss the relation between stability and symmetry, or stability and some kind of monotonicity.

In the area of nonlinear diffusion equations or heat equations, early studies in this direction can be found in Casten-Holland [1], and Matano [8]. Among many other things, they showed that if a bounded domain Ω is rotationally symmetric then any stable equilibrium solution of a semilinear diffusion equation

$$u_t = \Delta u + f(u), \qquad x \in \Omega,\ t > 0$$

inherits the same symmetry. Later, it was discovered that the same result holds in a much more general framework, namely in the class of equations in which the comparison principle holds in a certain strong sense. Such a class of equations form the so-called 'strongly order-preserving dynamical systems'. Mierczyński–Poláčik [11] (for the time-continuous case) and Takáč [14] (for the time-discrete case) showed that, in a strongly order-preserving dynamical system having a symmetry property corresponding to the action of a compact connected group G, any stable equilibrium point or stable periodic point is G-invariant.

The aim of this note is to establish a theory analogous to [11] and [14] for a wider class of systems. To be more precise, we will relax the requirement that the dynamical system be strongly order-preserving. This will allow us to deal with degenerate diffusion equations and equations on an unbounded domain. Secondly, we will relax the requirement that the acting group be compact. This will allow us to discuss the symmetry or monotonicity properties with respect to translation. As applications of the results, we will prove the monotonicity property of stable travelling wave solutions for nonlinear-diffusion equations and the instability of stationary solutions for an evolution equation of surfaces.

As the space is limited, we omit the proof of our theorems. See the forthcoming paper [10] for details.

2. Notation and main results

Let X be an ordered complete metric space, that is, a complete metric space with a closed partial order relation denoted by $\preceq$. Here, we say that a partial order relation in X is closed if, for any converging sequences $\{u_n\}$, $\{v_n\} \subset X$ satisfying $u_n \preceq v_n$, $\lim_{n\to\infty} u_n \preceq \lim_{n\to\infty} v_n$ holds. We also assume that, for any u, $v \in X$, the greatest lower bound of $\{u, v\}$ —denoted by $u \wedge v$— exists and that $(u, v) \mapsto u \wedge v$ is a continuous mapping from $X \times X$ into X. We write $u \prec v$ if $u \preceq v$ and $u \neq v$, and denote by d the metric of X.

Let $\{\Phi_t\}_{t\geq 0}$ be a semigroup of mappings Φ_t from X to X satisfying the following conditions (Φ1), (Φ2), (Φ3):

(Φ1) Φ_t is order-preserving (that is, $u \preceq v$ implies $\Phi_t u \preceq \Phi_t v$ for all $u, v \in X$) for all $t \geq 0$,

(Φ2) Φ_t is upper semicontinuous (that is, if a sequence $\{u_n\}$ in X converges to a point $u_\infty \in X$ and if the corresponding sequence $\{\Phi_t u_n\}$ also converges to some point $w \in X$, then $w \preceq \Phi_t(u_\infty))$ for all $t \geq 0$, (Φ3) any bounded monotone decreasing orbit (a bounded orbit $\{\Phi_t u\}_{t\geq 0}$ satisfying $\Phi_t u \succeq \Phi_{t'} u$ for $t \leq t'$) is relatively compact.

Let G be a metrizable topological group acting on X. We say G acts on X if there exists a continuous mapping $\gamma\colon G \times X \to X$ such that $g \mapsto \gamma(g, \cdot)$ is a group homomorphism of G into $Hom(X)$, the group of homeomorphisms of X onto itself. For brevity, we write $\gamma(g, u) = gu$ and identify the element $g \in G$ with its action $\gamma(g, \cdot)$. We assume that

(G1) γ is order-preserving (that is, $u \preceq v$ implies $gu \preceq gv$ for any $g \in G$),

(G2) γ commutes with Φ_t (that is, $g\Phi_t(u) = \Phi_t(gu)$ for all $g \in G$, $u \in X$) for all $t \geq 0$.

(G3) G is connected.

In what follows, e will denote the unit element of G, and $B_\delta(e)$ the δ-neighborhood of e.

Definition 2.1. An equilibrium point $u \in X$ of $\{\Phi_t\}_{t\geq 0}$ is *lower stable* if, for any $\varepsilon > 0$, there exists some $\delta > 0$ such that

$$d(\Phi_t v, u) < \varepsilon$$

for any $t \geq 0$, $v \in X$ satisfying $v \preceq u$ and $d(v, u) < \delta$.

Remark 2.2. It is easily seen that if u is stable in the sense of Ljapunov, then it is lower stable.

Main Theorem. *Let $\overline{u}$ be an equilibrium point of $\{\Phi_t\}_{t\geq 0}$ satisfying the following conditions*: (1) *$\overline{u}$ is lower stable*; (2) *for any equilibrium point $u \prec \overline{u}$, there exists some $\delta > 0$ such that $gu \prec \overline{u}$ for any $g \in B_\delta(e)$. Then, for any $g \in G$, the inequality $g\overline{u} \succeq \overline{u}$ or $g\overline{u} \preceq \overline{u}$ holds.*

If the group G is compact, one can easily show that neither the inequality $g\overline{u} \succ \overline{u}$ nor $g\overline{u} \prec \overline{u}$ holds (see Takač [14]). Thus we have the following corollary.

Corollary 2.3. *Under the hypotheses of Main Theorem, assume further that G is a compact group. Then $\overline{u}$ is G-invariant, that is, $\overline{u}$ is symmetric.*

Now let us consider the case where G is isomorphic to the additive group $\mathbb{R}$:

$$G = \{g_a \mid a \in \mathbb{R}\}, \quad g_a + g_b = g_{a+b}.$$

Then the following holds:

Corollary 2.4. *Under the hypotheses of Main Theorem, assume further that G is isomorphic to $\mathbb{R}$. Then one of the following holds*:

(i) $\overline{u}$ *is* G*-invariant*;

(ii) $g_a\overline{u}$ *is strictly monotone increasing in* a ($a < b$ *implies* $g_a\overline{u} \prec g_b\overline{u}$);

(iii) $g_a\overline{u}$ *is strictly monotone decreasing in* a ($a < b$ *implies* $g_a\overline{u} \succ g_b\overline{u}$).

Remark 2.5. If the mapping Φ_t is strongly order-preserving for some $t > 0$ (that is, $u \prec v$ implies $\Phi_t B_\delta(u) \preceq \Phi_t B_\delta(v)$ for sufficiently small $\delta > 0$), then clearly assumption (2) in Main Theorem is automatically fulfilled.

Remark 2.6. If G is not connected, then the conclusion of Main Theorem does not necessarily hold. See [8], [9] for detail.

3. Applications I—Rotational symmetry of stable equilibria

The first example has already been discussed in Mierczyński–Poláčik [11] and Takáč [14], but in view of its importance, we summarize their results.

Let G be a connected subgroup of the rotation group $SO(n)$ and $\Omega \subset \mathbb{R}^n$ be a bounded G-invariant domain with smooth boundary $\partial\Omega$. Here we say that a domain Ω is G-invariant if $gx \in \Omega$ for all $x \in \Omega$, $g \in G$. A typical example of such a domain is a disk or an annulus in the case of $n = 2$; a ball, a spherical shell, a solid torus or any other body of rotation in the case of $n = 3$.

First let us consider an initial boundary value problem for a nonlinear diffusion equation of the form

$$\begin{cases} u_t = \Delta u + f(u), & x \in \Omega,\ t > 0, \\ u = 0, & x \in \partial\Omega,\ t > 0, \\ u(\cdot, 0) = u_0, & x \in \Omega, \end{cases} \tag{3.1}$$

where $f\colon \mathbb{R} \to \mathbb{R}$ is a C^1 function satisfying $f(0) = 0$, $f'(0) \neq 0$.

Let $X = C_0(\overline{\Omega}) = \{w \in C(\overline{\Omega}) \mid w = 0 \text{ on } \partial\Omega\}$ and $\{\Phi_t\}_{t\in[0,\infty)}$ be the semiflow that (3.1) defines in X. Then the following holds:

Theorem 3.1. *Any stable equilibrium solution* $\overline{u}$ *of* (3.1) *is* G*-invariant, that is,* $\overline{u}(gx) = \overline{u}(x)$ *for all* $x \in \Omega$, $g \in G$.

Outline of the proof. Define an order relation in X by

$$u_1 \preceq u_2 \quad \text{if} \quad u_1(x) \le u_2(x) \text{ a.e. } x \in \Omega.$$

Then, it follows from the well-known 'maximum principle' ([13]) that (Φ1) holds. Applying Corollary 2.3, we obtain this theorem. □

Next we consider an initial boundary value problem for a degenerate equation of the form

$$\begin{cases} u_t = \Delta(u^m) + f(u), & x \in \Omega,\ t > 0, \\ u = 0, & x \in \partial\Omega,\ t > 0, \\ u(\cdot, 0) = u_0, & x \in \Omega, \end{cases} \tag{3.2}$$

where Ω and f are as above, and $m > 1$ is a constant so that equation (3.2) degenerate at $u = 0$. Here we consider only bounded nonnegative solutions. Given an equilibrium solution $\overline{u}$ of (3.2), we set

$$X = \{u \in L^1(\Omega) \mid \text{for some } g \in G,\ 0 \leq \overline{u}(x) + u(x) \leq \overline{u}(gx) \ \text{ a.e. } x \in \Omega\}$$

and $\{\Phi_t\}_{t \geq 0}$ being the semiflow that (3.2) in X. Then, by the same argument as in the proof of Theorem 3.1, we have the following:

Theorem 3.2. *Any stable equilibrium solution of* (3.2) *is G-invariant.*

Finally we apply our result in Section 2 to the case where the domain Ω is not bounded. Let G be a connected subgroup of the rotation group $SO(n)$ and $\Omega \subset \mathbb{R}^n$ be a G-invariant unbounded domain with smooth boundary $\partial\Omega$. We assume that there exists a constant L such that any points $x, y \in \Omega$ can be joined by a polygonal arc contained in Ω and of length $d \leq L|x - y|$. In the case where the domain Ω is bounded, this condition is automatically satisfied. An example of such a domain is the entire space $\mathbb{R}^n$ or an infinite cylinder. Let us again consider the initial boundary value problem of the form

$$\begin{cases} u_t = \Delta u + f(u), & x \in \Omega,\ t > 0, \\ u = 0, & x \in \partial\Omega,\ t > 0, \\ u(\cdot, 0) = u_0, & x \in \Omega. \end{cases} \tag{3.3}$$

Under the additional condition that $f'(0) < 0$, we obtain the following:

Theorem 3.3. *Any stable equilibrium solution $\overline{u}$ of* (3.3) *satisfying*

$$\overline{u}(x) \to 0 \quad \textit{as} \quad |x| \to \infty$$

is G-invariant.

Here we set $X = C_0(\overline{\Omega})$. Now, by $X = C(\overline{\Omega})$, we mean the space of bounded and uniformly continuous functions on $\overline{\Omega}$.

4. Applications II—Instability of solitary waves

We apply our theory to the so-called travelling wave solutions for systems of equations of the form

$$\begin{cases} u_t = & u_{xx} + f(u, v), \qquad x \in \mathbb{R},\ t > 0, \\ v_t = & dv_{xx} + g(u, v), \qquad x \in \mathbb{R},\ t > 0, \end{cases} \tag{4.1}$$

where $d > 0$ is a constant and $f,\ g\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are C^1 functions.

Here we assume $f_v \le 0$, $g_u \le 0$ so that system (4.1) be of competition type.

A solution (u, v) of (4.1) is called a travelling wave solution with speed $c \in \mathbb{R}$ if it can be written in the form

$$(u(x,t), v(x,t)) = (\phi(x - ct), \psi(x - ct)),$$

where $\phi(y)$, $\psi(y)$ are some functions. Here we restrict our attention to the travelling wave solutions that satisfy the condition

$$\lim_{x \to \pm\infty} (u(x, 0), v(x, 0)) = (u_\pm, v_\pm),$$

where u_+, u_-, v_+ and v_- are constants. A travelling wave solution is called a solitary wave (a travelling pulse) if $(u_+, v_+) = (u_-, v_-)$, and a travelling front if $(u_+, v_+) \ne (u_-, v_-)$. We assume that $(u_\pm, v_\pm)$ are both stable equilibrium solutions of the ordinary differential equation corresponding to (4.1), namely,

$$\begin{cases} u_t = & f(u, v), \qquad t > 0, \\ v_t = & g(u, v), \qquad t > 0. \end{cases}$$

Given a travelling wave solution $(\overline{u}, \overline{v})$ with speed c, let us define a metric space X by

$$X = \{(\overline{u}(\cdot, 0) + w_1, \overline{v}(\cdot, 0) + w_2) \mid w_1, w_2 \in H^1(\mathbb{R})\}$$

and a semigroup of mappings $\{\Phi_t\}_{t \ge 0}$ by

$$\Phi_t(u(x), v(x)) = \Psi_t(u(x + ct), v(x + ct))$$

with $\{\Psi_t\}_{t \ge 0}$ being the semiflow that equation (4.1) defines in X. It is easily seen that $\{\Phi_t\}_{t \in [0,\infty)}$ is the semiflow defined by the equation

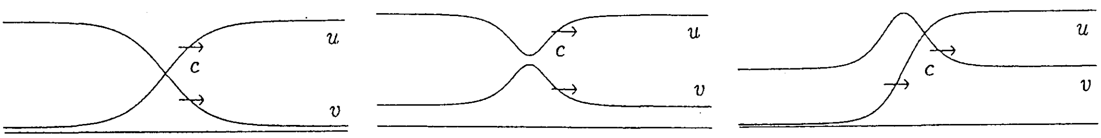
$$\begin{cases} u_t = & u_{xx} + cu_x + f(u, v), \qquad x \in \mathbb{R},\ t > 0, \\ v_t = & dv_{xx} + cv_x + g(u, v), \qquad x \in \mathbb{R},\ t > 0. \end{cases}$$

Clearly $(\overline{u}(\cdot, 0), \overline{v}(\cdot, 0))$ is an equilibrium point of $\{\Phi_t\}_{t \ge 0}$. A travelling wave solution $(\overline{u}, \overline{v})$ is called stable if $(\overline{u}(\cdot, 0), \overline{v}(\cdot, 0))$ is a stable equilibrium point of $\{\Phi_t\}_{t \ge 0}$.

We say that a travelling wave solution (u, v) is *monotone* if $u(x, 0)$ and $-v(x, 0)$ are both nonincreasing functions or both nondecreasing functions.

Theorem 4.1. *Any stable travelling wave solution of* (4.1) *is monotone.*

Corollary 4.2. *Solitary wave solutions of* (4.1) *are unstable.*



(monotone travelling wave) (solitary wave) (another type of travelling wave)

Outline of the proof of Theorem 4.1. Define an order relation in X by

$$(u_1, v_1) \preceq (u_2, v_2) \quad \text{if} \quad u_1(x) \leq u_2(x), \ v_1(x) \geq v_2(x) \text{ a.e. } x \in \mathbb{R}.$$

Letting G be the group of translations ($\cong \mathbb{R}$) and applying Corollary 2.4, we obtain this theorem. □

Remark 4.3. Results corresponding to Theorem 4.1 and Corollary 4.2 hold true if (4.1) is of cooperation type (that is, $f_v \geq 0$, $g_u \geq 0$). In this case, we call a travelling wave $(\overline{u}, \overline{v})$ *monotone* if $\overline{u}(x, 0)$ and $\overline{v}(x, 0)$ are both nondecreasing functions or both nonincreasing functions.

Remark 4.4. It is known that monotone travelling fronts are stable ([15], [5]).

Remark 4.5. The result in Theorem 4.1 is somewhat known; in the special case where $f(u, v) = u(1 - u - \gamma v)$, $g(u, v) = v(\alpha - \beta u - v)$, (4.1) is well-known as the Lotka-Volterra competition system. Under certain assumptions on coefficients α, β and γ, Kan-on [7], has proved the instability of a stationary solution $(\overline{u}, \overline{v})$ that satisfies $0 < \overline{u} < 1$, $0 < \overline{v} < \alpha$, $(u_\pm, v_\pm) = (0, \alpha)$. On the other hand, for general f, g, under the assumptions on the profile (ϕ, ψ) that there be at most a finite number of extrema (component wise), it was proved by Volpert et al. [15] that nonmonotone travelling waves are unstable. Our method is able to relax their conditions. Furthermore, our method works equally well for some degenerate diffusion equations. Degenerate equations will be discussed in a forthcoming paper ([12]).

5. Applications III—Instability of stationary surfaces

Let $\{\gamma(t)\}_{t \geq 0}$ be a family of time-dependent hypersurfaces embedded in $\mathbb{R}^n$. We assume that the motion of $\gamma(t)$ is subject to

$$V = f(\boldsymbol{n}, \nabla \boldsymbol{n}), \tag{5.1}$$

where $\boldsymbol{n} = \boldsymbol{n}(x,t)$ is the outward unit normal vector at each point of $\gamma(t)$ and V denotes the normal velocity of $\gamma(t)$ in the outward direction. A typical example of (5.1) is

$$V = \alpha(\boldsymbol{n})\kappa + g(\boldsymbol{n})$$

where $\kappa = (1/(n-1))\,\mathrm{trace}\nabla\boldsymbol{n}$ is the mean curvature at each point of $\gamma(t)$. In the case where $\alpha(\boldsymbol{n}) \equiv 1$ and $g(\boldsymbol{n}) \equiv 0$, this equation is known as the mean curvature flow equation.

We consider (5.1) in the framework of generalized solutions. The notion of such solutions was introduced by Evans and Spruck [4] and independently by Chen, Giga and Goto [2].

We assume that f is a smooth function and that equation (5.1) is strictly parabolic.

Let us define a metric space X by

$$X = \left\{ (\Gamma, D) \;\middle|\; \begin{array}{l} D \text{ is a bounded open set in } \mathbb{R}^n \text{ and} \\ \Gamma (\subset \mathbb{R}^n \setminus D) \text{ is a compact set containing } \partial D \end{array} \right\}$$

equipped with the metric d defined by

$$d((\Gamma, D), (\Gamma', D')) = h(\Gamma, \Gamma') + h(D \cup \Gamma, D' \cup \Gamma').$$

Here, for compact sets K_1 and K_2, $h(K_1, K_2)$ means the Hausdorff metric between K_1 and K_2 if $K_1, K_2 \neq \emptyset$, $h(K_1, K_2) = \infty$ if $K_1 \neq \emptyset$ and $K_2 = \emptyset$, and $h(K_1, K_2) = 0$ if $K_1, K_2 = \emptyset$. Then, define a mapping Φ_t on X by

$$\Phi_t(\Gamma, D) = (\Gamma_t, D_t),$$

where $(\Gamma_t, D_t)_{t\geq 0}$ denotes a solution of (5.1) with initial data $(\Gamma_0, D_0) = (\Gamma, D)$.

In this note, we will call a family of surfaces $\{\gamma(t)\}_{t\geq 0}$ *compact* if $\gamma(t)$ is a compact for each $t \geq 0$, and *smooth* if $\gamma(t)$ is a smooth hypersurface for each $t \geq 0$.

Theorem 5.1. *Any smooth compact stationary surface is unstable.*

Outline of the proof. Define an order relation in X by

$$(\Gamma_1, D_1) \preceq (\Gamma_2, D_2) \quad \text{if} \quad D_1 \subset D_2 \text{ and } D_1 \cup \Gamma_1 \subset D_2 \cup \Gamma_2.$$

Letting G be the group of translations and applying Main Theorem, we obtain this theorem. □

Remark 5.2. Giga and Yama-uchi [6], Ei and Yanagida [3] have shown the above result by using methods different from ours. However, unlike their methods which depend on linearization arguments or distant function arguments (thus smoothness assumptions are essential), our method may be extendable to generalized solutions of (5.1) if one is able to check condition (2) of Main Theorem holds for generalized solutions (which remains to be checked).

Remark 5.3. With minor modifications, most of the results in Section 2 carry over to time-discrete systems. Thus the results in Theorems 3.1–5.1 can be extended to nonautonomous equations (equations that are periodic in t). For example, an analogy of Theorem 5.1 holds for periodic solutions of

$$V = f(n, \nabla n, t) \qquad (f \text{ is periodic in } t).$$

REFERENCES

[1] R. G. Casten and C. J. Holland, Instability results for reaction diffusion equations with Neumann boundary conditions, J. Differential Equations **27** (1978), 266-273.

[2] Y-G. Chen, Y. Giga and S. Goto, Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations, J. Diff. Geometry, **33** (1991), 749–786.

[3] S. Ei and E. Yanagida, Stability of stationary interfaces in a generalized mean curvature flow, J. Fac. Sci. , Univ. Tokyo, Sec. IA **40** (1994), 651–661.

[4] L. C. Evans and J. Spruck, Motion by level sets by mean curvature, I, J. Diff. Geometry, **33** (1991), 635–681.

[5] Q. Fang and Y. Kan-on, Stability of monotone travelling waves for competition model with diffusion, in preparation.

[6] Y. Giga and K. Yama-uchi, On instability of evolving hypersurfaces, Diff. Integral Equations, **7** (1994), 863–872.

[7] Y. Kan-on, Instability of stationary solution for Lotka-Volterra

[8] H. Matano, Asymptotic behavior and stability of solutions of semilinear diffusion equations, Publ. RIMS, Kyoto Univ., **15** (1979), 401–454.

[9] H. Matano and M. Mimura, Pattern formation in competition-diffusion systems in nonconvex domains, Publ. Res. Inst. Math. Sci., **19** (1983), 1049–1079.

[10] H. Matano and T. Ogiwara, Stability analysis in order-preserving systems in the presence of symmetry, preprint.

[11] J. Mierczyński and P. Poláčik, Group actions on strongly monotone dynamical systems, Math. Ann., **283** (1989), 1–11.

[12] T. Ogiwara, Monotonicity properties of stable travelling wave solusions for some degenerate diffusion equations, in preparation.

[13] H. Protter and H. Weinberger, Maximum principles in differential equations, Prentice Hall, Englewood Cliffs, NJ, 1967.

[14] P. Takáč, Asymptotic behavior of strongly monotone time-periodic dynamical process with symmetry, J. Diff. Equations, **100** (1992), 355-378.

[15] A. I. Volpert, Vit. A. Volpert and Vl. A. Volpert, Traveling wave solutions of parabolic systems, Trans. Math. Monographs, **140**, Amer. Math. Soc., Providence, 1994.