Layer potentials for a bounded domain with fractal boundary

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Abstract

For a bounded domain with fractal boundary we define the double layer potentials of Hölder continuous functions and functions in a Besov space on the boundary and investigate the boundary behavior of the double layer potentials.

1. Introduction

Let D be a bounded domain in R^d with fractal boundary. We say that a domain D has a fractal boundary if the Hausdorff dimension β of ∂D is greater than $d-1$. There are many Jordan domains which have fractal boundaries. A typical example is the von Koch snowflake in R^2. In R^d $(d \geq 3)$ we can also construct many domains with fractal boundary using finite similitudes (cf. [Hu]).

We consider double layer potentials for these domains D. A double layer potential is an useful concept mathematically as well as physically. For example, it is well-known that the Dirichlet and Neumann problems for the Laplacian in a smooth domain can be solved by using double layer potentials. Let D be a bounded $C^{1,\alpha}$-domain in R^d. Recall that the double layer potential Φg of $g \in L^p(\partial D)$ is defined by

$$\Phi g(x) = -\int_{\partial D} \langle \nabla_y N(x-y), n_y \rangle g(y) d\sigma(y), \tag{1.1}$$

where $N(x-y)$ is the Newton kernel and n_y is the unit outer normal to ∂D. Furthermore if g is a C^1-function with compact support, then we see by the Green formula that

$$\Phi g(x) = \int_{\mathrm{R}^d \setminus \overline{D}} \langle \nabla_y g(y), \nabla_y N(x-y) \rangle dy \tag{1.2}$$

for $x \in D$ and

$$\Phi g(x) = -\int_{D} \langle \nabla_y g(y), \nabla_y N(x-y) \rangle dy \tag{1.3}$$

for $x \in \mathrm{R}^d \setminus \overline{D}$.

On the other hand if D is a domain with fractal boundary, then the integral in (1.1) can not be considered. But the integrals in (1.2) and (1.3) may be defined for sufficiently smooth functions g on R^d.

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J. Harrison and A. Norton introduced an abstract line integal over a fractal curve in the plane in [HN1] and a surface integral over a fractal surface in R^d in [HN]. In their theory the box dimension of the boundary plays an important role.

In [W] we considered double layer potentials for a bounded domain D in R^d such that ∂D has the following property (u).

(u) There are a positive Radon measure μ on ∂D and positive real numbers β, γ, r_0, b_1, b_2 such that $d-1 \leq \gamma \leq \beta < d$ and

$$b_1 r^\beta \leq \mu(B(z,r) \cap \partial D) \leq b_2 r^\gamma$$

for all $z \in \partial D$ and all $r \leq r_0$, where $B(z,r)$ stands for the open ball with center z and radius r in R^d.

We also investigated the boundary behavior of those double layer potentials.

In this paper we will study the further boundary behavior of layer potentials for a bounded domain D in R^d such that ∂D is a β-set ($d-1 \leq \beta < d$). According to [JW] we say that that a closed set F is a β-set if there are a positive Radon measure μ on F and positive real numbers, r_0, b_1, b_2 such that

$$b_1 r^\beta \leq \mu(B(z,r) \cap F) \leq b_2 r^\beta \tag{1.4}$$

for all $z \in F$ and all $r \leq r_0$.

We note that, if D is a bounded Lipschitz domain, then the boundary of D is a $(d-1)$-set with respect to the surface measure. Further if the boundary of D consists of finite self-similar sets, which satisfy the open set condition and whose similarlity dimension are β, then the boundary D is a β-set with respect to the β-dimensional Hausdorff measure (cf. [Hu]).

Under these conditions we will define the double layer potential of a function defined on ∂D. To do so, let $0 < \alpha \leq 1$ and F be a closed subset of R^d. We denote by $\Lambda_\alpha(F)$ the Banach space of all bounded α-Hölder continuous real-valued functions on F with norm

$$\|f\|_{\Lambda_\alpha(F)} = \sup\{|f(z)| : z \in F\} + \sup\{\frac{|f(z)-f(w)|}{|z-w|^\alpha} : z,\ w \in F, z \neq w\}.$$

In [S, Theorem 3 in Chapter 6] it is shown that there exists a linear bounded extension operator $\mathcal{E}_0$ from $\Lambda_\alpha(F)$ to $\Lambda_\alpha(\mathrm{R}^d)$ (cf. [S, Theorem 3 on p.174]). Multiplying $\mathcal{E}_0(f)$ by a fixed smooth function ϕ_0 such that $\phi_0 = 1$ on $B(0,R)$ and supp $\phi_0 \subset B(0,2R)$ we have

Theorem A. *Let $0 < \alpha \leq 1$ and F be a compact subset of R^d satisfying $F \subset B(O, \frac{R}{2})$. Then there exists a bounded linear operator $\mathcal{E}$ from $\Lambda_\alpha(F)$ to $\Lambda_\alpha(\mathrm{R}^d)$ such that supp $\mathcal{E}(f) \subset B(O,2R)$, $\mathcal{E}(f) = f$ on F and the restriction of $\mathcal{E}(f)$ to the complement of F is a C^∞-function satisfying*

$$|\nabla\mathcal{E}(f)(x)| \leq c\,dist(x,F)^{\alpha-1}\|f\|_{\Lambda_\alpha(F)},\ |\frac{\partial^2\mathcal{E}(f)}{\partial x_j \partial x_k}(x)| \leq c\,dist(x,F)^{\alpha-2}\|f\|_{\Lambda_\alpha(F)}$$

for all $x \in \mathrm{R}^d \setminus F$, *where* c *is a constant independent of* x *and* f, *and* $dist(x, A)$ *stands for the distance of* x *from* A.

Let $d \geq 2$ and $0 \leq \beta - (d-1) < \alpha < 1$. We define *the double layer potential* of $f \in \Lambda_\alpha(\partial D)$ by

$$\Phi f(x) = \int_{\mathrm{R}^d \setminus \overline{D}} \langle \nabla_y \mathcal{E}(f)(y), \nabla_y N(x-y) \rangle dy \tag{1.5}$$

for $x \in D$ and

$$\Phi f(x) = -\int_D \langle \nabla_y \mathcal{E}(f)(y), \nabla_y N(x-y) \rangle dy \tag{1.6}$$

for $x \in \mathrm{R}^d \setminus \overline{D}$, where

$$N(x-y) = \begin{cases} \frac{1}{\omega_d (d-2)|x-y|^{d-2}} & \text{if } d \geq 3 \\ -\frac{3R}{2\pi} \log \frac{|x-y|}{3R} & \text{if } d = 2 \end{cases}$$

and ω_d stands for the surface area of the unit ball in R^d.

We will prove the following theorem in §3.

Theorem 1. *Suppose* D *is a bounded domain in* R^d $(d \geq 2)$ *such that* ∂D *is a* β*-set. Furthermore, assume that* $0 \leq \beta - (d-1) < \alpha < 1$. *Then for every* $f \in \Lambda_\alpha(\partial D)$ Φf *is harmonic in* $\mathrm{R}^d \setminus \partial D$ *and for every* $z \in \partial D$

$$\lim_{x \to z, x \in D} \Phi f(x) = Kf(z) + \frac{f(z)}{2}$$

and

$$\lim_{x \to z, x \in \mathrm{R}^d \setminus \overline{D}} \Phi f(x) = Kf(z) - \frac{f(z)}{2},$$

where K *is a bounded operator from* $\Lambda_\alpha(\partial D)$ *to* $\Lambda_\alpha(\partial D)$ *defined by*

$$\begin{aligned} Kf(z) = &\frac{1}{2} \int_{\mathrm{R}^d \setminus \overline{D}} \langle \nabla_y \mathcal{E}(f)(y), \nabla_y N(z-y) \rangle dy \\ &- \frac{1}{2} \int_D \langle \nabla_y \mathcal{E}(f)(y), \nabla_y N(z-y) \rangle dy. \end{aligned} \tag{1.7}$$

We next consider a Besov space on ∂D. More generally, let $p \geq 1$, $0 < \alpha \leq 1$ and F be a closed set satisfying (1.4). We denote by the space $\Lambda_\alpha^p(\mu)$ of all μ-measurable functions in $L^p(\mu)$ such that

$$\iint \frac{|f(x) - f(y)|^p}{|x-y|^{\beta + p\alpha}} d\mu(x) d\mu(y) < \infty,$$

and define the norm of a function $f \in \Lambda^p_\alpha(\mu)$ by

$$\|f\|_{p,\alpha} = \left(\int |f(x)|^p d\mu(x) + \iint \frac{|f(x)-f(y)|^p}{|x-y|^{\beta+p\alpha}} d\mu(x)d\mu(y)\right)^{1/p}.$$

Let $1 \geq \alpha > \beta-(d-1) \geq 0$ and $f \in \Lambda^p_\alpha(\mu)$. Using an extention operator $\mathcal{E}$ similar to that in [JW], we will define the double layer potential Φf of f by (1.5) and (1.6).

Furthermore let $z \in \partial D$ and τ be a positive real number. The nontangential approach regions at z are defined as follows:

$$\Gamma_\tau(z) = \{x \in D : |x-z| < (1+\tau)\mathrm{dist}\,(x, \partial D)\}$$

and

$$\Gamma^e_\tau(z) = \{x \in \mathrm{R}^d \setminus \overline{D} : |x-z| < (1+\tau)\mathrm{dist}\,(x, \partial D)\}.$$

In §4 we will sketch the following theorem.

Theorem 2. *Suppose D is a bounded domain in R^d $(d \geq 2)$ such that ∂D is a β-set, and assume that $0 \leq \beta-(d-1) < \alpha < 1$ and $p > 1$. Furthermore assume that $\Gamma_\tau(z) \cap B(z,r) \neq$ set and $\Gamma^e_\tau(z) \cap B(z,r) \neq \emptyset$ for μ-a.e. $z \in \partial D$ and for every $r \leq \epsilon_0$. Then for every $f \in \Lambda^p_\alpha(\mu)$ Φf is harmonic in $\mathrm{R}^d \setminus \partial D$ and*

$$\lim_{x\to z, x\in\Gamma_\tau(\mu)} \Phi f(x) = Kf(z) + \frac{f(z)}{2}$$

and

$$\lim_{x\to z, x\in\Gamma^e_\tau(\mu)} \Phi f(x) = Kf(z) - \frac{f(z)}{2}$$

at μ-a.e. $z \in \partial D$.

2. Fundamental lemmas

Hearafter we assume that D is a bounded domain in R^d such that ∂D is a β-set $(d-1 \leq \beta < d)$, and fix a positive Radon measure μ on ∂D satisfying (1.4) for $F = \partial D$. Further fix a positive real number R satisfying $\overline{D} \subset B(O, R/2)$. We may assume that (1.4) holds for $r_0 = 12R$ and $F = \partial D$. The following fundamental lemma was obtained in [W, Lemma 2.2] using a covering lemma.

Lemma B. *Let $0 \leq \beta-(d-1) < \alpha < 1$ and $R_0 > 0$. Then there exists a constant c such that*

$$\int_{B(z,r)} dist(x, \partial D)^{\alpha-1} dx \leq cr^{d-1+\alpha}$$

for all $z \in \partial D$ and all positive real number $r \leq R_0$.

Lemma 2.1. *Let $k > 0$ and $0 \leq \beta - (d-1) < \alpha < 1$.*

(i) *If $d + \alpha - 1 - k > 0$, then*

$$\int_{B(z,r)} dist(x, \partial D)^{\alpha-1} |x - z|^{-k} dx \leq c r^{d+\alpha-1-k}$$

for all $r \leq 4R$ and $z \in \partial D$.

(ii) *If $d + \alpha - 1 - k < 0$, then*

$$\int_{\mathbf{R}^d \setminus B(z,r)} dist(x, \partial D)^{\alpha-1} |x - z|^{-k} dx \leq c r^{d+\alpha-1-k}$$

for all $r > 0$ and $z \in \partial D$.

Proof. (i) Set

$$F_n = \{x \in B(z,r) : |x - z|^{-k} > 2^n\}$$

and $r_n = 2^{-n/k}$. On account of Lemma B we have

$$\int_{B(z,r)} \text{dist}(x, \partial D)^{\alpha-1} |x - z|^{-k} dx \leq \sum_{n=m}^{\infty} 2^n \int_{B(z,r_n)} \text{dist}(x, \partial D)^{\alpha-1} dx$$

$$\leq c_1 \sum_{n=m}^{\infty} 2^n r_n^{d-1+\alpha} \leq c_1 \sum_{n=m}^{\infty} 2^{((k-d+1-\alpha)/k)n},$$

where m is the integer satisfying $r_m \geq r > r_{m+1}$. Since $k - d + 1 - \alpha < 0$, we have the conclusion (i).

(ii) Similarly we have

$$\int_{\mathbf{R}^d \setminus B(z,r)} \text{dist}(x, \partial D)^{\alpha-1} |x - z|^{-k} dx$$

$$= \int_{(\mathbf{R}^d \setminus B(z,r)) \cap \overline{B(z,4R)}} + \int_{\mathbf{R}^d \setminus (B(z,r) \cup \overline{B(z,4R)})}$$

$$\leq c_2 \sum_{n=-\infty}^{l} 2^{((k-d+1-\alpha)/k)n} + c_3 \int_{|x-z|>4R} |x - z|^{\alpha-1-k} dx,$$

where l is the integer satisfying $r_l \leq r < r_{l-1}$. Since $k - d + 1 - \alpha > 0$, we also have (ii).

□

Using Lemma 2.1, (i), we can show the following lemma (cf. [W, Lemma 2.4]).

Lemma 2.2. *Let $r > 0$, $0 \leq \beta - (d-1) < \alpha < 1$ and $d + \alpha - 1 - k > 0$. Then the function*

$$x \mapsto \int_{B(O,r)} dist(y, \partial D)^{\alpha-1} |x - y|^{-k} dy$$

is bounded on $\mathbf{R}^d$.

3. Properties of double layer potentials

We first show that for $f \in \Lambda_\alpha(\partial D)$ the double layer potential Φf is defined in $\mathbf{R}^d \setminus \partial D$.

Lemma 3.1. *Let* $1 > \alpha > \beta - (d-1) \geq 0$ *and* $f \in \Lambda_\alpha(\partial D)$. *Then the double layer potential* Φf *defined by (1.5) and (1.6) is harmonic in* $\mathbf{R}^d \setminus \partial D$.

Proof. Let F be a compact subset of $\mathbf{R}^d \setminus \overline{D}$. Since Theore A yields

$$|\langle \nabla_y \mathcal{E}(f)(y), \nabla_y N(x-y)\rangle| \leq c_1 \|f\|_{\Lambda_\alpha(\partial D)} \mathrm{dist}(y, \partial D)^{\alpha-1} \mathrm{dist}\,(F, \overline{D})^{1-d}$$

and

$$|\langle \nabla_y \mathcal{E}(f)(y), \nabla_y \Delta_x N(x-y)\rangle| \leq c_1 \|f\|_{\Lambda_\alpha(\partial D)} \mathrm{dist}(y, \partial D)^{\alpha-1} \mathrm{dist}\,(F, \overline{D})^{-1-d}$$

for $x \in F$ and $y \in D$, we see by Lemma B that

$$\int_D \langle \nabla_y \mathcal{E}(f)(y), \nabla_y N(x-y)\rangle dy$$

converges for $x \in \mathbf{R}^d \setminus \overline{D}$ and Φf is harmonic in $\mathbf{R}^d \setminus \overline{D}$.

Analogously we can show that Φf is harmonic in D. □

Lemma 3.2 Let $0 \leq \beta - (d-1) < \alpha < 1$. *Then the operator* K *defined by (1.7) is bounded on* $\Lambda_\alpha(\partial D)$.

Proof. For $f \in \Lambda_\alpha(\partial D)$ and $z \in \partial D$ we define

$$K_1 f(z) = \int_{\mathbf{R}^d \setminus \overline{D}} \langle \nabla_y \mathcal{E}(f)(y), \nabla_y N(z-y)\rangle dy$$

and

$$K_2 f(z) = -\int_D \langle \nabla_y \mathcal{E}(f)(y), \nabla_y N(z-y)\rangle dy.$$

To see that K is bounded, it suffices to show that K_j $(j = 1, 2)$ are bounded. To do so, let $f \in \Lambda_\alpha(\partial D)$. Then we have, by Theorem A,

$$\begin{aligned} |K_1 f(z)| &\leq \int_{\mathbf{R}^d \setminus \overline{D}} |\nabla \mathcal{E}(f)(y)||\nabla_y N(z-y)| dy \\ &\leq c_1 \|f\|_{\Lambda_\alpha(\partial D)} \int_{B(0,2R)\setminus \overline{D}} \mathrm{dist}(y, \partial D)^{\alpha-1} |z-y|^{1-d} dy \\ &\leq c_1 \|f\|_{\Lambda_\alpha(\partial D)} \int_{B(z,3R)\setminus \overline{D}} \mathrm{dist}(y, \partial D)^{\alpha-1} |z-y|^{1-d} dy. \end{aligned}$$

Noting that $d-1+\alpha > d-1$ and using Lemma 2.1, we conclude that

$$|K_1 f(z)| \le c_2 \|f\|_{\Lambda_\alpha(\partial D)} R^\alpha \text{ for every } z \in \partial D.$$

Next, let z, $w \in \partial D$. We write

$$\begin{aligned}|K_1 f(z) - K_1 f(w)| &\le \int_{\mathbf{R}^d \setminus \overline{D}} |\nabla \mathcal{E}(f)(y)||\nabla_y N(z-y) - \nabla_y N(w-y)| dy \\ &= \int_A + \int_B \equiv I_1(z,w) + I_2(z,w),\end{aligned}$$

where $A = \{y \in B(O,2R) \setminus \overline{D} : |z-y| \le 3|z-w|\}$ and $B = \{y \in B(O,2R) \setminus \overline{D} : |z-y| > 3|z-w|\}$.

Take $\epsilon > 0$ satisfying $\alpha - \epsilon > 0$. On account of Theorem A and Lemma 2.1 we obtain

$$\begin{aligned}&I_1(z,w) \le c_3 \|f\|_{\Lambda_\alpha(\partial D)} |z-w|^{\alpha-\epsilon} \\ &\times \int_A \operatorname{dist}(y, \partial D)^{\alpha-1} \left(|z-y|^{1-d-\alpha+\epsilon} + |w-y|^{1-d-\alpha+\epsilon}\right) dy \\ &\le c_3 \|f\|_{\Lambda_\alpha(\partial D)} |z-w|^{\alpha-\epsilon} \int_{|z-y| \le 3|z-w|} \operatorname{dist}(y, \partial D)^{\alpha-1} |z-y|^{1-d-\alpha+\epsilon} dy \\ &+ c_3 \|f\|_{\Lambda_\alpha(\partial D)} |z-w|^{\alpha-\epsilon} \int_{|w-y| \le 4|z-w|} \operatorname{dist}(y, \partial D)^{\alpha-1} |w-y|^{1-d-\alpha+\epsilon} dy \\ &= c_4 \|f\|_{\Lambda_\alpha(\partial D)} |z-w|^\alpha.\end{aligned}$$

We next estimate $I_2(z,w)$. Using Theorem A again, we have

$$\begin{aligned}&I_2(z,w) \\ &\le c_5 \|f\|_{\Lambda_\alpha(\partial D)} |z-w| \int_B \operatorname{dist}(y, \partial D)^{\alpha-1} \left(|z-y|^{-d} + |w-y|^{-d}\right) dy \\ &\le c_5 \|f\|_{\Lambda_\alpha(\partial D)} |z-w| \int_{|z-y| > 3|z-w|} \operatorname{dist}(y, \partial D)^{\alpha-1} |z-y|^{-d} dy \\ &+ \le c_5 \|f\|_{\Lambda_\alpha(\partial D)} |z-w| \int_{|z-y| > 2|z-w|} \operatorname{dist}(y, \partial D)^{\alpha-1} |w-y|^{-d} dy,\end{aligned}$$

whence, together with Lemma 2.1,

$$I_2(z,w) \le c_6 \|f\|_{\Lambda_\alpha(\partial D)} |z-w||z-w|^{\alpha-1} \le c_6 \|f\|_{\Lambda_\alpha(\partial D)} |z-w|^\alpha.$$

Therefore we have

$$|K_1 f(z) - K_1 f(w)| \le c_7 \|f\|_{\Lambda_\alpha(\partial D)} |z-w|^\alpha$$

for every z, $w \in \partial D$.

Analogusly we can estimate $K_2 f$.

□

To prove our theorem, we use the Whitney decomposition. More precisely, let G be an open set in R^d. A cube Q is called a k-cube if it is of the form

$$[l_1 2^{-k}, l_1 + 2^{-k}] \times \cdots \times [l_d 2^{-k}, l_d + 2^{-k}],$$

where k, $l_1, \cdots, l_d$ are integers. We denote by $\mathcal{W}_k(G)$ the family of all k-cubes in G and set $\mathcal{W}(G) = \sum_{k=-\infty}^{\infty} \mathcal{W}_k(G)$. The following theorem is well-known (cf. [S, Theorem 1 in Chapter 6]).

Theorem C. *Let G be an open set in R^d. Then there exists a family $\mathcal{V}(G) = \{Q_j\}$ of cubes in $\mathcal{W}(G)$ having the following properties:*

(i) $\sum_j Q_j = G$,

(ii) $\mathrm{int}\, Q_j \cap \mathrm{int}\, Q_k = \emptyset$ $(j \neq k)$,

(iii) $\mathrm{diam}\, Q_j \leq \mathrm{dist}\,(Q_j, \mathrm{R}^d \setminus G) \leq 4 \mathrm{diam}\, Q_j$,

where $\mathrm{int}\, A$ *and* $\mathrm{diam}\, A$ *stand for the interior of A and the diameter of A, respectively.*

Lemma 3.3. *Let $0 \leq \beta - (d-1) < \alpha < 1$ and $f \in \Lambda_\alpha(\partial D)$. Then*

$$\begin{aligned} (3.1) \qquad & \int_{\mathrm{R}^d \setminus \overline{D}} \langle \nabla \mathcal{E}(f)(y), \nabla_y N(x-y) \rangle dy \\ & = - \int_D \langle \nabla \mathcal{E}(f)(y), \nabla_y N(x-y) \rangle dy + \mathcal{E}(f)(x) \end{aligned}$$

for every $x \in \mathrm{R}^d$.

Proof. To show (3.1), let $x \in \mathrm{R}^d \setminus D$. We denote by $\mathcal{V}_k(D)$ the union of k-cubes in $\mathcal{V}(D)$ and set

$$A_n = \cup_{k=-\infty}^{n} \cup_{Q \in \mathcal{V}_k(D)} Q.$$

We take a family $\{v_m\}$ of mollifiers on R^d such that $\mathrm{supp}\, v_m \subset B(0, 1/m)$, and set $g_m = \mathcal{E}(f) * v_m$. Then g_m is a C^1-function on R^d. The Green formula yields, for a sufficient small number $\delta > 0$ and for a large number r,

(3.2)

$$\begin{aligned} & \int_{B(O,r) \setminus (A_n \cup \overline{B(x,\delta)})} \langle \nabla g_m(y), \nabla_y N(x-y) \rangle dy \\ & = \int_{|y|=r} g_m(y) \langle \nabla_y N(x-y), n_y \rangle d\sigma(y) \\ & - \int_{\partial A_n} g_m(y) \langle \nabla_y N(x-y), n_y \rangle d\sigma(y) - \int_{|x-y|=\delta} g_m(y) \langle \nabla_y N(x-y), n_y \rangle d\sigma(y). \end{aligned}$$

Using the Green formula again, we have

$$(3.3)\quad -\int_{\partial A_n} g_m(y)\langle \nabla_y N(x-y), n_y\rangle d\sigma(y) = -\int_{\operatorname{int} A_n} \langle \nabla g_m(y), \nabla_y N(x-y)\rangle dy.$$

As $r \to \infty$ and $\delta \to 0$, we deduce from (3.2) and (3.3)

$$\int_{\mathbf{R}\setminus A_n} \langle \nabla g_m(y), \nabla_y N(x-y)\rangle dy = -\int_{\operatorname{int} A_n} \langle \nabla g_m(y), \nabla_y N(x-y)\rangle dy + g_m(x)$$

As $n \to \infty$, we have

$$(3.4)\quad \int_{\mathbf{R}^d\setminus D} \langle \nabla g_m(y), \nabla_y N(x-y)\rangle dy = -\int_D \langle \nabla g_m(y), \nabla_y N(x-y)\rangle dy + g_m(x).$$

We claim that

$$\int_{\mathbf{R}^d\setminus D} \langle \nabla g_m(y), \nabla_y N(x-y)\rangle dy \to \int_{\mathbf{R}^d\setminus D} \langle \nabla \mathcal{E}(f)(y), \nabla_y N(x-y)\rangle dy$$

as $m \to \infty$.

To show the claim we write

$$\begin{aligned}&\int_{\mathbf{R}^d\setminus \overline{D}} |\nabla(g_m(y) - \mathcal{E}(f)(y))||\nabla_y N(x-y)|dy\\ &= \int_{\operatorname{dist}(y,\partial D)\le 2/m} + \int_{\operatorname{dist}(y,\partial D)> 2/m} \equiv I_1(x) + I_2(x)\end{aligned}$$

and

$$\begin{aligned}I_1(x) &\le \int_{\mathbf{R}^d\setminus \overline{D}} |\nabla(g_m(y)||\nabla_y N(x-y)|dy\\ &+ \int_{\mathbf{R}^d\setminus \overline{D}} |\nabla \mathcal{E}(f)(y)||\nabla_y N(x-y)|dy \equiv I_{11}(x) + I_{12}(x).\end{aligned}$$

We first estimate $I_{11}(x)$. We choose $\epsilon > 0$ satisfying $\alpha - \epsilon > \beta - (d-1)$. Noting that

$$\begin{aligned}|\nabla_y g_m(y)| &\le c_1 \|f\|_{\Lambda_\alpha(\partial D)} \int \operatorname{dist}(y-z, \partial D)^{\alpha-1} v_m(z)dz\\ &\le c_1 \|f\|_{\Lambda_\alpha(\partial D)} \int_{B(w,3/m)} \operatorname{dist}(u, \partial D)^{\alpha-1} v_m(y-u)du,\end{aligned}$$

where w is a point on ∂D such that $\operatorname{dist}(y, \partial D) = |y-w|$. Hence, together with Lemmas B and 2.2,

$$\begin{aligned}I_{11}(x) &\le c_2 \|f\|_{\Lambda_\alpha(\partial D)} \left(\frac{3}{m}\right)^{\alpha-1} \int_{\operatorname{dist}(y,\partial D)\le 2/m} |x-y|^{1-d} dy\\ &\le c_3 \|f\|_{\Lambda_\alpha(\partial D)} m^{-\epsilon} \int_{B(O,R+1)} \operatorname{dist}(y, \partial D)^{\alpha-1-\epsilon} |x-y|^{1-d} dy\\ &\le c_4 \|f\|_{\Lambda_\alpha(\partial D)} m^{-\epsilon}.\end{aligned}$$

Using Lemma 2.2 again, we also have

$$\begin{aligned} I_{12}(x) &\le c_5 \|f\|_{\Lambda_\alpha(\partial D)} m^{-\epsilon} \int_{B(O,R+1)} \operatorname{dist}(y,\partial D)^{\alpha-1-\epsilon}|x-y|^{1-d}dy \\ &\le c_6 \|f\|_{\Lambda_\alpha(\partial D)} m^{-\epsilon}. \end{aligned}$$

Thus we see that $I_1(x) \to 0$ as $m \to \infty$.

We next estimate $I_2(x)$. To do so, suppose $\operatorname{dist}(y, \partial D) > 2/m$. Noting that

$$\left|\frac{\partial^2 \mathcal{E}(f)}{\partial y_j \partial y_k}(y)\right| \le c_7 \|f\|_{\Lambda_\alpha(\partial D)} \operatorname{dist}(y,\partial D)^{\alpha-2}$$

by Lemma A, we have

$$\begin{aligned} &\left|\frac{\partial g_m}{\partial y_j}(y) - \frac{\partial \mathcal{E}(f)}{\partial y_j}(y)\right| \\ &\le c_8 \|f\|_{\Lambda_\alpha(\partial D)} \int \left|\frac{\partial \mathcal{E}(f)}{\partial y_j}(y-z) - \frac{\partial \mathcal{E}(f)}{\partial y_j}(y)\right| v_m(z)dz \\ &\le c_9 \|f\|_{\Lambda_\alpha(\partial D)} \frac{1}{m} \operatorname{dist}(y,\partial D)^{\alpha-2}, \end{aligned}$$

whence, by Lemma 2.2,

$$\begin{aligned} I_2(x) &\le c_{10} \|f\|_{\Lambda_\alpha(\partial D)} \frac{1}{m} \\ &\quad \times \int_{\{\operatorname{dist}(y,\partial D) > 2/m\} \cap B(O,2R+1)} \operatorname{dist}(y,\partial D)^{\alpha-2}|x-y|^{1-d}dy \\ &\le c_{11} \|f\|_{\Lambda_\alpha(\partial D)} m^{-\epsilon} \int_{B(O,2R+1)} \operatorname{dist}(y,\partial D)^{\alpha-1-\epsilon}|x-y|^{1-d}dy \\ &\le c_{12} \|f\|_{\Lambda_\alpha(\partial D)} m^{-\epsilon}. \end{aligned}$$

Therefore we also see that $I_2(x) \to 0$ as $m \to \infty$. Thus we see that the claim is true.

Similarly we can show that

$$\int_D \langle \nabla g_m(y), \nabla_y N(x-y)\rangle dy \to \int_D \langle \nabla \mathcal{E}(f)(y), \nabla_y N(x-y)\rangle dy$$

as $m \to \infty$.

As $m \to \infty$ in (3.4), we obtain (3.1) for every $x \in \mathrm{R}^d \setminus D$.

We can show that (3.1) holds for every $x \in D$, by using $\mathcal{V}(\mathrm{R}^d \setminus \overline{D})$. we can show (3.1) for every $x \in D$. □

Proof of Theorem 1. Let $f \in \Lambda_\alpha(\partial D)$. In [W, Theorem] we proved that

$$\lim_{x \to z, x \in D} \Phi f(x) = \int_{\mathrm{R}^d \setminus \overline{D}} \langle \nabla \mathcal{E}(f)(y), \nabla_y N(z-y)\rangle dy$$

and

$$\lim_{x\to z, x\in \mathbf{R}^d\setminus\overline{D}} \Phi f(x) = -\int_D \langle \nabla\mathcal{E}(f)(y), \nabla_y N(z-y)\rangle dy$$

for every $z \in \partial D$. Using Lemma 3.3 we see that

$$\int_{\mathbf{R}^d\setminus\overline{D}} \langle \nabla\mathcal{E}(f)(y), \nabla_y N(z-y)\rangle dy = Kf(z) + \frac{f(z)}{2}$$

and

$$-\int_D \langle \nabla\mathcal{E}(f)(y), \nabla_y N(z-y)\rangle dy = Kf(z) - \frac{f(z)}{2}.$$

Therefore we have the conclusion.

□

4. Layer potentials of functions in a Besov space

Let $p \geq 1$ and μ be a measure satisfying (1.4). To extend functions in $L^p(\mu)$ to be functions on $\mathbf{R}^d$, we use the Whitney decomposition. Fix a positive real number η satisfying $\eta < 1/4$ and choose a C^∞-function ϕ on $\mathbf{R}^d$ such that

$$\phi = 1 \text{ on } Q_0, \quad \operatorname{supp}\phi \subset (1+\eta)Q_0, \quad 0 \leq \phi \leq 1,$$

where Q_0 is the closed cube of unit length centered at origin and $(1+\eta)Q_0$ stands for the set $\{(1+\eta)x : x \in Q_0\}$.

We simply denote by $\mathcal{V} = \{Q_j\}$ the family $\mathcal{V}(\mathbf{R}^d \setminus \partial D)$. Further let $q^{(j)}$, l_j be the center of Q_j and the common length of its side, respectively. For each j pick a point $a^{(j)} \in \partial D$ satisfying $\operatorname{dist}(\partial D, Q_j) = \operatorname{dist}(a^{(j)}, Q_j)$ and fix it. Set

$$t(x) = \sum_j \phi(\frac{x-q^{(j)}}{l_j}) \quad \text{and} \quad \phi_j^*(x) = \frac{\phi((x-q^{(j)})/l_j)}{t(x)}.$$

We define, for $f \in L^p(\mu)$,

$$\mathcal{E}_0(f)(x) = \sum_j \frac{1}{\mu(B(a^{(j)}, \eta_j))}\left(\int_{B(a^{(j)},\eta l_j)} f(x)d\mu(x)\right)\phi_j^*(x)$$

if $x \in \mathbf{R}^d \setminus \partial D$ and $\mathcal{E}_0(f)(x) = f(x)$ if $x \in \partial D$. Choose a C^∞-function ϕ_0 such that

$$\phi_0 = 1 \text{ on } B(O,R), \quad \operatorname{supp}\phi_0 \subset B(O,2R), \quad 0 \leq \phi_0 \leq 1$$

and define

$$\mathcal{E}(f)(x) = \mathcal{E}_0(f)(x)\phi_0(x).$$

Then $\mathcal{E}(f)$ is a C^∞-function in $\mathbf{R}^d \setminus \partial D$. Furthermore $\mathcal{E}(f)$ has following property.

Lemma 4.1. *Let* $p > 1$, $1 > \alpha > 0$, $\delta \in \mathbf{R}^d$ *and* $f \in \Lambda^p_\alpha(\mu)$. *If* $p(\alpha-1)+d-\beta+p\delta > 0$, *then*

$$\int_{\mathbf{R}^d \setminus \partial D} |\nabla \mathcal{E}(f)(y)|^p \, dist(y, \partial D)^{\delta p} dy \le c\|f\|^p_{p,\alpha}.$$

Using this lemma, we can show the following two lemmas.

Lemma 4.2. *Let* $p > 1$ *and* $1 > \alpha > \beta - (d-1) \ge 0$ *and* $f \in \Lambda^p_\alpha(\mu)$. *Then* K *is a bounded operator from* $\Lambda^p_\alpha(\mu)$ *to* $L^p(\mu)$.

Lemma 4.3 Let $p > 1$ and $1 > \alpha > \beta - (d-1) \ge 0$ and $f \in \Lambda^p_\alpha(\mu)$. *Define, for* $z \in \partial D$,

$$(\phi f)^*(z) = \sup\{|\Phi f(x)| : x \in \Gamma_\tau(z) \cap B(z, e_0)\}$$

and

$$(\phi f)^{**}(z) = \sup\{|\Phi f(x)| : x \in \Gamma^e_\tau(z) \cap B(z, e_0)\}.$$

Then

$$\|(\Phi f)^*\|_p \le c\|f\|_{p,\alpha} \quad and \quad \|(\Phi f)^{**}\|_p \le c\|f\|_{p,\alpha}.$$

On the other hand, by constucting a mollifier on ∂D, we see that the set of all Lipschitz functions on ∂D is dense in $\Lambda^p_\alpha(\mu)$. So, using Theorem 1, Lemma 4.2 and Lemma 4.3, we can prove Theorem 2.

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