Topological property of an invariant set with respect to a family of functions

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ABSTRACT

We investigate the topological property of invariant sets with respect to a family of functions by using the quotient space of infinite sequences. We give some results concerning the number of end points.

§1. Introduction

For a family of contraction functions $\{f_1, \ldots, f_m\}(m \geq 2)$ on a complete metric space, there is an invariant set K [3] satisfying the following

$$K = f_1(K) \cup \ldots \cup f_m(K).$$

M. Hata [2] investigated the topological property of the invariant set K and gave some results concerning the number of end points. Consider the set $E^{(\omega)}$ of infinite sequences, where $E = \{1, 2, \ldots, m\}$. Then there is a map ψ of $E^{(\omega)}$ onto K such that $\psi(x_1 x_2 \ldots) = \lim_{n\to\infty} f_{x_1} f_{x_2} \cdots f_{x_n}(K)$ [3]. The space $E^{(\omega)}$ with product topology is totally disconnected and perfect. If ψ is one to one, then the set K is also totally disconnected and perfect [e.g. the Cantor set]. If ψ is not one to one, then the topology of K shows various aspects and K is considered to be isomorphic to the quotient space induced by the equivalence relation $\sim$ on $E^{(\omega)}$. The topology of the quotient space has been studied by some people. A. Kameyama [4] studied the topology of the quotient space $E^{(\omega)}/_\sim$ and considered the condition that $E^{(\omega)}/_\sim$ is connected or $E^{(\omega)}/_\sim$ is metrizable. C.Bandt and K.Keller [1] also studied the topology of the quotient space and considered connectivity and ramification properties. In [5], we investigated the topological property of the quotient space in case $\sharp(E) = 2$ and examined the number of end points. In this paper, we shall investigate the topological property of $E^{(\omega)}/_\sim$ for the case that the number of E is any finite number and give some results concerning the number of end points. In §4, we shall show some examples.

§2. The topology of the quotient space

For $m \geq 2$, let $E = \{1, 2, \ldots, m\}$ and $E^{(\omega)}$ be the set of infinite sequences from E. Let $E^{(n)}$ be the set of sequences from E of length n for $n \in \mathbf{N}$, $E^{(0)}$ be the empty set and $E^{(*)}$ be the set of finite sequences from E, i.e. $E^{(*)} = \cup_{n=0}^{\infty} E^{(n)}$. For $n \in \mathbf{N} \cup \{0\}$, let the map $P_n : E^{(\omega)} \to E^{(*)}$ be the projection such as

$$P_n x = x_1 x_2 \ldots x_n, \quad \text{where } x = x_1 x_2 \ldots \in E^{(\omega)}.$$

For $s \in E$ and $x = x_1 x_2 \ldots$, $y = y_1 y_2 \ldots \in E^{(\omega)}$, let

$$sx = s x_1 x_2 \ldots$$

$$(P_n x) y = x_1 x_2 \ldots x_n y_1 y_2 \ldots .$$

Let the map $\sigma : E^{(\omega)} \to E^{(\omega)}$ be a shift operator, i.e.

$$\sigma(x_1 x_2 \ldots) = x_2 x_3 \ldots$$

An equivalence relation $\sim$ on $E^{(\omega)}$ is called to be *invariant* if the following (1) and (2) are satisfied:

(1) $x \sim y$ implies $sx \sim sy$ $\quad (\forall s \in E)$

(2) $sx \sim sy$ implies $x \sim y$ $\quad (\forall s \in E)$.

For $x \in E^{(\omega)}$, let Qx be the equivalence class of x, i.e. $Qx = \{y \in E^{(\omega)} | x \sim y\}$. Let A, A_s, E_s and F_n be the sets as follows:

$$A := \{x \in E^{(\omega)} |\ \exists y \in Qx \text{ s.t. } P_1 x \neq P_1 y\}, \qquad A_s := \{x \in A \mid P_1 x = s\},$$

$$E_s := \{x \in E^{(\omega)} \mid P_1 x = s\}, \qquad F_n = \{s \in E \mid \sharp(q(A_s)) = n\},$$

where $\sharp(q(A_s))$ is the number of elements of $q(A_s)$.

Hereafter, we assume that the equivalence relation $\sim$ is invariant and $\sharp A < \infty$. By using the equivalence relation $\sim$ on $E^{(\omega)}$, we shall investigate the topology of the quotient space.

Lemma 1 *When $\sharp A < \infty$, $x \in A$ is non-cyclic.*

Proof. Suppose $x \in A$ is cyclic, i.e. there exists $n \in \mathbf{N}$ such that $x = \sigma^n x$. Since $x \in A$ implies that there exists $y \in E^{(\omega)}$ such that $x \sim y$ and $P_1 x \neq P_1 y$, the relation $(P_n x)^l y \sim x$ holds for any $l \in \mathbf{N}$, which implies $\sharp A = \infty$. □

For the equivalence class of $x \in E^{(\omega)}$, we have the following lemma.

Lemma 2 *For $x \in E^{(\omega)}$, it holds that either*

$Qx = \{x\}$ *or*

$Qx = \{(P_j x)v \mid v \in Qa\}$ *with some $j \in \mathbf{N} \cup \{0\}$ and some $a \in A$.*

Proof. If there exists no $y \in E^{(\omega)}$ such that $y \sim x$, then $Qx = \{x\}$. If there exists $y \in E^{(\omega)}$ such that $y \sim x$, then there exists n_y such that $P_{n_y} x = P_{n_y} y$ and $x_{n_y+1} \neq y_{n_y+1}$. Then $\sigma^{n_y} x \sim \sigma^{n_y} x$ and $\sigma^{n_y} x \in A$. Put $j = \min \{n_y \mid x \sim y\}$ and $a = \sigma^j x$. Then $a \in A$ since there exists $y \in E^{(\omega)}$ such that $y \sim x$, $P_j y = P_j x$ and $y_{j+1} \neq x_{j+1}$. Hence $Qx = \{(P_j x)v \mid v \in Qa\}$. ($v \in Qa$ implies $(P_j x)v \sim (P_j x)a = x$ and so $\{(P_j x)v \mid v \in Qa\} \subset Qx$. On the other hand, $x \sim y$ implies $P_j x = P_j y$ and so $\sigma^j y \in Qa$ by the relation $\sigma^j x \sim \sigma^j y$.) □

By Lemma 2, we can define the number $l(x)$ for $x \in E^{(\omega)}$ as follows:

$$l(x) = \begin{cases} n & \text{if } Qx = \{(P_n x)v \mid v \in Qa\} \text{ for some } a \in A \\ \infty & \text{if } Qx = \{x\}. \end{cases}$$

When we consider the boundary of open sets, the number $l(x)$ plays an important role.

Let $U_n(x)$ and $V_n(x)$ be subsets of $E^{(\omega)}$ as follows:

$$U_n(x) = \{y \in E^{(\omega)} \mid P_n y = P_n x\}$$

$$V_n(x) = \{y \in U_n(x) \mid P_n Q y \subset P_n Q x\}.$$

Let $q : E^{(\omega)} \to E^{(\omega)}/_\sim$ be the natural quotient map.

Let $\tilde{U}_n(q(x))$ be the subset of the quotient space $E^{(\omega)}/_\sim$ as follows:

$$\tilde{U}_n(q(x)) = \{q(y) \in E^{(\omega)}/_\sim \mid P_n Q y \subset P_n Q x\}.$$

As for these sets, the following lemma holds.

Lemma 3 *1.* $U_n(x)\backslash V_n(x) \subset \{y \mid l(y) \leq n-1\})$

2. $V_n(x)$ *is open.*

3. $\tilde{U}_n(q(x)) = \cup\{q(V_n(x')) \mid x' \in Qx\}$

Proof. 1) It is clear by definiton.
2) By $\sharp(A) < \infty$, the set $\{y \mid l(y) \leq n-1\}$ is a finite set. So $U_n\backslash V_n$ is a finite set and a closed set. So $V_n(x)$ is open.
3) It is easily seen by definition. □

By using lemma 3, we get the following proposition.

Proposition 1 *The family* $\{\tilde{U}_n(q(x)) \mid n \in \mathbf{N},\ q(x) \in E^{(\omega)}/_\sim\}$ *is a basis for the quotient topology in* $E^{(\omega)}/_\sim$.

Proof. In order to show that $q^{-1}(\tilde{U}_n(q(x))$ is open in $E^{(\omega)}$, we shall show that

$$q^{-1}(\tilde{U}_n(q(x))) = \cup\{V_n(x') \mid x' \in Qx\}.$$

By Lemma 3, it is obvious that $q^{-1}(\tilde{U}_n(q(x))) \supset \cup\{V_n(x') \mid x' \in Qx\}$.
On the other hand, let $y \in q^{-1}(\tilde{U}_n(q(x)))$. Then $q(y) \in \tilde{U}_n(q(x))$. By Lemma 3, there exists $z \in V_n(x')$ such that $q(y) = q(z)$. Then $P_nQy = P_nQz \subset P_nQx$. So there exists $x'' \in Qx$ such that $y \in V(x'')$. Hence $q^{-1}(\tilde{U}_n(q(x))$ is open, since $V_n(x)$ is open.

Next suppose W is a subset of $E^{(\omega)}$ such that $q^{-1}(W)$ is open in $E^{(\omega)}$ and $q(x) \in W$. Then we shall show that there exists $n_0 \in \mathbf{N}$ such that $\tilde{U}_{n_0}(q(x)) \subset W$. For any $x' \in Qx$ there exists $n_{x'} \in \mathbf{N}$ such that $U_{n_{x'}}(x') \subset q^{-1}(W)$. Since $\sharp(A) < \infty$, put $n_0 = \max\{n_{x'} \mid x' \in Qx\}$. Then $V_{n_0}(x') \subset U_{n_0}(x') \subset q^{-1}(W)$ and so $\cup\{q(V_{n_0}(x') \mid x' \in Qx\} \subset W$, which implies $\tilde{U}_{n_0}(q(x)) \subset W$.

Hence $\{\tilde{U}_n(q(x))\}$ is a basis for the quotient topology in $E^{(\omega)}/_\sim$.□

Lemma 4 *The boundary* $\partial\tilde{U}_n(q(x))$ *of the set* $\tilde{U}_n(q(x))$ *is as follows:*

$$\partial\tilde{U}_n(q(x)) = \{q(y) \mid P_ny = P_nx' \text{ for some } x' \in Qx \text{ and } P_nQy \not\subset P_nQx\}$$

$$\subset \{q(y) \mid l(y) \leq n-1\}$$

Proof. Suppose $P_n x' = P_n y$ for some $x' \in Qx$ and $P_n Qy \not\subset P_n Qx$. Then $q(y) \notin \tilde{U}_n(q(x))$. Since for any $k > n$, there exists $z \in E^{(\omega)}$ such that $Qz = \{z\}$ and $P_k z = P_k y$, the relation $\tilde{U}_k(q(y)) \cap \tilde{U}_n(q(x)) \neq \phi$ holds. So $q(y) \in \partial\tilde{U}_n(q(x))$. On the other hand, suppose $q(y)$ does not belong to the set $\{q(y) \mid P_n y = P_n x'$ for some $x' \in Qx$ and $P_n Qy \not\subset P_n Qx\}$. If for any $x' \in Qx$, $P_n y \neq P_n x'$ holds, then $\tilde{U}_n(q(y)) \cap \tilde{U}_n(q(x)) = \phi$, which implies $q(y) \notin \partial\tilde{U}_n(q(x))$. If $P_n Qy \subset P_n Qx$ holds, then $q(y) \in \tilde{U}_n(q(x))$, which implies $q(y) \notin \partial\tilde{U}_n(q(x))$.

$\partial\tilde{U}_n(q(x)) \subset \{q(y) \mid l(y) \leq n-1\}$ follows from the definition. □

Remark. If $E^{(\omega)}/_\sim$ is connected, then $P_1 A = E$ holds.

§3. End points of the quotient space

Hereafter, we consider the case that $E^{(\omega)}/_\sim$ is connected. In this paper, we discuss the number of end points of $E^{(\omega)}/_\sim$. So at first we shall define the end point using the basis $\{\tilde{U}_n(q(x))\}$ of the quotient space.

Definition. We shall call $q(x) \in E^{(\omega)}/_\sim$ to be an *end point* of $E^{(\omega)}/_\sim$ if there exists $N \in \mathbf{N}$ such that $\partial\tilde{U}_n(q(x))$ is a singleton for any $n \geq N$.

Theorem 1 *1. The following (a) and (b) are equivalent.*

(a) $q(x) \in E^{(\omega)}/_\sim$ is an end point of $E^{(\omega)}/_\sim$.

(b) i. $Qx = \{x\}$ and

ii. There exists $N \in \mathbf{N}$ such that $n \geq N$ implies

$$x_n x_{n+1} \notin P_2 A, \qquad x_n \in F_1.$$

2. If $q(x)$ is an end point, then $q(\sigma x)$ is also an end point.

3. If $q(x)$ is an end point, then for $s \in E$ either $sx \in A$ holds or $q(sx)$ is an end point.

Proof. 1) b) $\to$ a): Since $Qx = \{x\}$, there exists $N' \in \mathbf{N}$ such that $P_{N'}y = P_{N'}x$ implies $l(y) \geq N$. For $n \geq N'$, let $q(y) \in \partial\tilde{U}_n(q(x))$. Then by Lemma 4, $l(y) \leq n-1$ and $P_n y = P_n x$, which implies $l(y) > N$. If $N < l(y) \leq n-2$, then $y = wv$ with $w \in E^{l(y)}, v \in A$. So $wv_1v_2 = P_{l(y)+2}y = P_{l(y)+2}x$ implies $v_1v_2 = x_{l(y)+1}x_{l(y)+2} \notin P_2A$. This is a contradiction to $v \in A$. Therefore $l(y) = n-1$. $P_1(\sigma^n x) \in F_1$ implies that $q(y)$ is a singleton.

a) $\to$ b): If $Qx \neq \{x\}$, there exist $k \in \mathbf{N}, w \in E^k$ and $u \in A$ such that $x = wu$. Hence $\partial\tilde{U}_n(q(x))$ is not a singleton for $n > k$.

If $P_2\sigma^n x \in P_2A$ with some $n \in \mathbf{N}$, then $\partial\tilde{U}_{n+2}(q(x))$ is not a singleton.

If $P_1\sigma^n x \in \{s \mid \sharp(q(A_s)) \geq 2\}$ holds, there exist $u^1, u^2 \in A$ such that $P_1u^1 = P_1u^2 = s$ and $q(u^1) \neq q(u^2)$. Then $q((P_nx)u^1), q((P_nx)u^2) \in \partial\tilde{U}_{n+1}(q(x))$ and $\partial\tilde{U}_{n+1}(q(x))$ is not a singleton. So if for any $N \in \mathbf{N}$ there exists $n > N$ such that $P_2\sigma^n x \in P_2A$ or $P_1\sigma^n x \in \{s \mid \sharp(q(A_s)) \geq 2\}$, $q(x)$ is not an end point of $E^{(\omega)}/_\sim$.

2) is obtained by 1).

3) If $sx \notin A$, then $Q(sx) = \{sx\}$ holds and so $q(sx)$ is an end point by 1). □

In $E^{(\omega)}/_\sim$, end points do not necessarily exist. So we shall give the condition for the existence of end points.

Theorem 2 *The following are equivalent.*

1. *There exists an end point of $E^{(\omega)}/_\sim$.*

2. *$F_1 \neq \phi$ and there exists $\{s_1, s_2, \ldots, s_n\} \subset F_1$ $(n \geq 1)$ such that*

$$s_js_{j+1} \notin P_2A (j = 1, 2, \ldots, n-1), \quad s_ns_1 \notin P_2A.$$

Proof. 1) $\to$ 2): Suppose $q(x)$ is an end point of $E^{(\omega)}/_\sim$. Then by Theorem 1, there exists $N \in \mathbf{N}$ such that $n \geq N$ implies

$$x_nx_{n+1} \notin P_2A, \qquad x_n \in F_1.$$

Put $k_1 = x_N, k_2 = x_{N+1}, \ldots,$. Then $k_jk_{j+1} \notin P_2A (j \geq 1)$ and $k_j \in F_1$. Since F_1 is a finite set, there exist $n, l \in \mathbf{N}$ such that $k_{n+l} = k_l$. So $s_j = k_{l+j-1} (j = 1, 2, \ldots, n)$

are desired ones.

2) $\to$ 1): Put $x = s_1 s_2 \dots s_n s_1 s_2 \dots s_n \dots$ be the iterated sequence of $s_1 s_2 \dots s_n$. Then $Qx = \{x\}, P_1\sigma^k x \in F_1 (\forall k \in \mathbf{N})$ and $P_2\sigma^k x \notin P_2A(\forall k \in \mathbf{N})$. So $q(x)$ is an end point of $E^{(\omega)}/_\sim$ by Theorem 1. □

To consider the number of end points, we shall show the following lemma.

Lemma 5 *Let s^1, s^2, s^3 be distinct elements of F_1. Then the following holds:*

$$\sharp\{rt|\ r,t \in \{s^1, s^2, s^3\}, rt \notin P_2A\} \geq 5.$$

Proof. Let i, j, k be distinct elements of $\{1,2,3\}$.

1) At first we shall show that if $s^i s^i$ belongs to P_2A, then both $s^i s^j$ and $s^i s^k$ do not belong to P_2A.

If we suppose both $s^i s^i$ and $s^i s^j$ belong to P_2A, then there exist $x, y \in E^{(\omega)}$ such that $s^i s^i x, s^i s^j y \in A$ and $q(s^i s^i x) = q(s^i s^j y)$. So $q(s^i x) = q(s^j y)$. Since $s^i \in F_1$, $q(s^i s^i x) = q(s^i x)$ holds, which implies $(s^i)^l s^j y \in A(\forall l)$. This is a contradiction to $\sharp(A) < \infty$.

2) Next we shall show that $\sharp(\{s^i s^i, s^i s^j, s^i s^k\} \cap (P_2A)^c) = 1$ implies

$$\sharp(\{s^j s^j, s^j s^i, s^j s^k\} \cap (P_2A)^c) \geq 2.$$

$\sharp(\{s^i s^i, s^i s^j, s^i s^k\} \cap (P_2A)^c) = 1$ and 1) imply that both $s^i s^j$ and $s^i s^k$ belong to P_2A. So there exist $a, b \in E^{(\omega)}$ such that $s^i s^j a, s^i s^k b \in A$ and $q(s^i s^j a) = q(s^i s^k b)$. So $q(s^j a) = q(s^k b)$.

If $s^j s^j$ belongs to P_2A, then by 1), $s^j s^k, s^j s^i \notin P_2A$ holds. If both $s^j s^k$ and $s^k s^i$ belong to P_2A, then there exist $c, d \in E^{(\omega)}$ such that $s^j s^k c, s^j s^i d \in A$ and $q(s^j s^k c) = q(s^j s^i d)$. So $q(s^k c) = q(s^i d)$. Since $s^i, s^j, s^k \in F_1$, the following holds:

$$q(s^i d) = q(s^k c) = q(s^k b) = q(s^j a = q(s^j s^i d),$$

which implies $(s^j)^l s^i d \in A(\forall l)$. This is a contradiction to $\sharp(A) < \infty$. So $\sharp(\{s^j s^j, s^j s^i, s^j s^k\} \cap (P_2A)^c) \geq 2$ holds. Therefore $\sharp\{rt|\ r,t \in \{s^1, s^2, s^3\}, rt \notin P_2A\} \geq 5$ holds. □

By using lemma, we get the following proposition in case $\sharp(F_1) \geq 3$.

Proposition 2 *If $\sharp(F_1) \geq 3$, then there exist infinitely many end points of $E^{(\omega)}/_\sim$.*

Proof. Suppose s^1, s^2, s^3 are distinct elements of F_1. Put $B = \{rt| \ r,t \in \{s^1,s^2,s^3\}, rt \notin P_2A\}$. Then by lemma 5, $\sharp(B) \geq 5$ holds. Let $x \in E^{(\omega)}$ satisfy $x_n x_{n+1} \in B$ for any $n \in \mathbf{N}$. Then $q(x)$ is an end point of $E^{(\omega)}/_\sim$ since x satisfies the condition of (b) of Theorem 1. Since there are infinitely many $x \in E^{(\omega)}$ such that $x_n x_{n+1} \in B$ for any $n \in \mathbf{N}$, we get the conclusion. □

Now we shall consider the case $\sharp(F_1) = 2$.

Lemma 6 *If $\sharp(F_1) = 2$, then there exists an end point $q(e_s)$ such that $P_1 e_s = s$ for $s \in F_1$.*

Proof. Put $F_1 = \{s,t\}$. If we suppose both ss and st belong to P_2A, then there exist $x, y \in E^{(\omega)}$ such that $ssx, sty \in A$ and $q(ssx) = q(sty)$. So $q(sx) = q(ty)$. Since $s \in F_1$, $q(sty) = q(sx) = q(ty)$ holds, which implies $(s)^l ty \in A$ $(\forall l)$. This is a contradiction to $\sharp(A) < \infty$. So one of $\{ss, st\}$ does not belong to P_2A. So there exists an end point $q(e_s)$ such that $P_1 e_s = s$. □

Proposition 3 *If $\sharp(F_1) = 2$, then the number of end points of $E^{(\omega)}/_\sim$ is 2 or infinity.*

Proof. Let $F_1 = \{s,t\}$. Then by lemma 6 there exist end points $q(e_s)$, $q(e_t)$ such that $P_1 e_s = s$ and $P_1 e_t = t$.

If there exists $r \in E\backslash\{s, \ t\}$ such that $re_s \notin A$ or $re_t \notin A$, then either $\{s^k re_s \mid k \in \mathbf{N}\}$ or $\{s^k re_t \mid k \in \mathbf{N}\}$ does not contain any element of A. So the number of end points are infinity.

If there exists $t \in E\backslash\{s,r\}$ such that $t\overline{s} \notin A$ [resp. $pr\overline{s} \notin A$], then we see that there exist infinitely many end points in the same way as Proposition 2.

If there exists no $r \in E\backslash\{s, \ t\}$ such that $re_s, \ re_t \notin A$, then the only $q(e_s)$ and $q(e_t)$ are end points. □

When the number of end points is 2, we shall consider the condition that it is isomorphic to the unit interval.

Lemma 7 *For $x, y \in E^{(\omega)} \backslash A$ satisfying $P_1x,\ P_1y \in F_1$ ($P_1x \neq P_1y$), suppose*

$$\sigma x,\ \sigma y,\ \sigma z \in \{x, y\} \ \ (\forall z \in A).$$

Then

1. *for $s, t \in \{P_1x, P_1y\}$, the following (a) and (b) are equivalent.*
   (a) $st \in P_2A$
   (b) $st \notin \{P_2x, P_2y\}$
2. *The following (a) and (b) are equivalent.*
   (a) $z_n z_{n+1} \notin P_2A$ ($\forall n$) *and* $z_n \in \{P_1x, P_1y\}$ ($\forall n$)
   (b) $z \in \{x, y\}$

Proof. 1. Suppose that $st \in P_2A \cap \{P_2x,\ P_2y\}$. Then there exists $z \in A$ such that $P_2z = st$. $\sigma x,\ \sigma y,\ \sigma z \in \{x, y\}$ implies $x = z$ or $y = z$, which is a contradiction.
2. Since $\sharp\{st \in P_2A \mid s, t \in \{P_1x, P_1y\}\} = 2$, $\sharp\{P_2x, P_2y\} = 2$ and $\sharp\{st \in P_2E^{(\omega)} \mid s, t \in \{P_1x, P_1y\}\} = 4$ hold, $P_2A \cap \{P_2x,\ P_2y\} \neq \phi$ implies (a) $\Leftrightarrow$ (b). □

Remark. Suppose $\sigma x,\ \sigma y \in \{x, y\}$. Then the pair $\{x,\ y\}$ is one of the following four cases, where $s = P_1x,\ t = P_1y$:

$$\begin{cases} x = \overline{s} \\ y = \overline{t} \end{cases} \quad \begin{cases} x = \overline{s} \\ y = t\overline{s} \end{cases} \quad \begin{cases} x = s\overline{t} \\ y = \overline{t} \end{cases} \quad \begin{cases} x = \overline{st} \\ y = \overline{ts}. \end{cases}$$

Lemma 8 *Suppose that $\sharp(F_1) = 2$ and $\sharp(F_2) = m - 2$. Then the following are equivalent.*

1. *There exist distinct $x, y \in E^{(\omega)}$ satisfying*
$$\cup_{j=1}^{m} \{jx, jy\} = A \cup \{x, y\} \qquad (**)$$
2. *Let $F_1 = \{s, t\}$. Then there exist $x, y \in E^{(\omega)} \backslash A$ satisfying*
$$P_1x = s,\ P_1y = t \qquad \textit{and} \qquad \sigma x,\ \sigma y,\ \sigma z \in \{x, y\} \ \ (\forall z \in A).$$

3. *Let* $F_1 = \{s, t\}$. *Then there exist* $x, y \in E^{(\omega)}$ *satisfying*

   (a) $q(x)$ *and* $q(y)$ *are end points.*

   (b) $P_1 x = s$, $P_1 y = t$ *and* σx, σy, $\sigma z \in \{x, y\}$ $(\forall z \in A)$.

Proof. By the assumption, we have $\sharp(A) = 2 \times (m-2) + 1 \times 2 = 2m - 2$.
(1) $\to$ (2): Considering the number of elements of both sides of (**), we have $x, y \notin A$. For $s \in F_1$, both sx and sy do not belong to A and so either sx or sy belongs to $\{x, y\}$, which implies $s = P_1 x$ or $P_1 y$. In the same way, $t = P_1 x$ or $P_1 y$ for $t \in F_1$. So $\{s, t\} = \{P_1 x, P_1 y\}$. The relation (**) implies σx, σy, $\sigma z \in \{x, y\}$ $(\forall z \in A)$.

(2) $\to$ (3): $\sigma x \in \{x, y\}$ implies $Qx = \{x\}$ and $x_n \in \{s, t\} = F_1$ for all $n \in \mathbf{N}$. $x_n x_{n+1} \notin P_2 A$ follows from lemma 7. So $q(x)$ and $q(y)$ are end points.

(3) $\to$ (1): (3) implies $A \cup \{x, y\} \subset \cup_{j=1}^{m} \{jx, jy\}$. Since $q(x)$ is an end point, we have $Qx = \{x\}$, which implies $x \notin A$. By counting the number of elements of both sides, we have $A \cup \{x, y\} = \cup_{j=1}^{m} \{jx, jy\}$. □

Theorem 3 *The following 1 and 2 are equivalent.*

1. $E^{(\omega)}/_{\sim}$ *is homeomorphic to the unit interval* $[0, 1]$.

2. (a) $\sharp(F_1) = 2$, $\sharp(F_2) = m - 2$

   (b) $\sharp(Qa) = 2$ *for any* $a \in A$.

   (c) *There exist* $x, y \in E^{(\omega)}$ *satisfying*

$$\cup_{j=1}^{m} \{jx, jy\} = A \cup \{x, y\}. \qquad (*)$$

Proof. (1) $\to$ (2): Let $\psi : E^{(\omega)}/_{\sim} \to [0, 1]$ be a homeomorphism. Let $\tilde{j} : E^{(\omega)}/_{\sim} \to E^{(\omega)}/_{\sim}$ be defined by $\tilde{j}(q(x)) = q(jx)$ for any $q(x) \in E^{(\omega)}/_{\sim}$. Then $\tilde{j}$ is well-defined and continuous. So $\psi(q(E_j)) = \psi(\tilde{j}(E^{(\omega)}/_{\sim}))$ is a compact continuous subset of [0,1] and a closed interval, say $[\alpha_j, \beta_j]$ for any $j \in E$ $(0 \le \alpha_j < \beta_j \le 1)$. Since $\sharp(A) < \infty$,

$$q(E_j) \cap q(E_i) \neq \phi \text{ implies } \beta_j = \alpha_i \text{ or } \alpha_j = \beta_i. \qquad (**)$$

Let $q(x)$ and $q(y)$ be $\psi^{-1}(0)$ and $\psi^{-1}(1)$ respectively. Then $q(x)$ and $q(y)$ are end points of $E^{(\omega)}/_\sim$. Put $s = P_1x$. Then $\psi(q(E_s)) = [0, \beta_s]$. Since $E^{(\omega)}/_\sim$ is connected and $\sharp(A) < \infty$, there exists only one $r \in E$ such that $q(E_s) \cap q(E_r) \neq \phi$ and so $s \in F_1$. In the same way, $t = P_1 y \in F_1$. Then by the relation (**) and $\sharp(A) < \infty$, there exists a permutation $\pi : E \to E$ such that $\pi(1) = s$, $\pi(m) = t$, $q(E_{\pi(j)}) \cap q(E_{\pi(j+1)})$ is a singleton, say, $\{q(a^j)\}$ $(j = 1, \ldots, m-1)$ and $q(E_{\pi(l)}) \cap q(E_{\pi(k)}) = \phi$ for $|k - l| \geq 2$. So $F_1 = \{s,\ t\}$ and $\sharp(F_2) = m - 2$. $a \in A$ implies $q(a) \in \{q(a^1), \ldots, q(a^{m-1})\}$ and so $\sharp(Qa) = 2$. Since $E^{(\omega)}/_\sim$ is homeomorphic to [0,1], the number of end points of $E^{(\omega)}/_\sim$ is 2 and $q(x), q(y)$ are end points. By Theorem 1 (2), $q(\sigma x), q(\sigma y)$ are end points and $\sigma x, \sigma y \in \{x, y\}$. By Theorem 1 (3), $jx \in A$ or $q(jx)$ is an end point. So $\cup_{j=1}^{m}\{jx, jy\} \subset A \cup \{x, y\}$. Since the numbers of elements of both sides are $2m$, the equality holds and we get (*).

(2) $\to$ (1): By lemma 6, $q(x)$ and $q(y)$ are end points of $E^{(\omega)}/_\sim$. By (a) and (b), there exists a permutation π of E satisfying

$$\pi(1),\ \pi(m) \in F_1,\ \ \pi(2), \ldots, \pi(m-1) \in F_2$$

and

$$q(E_{\pi(j)}) \cap q(E_{\pi(j+1)}) \neq \phi \quad (j = 1, \ldots, m-1).$$

So we may suppose that $P_1x = 1, P_1y = m, F_1 = \{1, m\}$ and

$q(E_j) \cap q(E_{j+1}) \neq \phi \quad (j = 1, \ldots, m-1)$.

Put $\lambda_j := \begin{cases} 1 & \text{if } \ q(jx) \in q(E_{j-1}) \\ -1 & \text{if } \ q(jx) \in q(E_{j+1}) \end{cases}$ for $j (2 \leq j \leq m-1)$,

$\lambda_1 := \begin{cases} 1 & \text{if } \ 1x = x \\ -1 & \text{if } \ 1y = x \end{cases}$ and $\lambda_m := \begin{cases} 1 & \text{if } \ my = y \\ -1 & \text{if } \ mx = y \end{cases}$.

Put $\gamma_1(u) = \frac{u_1 - 1}{m}$

$$\gamma_n(u) = \frac{\frac{\Pi_{k=1}^{n}\lambda_{u_k}+1}{2}(u_n - 1) + \frac{1-\Pi_{k=1}^{n}\lambda_{u_k}}{2}(m - u_n)}{m^n} \quad \text{for } \ n \geq 2$$

and $\gamma(u) = \sum_{n=1}^{\infty} \gamma_n(u)$ for $u \in E^{(\omega)}$. Then for $n \geq 2$ and $j \in E, u \in E^{(\omega)}$,

$$\gamma_n(ju) = \frac{\lambda_j}{m}\gamma_{n-1}(u) + \frac{(1-\lambda_j)(m-1)}{2m^n}. \qquad (***)$$

It is obvious that $\gamma_1(x) = \gamma_2(x) = 0$ and $\gamma_1(y) = \frac{m-1}{m}, \gamma_2(y) = \frac{m-1}{m^2}$. By using the relation (***), we get $\gamma_n(x) = 0$ and $\gamma_n(y) = \frac{m-1}{m^n}$ for all $n \in \mathbf{N}$. So $\gamma(x) = 0$ and

$\gamma(y) = 1$. By using the relation (***), we also get that $\gamma(u) = \gamma(v)$ holds if and only if $q(u) = q(v)$. □

Now we shall consider the case $\sharp(F_1) = 1$.

Proposition 4 *If $\sharp(F_1) = 1$, then the following hold.*

1. *The number of elements of E is greater than 3, i.e. $m \geq 3$.*
2. *The number of end points of $E^{(\omega)}/_\sim$ is 0, 1, 2 or infinity.*

Proof. 1. Suppose that $m = 2$, i.e. $E = \{s, t\}$. If $F_1 = \{s\}$, then $\sharp(A_t) \geq 2$, that is, there exist sx, ty, $tz \in A(y \neq z)$ such that $sx \sim ty \sim tz$. $y \neq z$ and $E = \{s, t\}$ imply that there exists $w \in E^{(*)}$ such that $y = wsy'$, $z = wtz'$ or vice versa. So $sy' \sim tz'$. Since $\sharp(A_s) = 1$, then $x = y'$. So $sx \sim tz' \sim twtz'$, from which $z' \sim (wt)^l z'$ follows for any $l \in \mathbf{N}$. This is a contradiction to $\sharp(A) < \infty$. So $m \geq 3$.

2. Suppose the number of end points is $n (3 \leq n < \infty)$. Since $\sharp(F_1) = 1$, we can put $F_1 = \{s\}$. So by Theorem 1, there exists $w \in E^{(*)}$ such that $q(w\overline{s})$ is an end point. Theorem 1 2) implies $q(\overline{s})$ is also an end point. Since the number of end points is not 1, there exists $r \in E\backslash\{s\}$ such that $r\overline{s} \notin A$ by Theorem 5. If there exists $t \in E\backslash\{s, r\}$ [resp. $p \in E\backslash\{s\}$] such that $t\overline{s} \notin A$ [resp. $pr\overline{s} \notin A$], then either $\{s^k r\overline{s} \mid k \in \mathbf{N}\}$ or $\{s^k t\overline{s} \mid k \in \mathbf{N}\}$ [resp. $\{s^k pr\overline{s} \mid k \in \mathbf{N}\}$] does not contain any element of A and we see that there exist infinitely many end points in the same way as Proposition 3. So $t\overline{s} \in A$ for any $t \in E\backslash\{r, s\}$ [resp. $pr\overline{s} \in A$ for any $t \in E\backslash\{s\}$] and end points are either $\overline{s}$ or $\underbrace{s \dots s}_{l} r\overline{s}$ $(0 \leq \forall l \leq n-2)$. Therefore $\underbrace{s \dots s}_{n-1} r\overline{s} \in A$. If $n > 2$ holds, then $ss \in P_2 A$, which implies that $q(\overline{s})$ is not an end point. This is a contradiction. So the number of end points of $E^{(\omega)}/_\sim$ is 0, 1, 2 or infinity. □

By Propositions 2, 3 and 4, we get the following theorem.

Theorem 4 *If $\sharp A < \infty$, then the number of end points of $E^{(\omega)}/_\sim$ is 0, 1, 2 or infinity.*

We shall give a condition that the number of end points is 1.

Theorem 5 *The following are equivalent.*

1. *The number of end points is 1.*

2. *(a) $\sharp(F_1) = 1$, i.e. $F_1 = \{s\}$, and $ss \notin P_2A$,*

   (b) $r\bar{s} \in A$ for any $r \in E\backslash\{s\}$.

Proof. 1) $\to$ 2): Since there exists an end point of $E^{(\omega)}/_\sim$, $\sharp(F_1) \geq 1$ by Theorem 2. By Propositions 2 and 3, $\sharp(F_1) \leq 1$. So $\sharp(F_1) = 1$ i.e. $F_1 = \{s\}$, which implies that if $q(x)$ is an end point, then $x = w\bar{s}$ with some $w \in E^{(*)}$. Hence $ss \notin P_2A$. If $r\bar{s} \notin A$, then $q(r\bar{s})$ is also en end point. So $r\bar{s} \in A$ for any $r \in E\backslash\{s\}$.

2) $\to$ 1): If $q(x)$ is an end point, then $\sharp(A_s) = 1$ implies $x = w\bar{s}$ with some $w \in E^{(*)}$. The condition (b) implies that $q(r\bar{s})$ is not an end point of $E^{(\omega)}/_\sim$ for $r \in E\backslash\{s\}$. So $q(\bar{s})$ is the only end point. □

To consider a condition that the number of end points is 2, we give some definitions.

Let T be the set

$$\{(r,s)|r,s \in F_1, rs \notin P_2A \text{ and } sr \notin P_2A\}$$

and let $EN(j)$ (j=0,1,2) be the following:

$EN(0) = \{e = \overline{rs}|(r,s) \in T\}$

$EN(1) = \{t\overline{rs}|t \in E, (r,s) \in T, t \neq s \text{ and } t\overline{rs} \notin A\}$

$EN(2) = \{te|t \in E, e \in EN(1) \text{ and } te \notin A\}$.

Then the following theorem 6 shows a condition that the number of end points is 2.

Lemma 9 *If x belongs to $EN(j)$ $(2 \geq j \geq 0)$, then $q(x)$ is an end point.*

Proof. If $x \in EN(j)$, then $Qx = \{x\}$ holds. So by Theorem 1, $q(x)$ is an end point. □

Theorem 6 *The following are equivalent.*

1. *The number of end points is 2.*

2. $1 \leq \sharp(F_1) \leq 2$, $EN(2) = \phi$ *and* $\sharp(EN(0)) + \sharp(EN(1)) = 2$.

Proof. 1) → 2): $1 \leq \sharp(F_1) \leq 2$ follows from Theorem 2 and Proposition 2. By definition of $EN(j)$, $EN(i) \cap EN(j) = \phi$ $(i \neq j)$ follows. If there exists $x \in EN(2)$, then $q(x), q(\sigma x), q(\sigma^2 x)$ are distinct end points by Lemma 9 and Theorem 1. So $EN(2) = \phi$ and $\sharp(EN(0)) + \sharp(EN(1)) = 2$ follows.

2) → 1): If $q(x)$ is an end point, there exist $n \in \mathbf{N} \cup \{0\}$, $w \in E^{(n)}$ and $(r, s) \in T$ such that $x_n \neq s$ (if $n \neq 0$) and $x = w\overline{rs}$, since $\sharp(F_1) \leq 2$. Since $q(\sigma^j x)$ is an end point for any $j \in \mathbf{N}$, $n \leq 2$ follows from $EN(2) = \phi$, which implies $q(x) \in EN(0) \cup EN(1)$. So $\sharp(EN(0)) + \sharp(EN(1)) = 2$ implies that the number of end points is 2. □

§4. The invariant set and examples

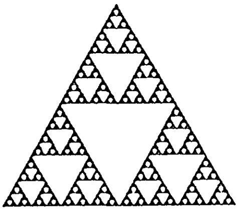
For a family of contraction functions $\{f_1, \ldots, f_m\}(m \geq 2)$ on a complete metric space, let K be the invariant set, that is, $K = f_1(K) \cup \ldots f_m(K)$ and ψ be a map of $E^{(\omega)}$ onto K such that $\psi(x_1 x_2 \ldots) = \lim_{n\to\infty} f_{x_1} f_{x_2} \cdots f_{x_n}(K)$ [3]. Define the equivalence relation $\sim$ on $E^{(\omega)}$ by

$$x \sim y \quad \text{if and only if} \quad \psi(x) = \psi(y) \qquad\qquad \text{for } x, y \in E^{(\omega)}.$$

Then the equivalence relation $\sim$ is invariant and K is isomorphic to $E^{(\omega)}/_\sim$. Let A be the set $\{x \in E^{(\omega)} \mid \exists y \in E^{(\omega)}$ such that $x \sim y$ and $P_1 x \neq P_1 y\}$. When $\sharp A < \infty$, the results in §3 can be applied to K. We shall show some examples for $\{f_1, \ldots, f_m\}(m \geq 2)$ on $\mathbf{C}$. Let $\sharp(EP)$ denote the number of end points.

Example 1. $[\sharp(EP) = 0]$

Let $f_1(z) = \frac{1}{2}z$, $f_2(z) = \frac{1}{2}z + \frac{1}{2}$,
$f_3(z) = \frac{1}{2}z + \frac{1}{4} + \frac{\sqrt{3}}{4}i$.
Then $E = \{1,2,3\}$, $F_1 = \phi$ and

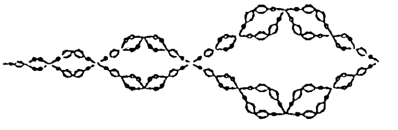


$$A = \{1\bar{2}, 1\bar{3}, 2\bar{1}, 2\bar{3}, 3\bar{1}, 3\bar{2}\}.$$

Since $F_1 = \phi$, by Theorem 2 the number of end points of K is 0.

Example 2. $[\sharp(EP) = 1]$

Put $\theta = \frac{\pi}{6}$ and $r = \frac{1}{4 * cos\theta}$.
Let $f_1(z) = \frac{1}{2}z$, $f_2(z) = r * e^{\theta i}z + 1$,
$f_3(z) = r * e^{-\theta i}z + 1$, $f_4(z) = r * e^{(\pi-\theta)i}z + 2$,
$f_5(z) = r * e^{(\pi+\theta)i}z + 2$.
Then $E = \{1,2,3,4,5\}$, $F_1 = \{1\}$ and

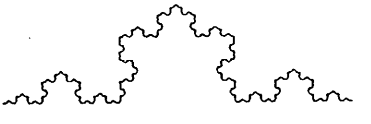


$$A = \{14\bar{1}, 15\bar{1}, 2\bar{1}, 24\bar{1}, 25\bar{1}, 3\bar{1}, 34\bar{1}, 35\bar{1}, 4\bar{1}, 44\bar{1}, 45\bar{1}, 5\bar{1}, 54\bar{1}, 55\bar{1}\}.$$

So $11 \notin P_2A$ and $r\bar{1} \in A$ for any $r \in E\backslash\{1\}$. Hence by Theorem 5 the number of end points of K is 1.

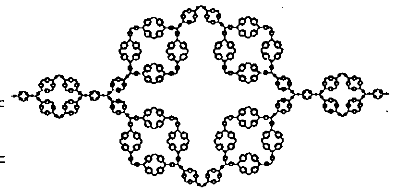
Example 3. $[\sharp(EP) = 2$, $\sharp(F_1) = 2$ and K is homeomorphic to $[0,1]]$

Put $\theta = \frac{\pi}{3}$ an $r = \frac{1}{3}$.
Let $f_1(z) = r * z$, $f_2(z) = r * e^{\theta i}z + \frac{1}{3}$,
$f_3(z) = r * e^{-\theta i}z + \frac{1}{2} + \frac{\sqrt{3}}{6}i$,
$f_4(z) = r * z + \frac{2}{3}$.



Then $E = \{1,2,3,4\}$, $F_1 = \{1,4\}$ $A = \{1\bar{4}, 2\bar{1}, 2\bar{4}, 3\bar{1}, 3\bar{4}, 4\bar{1}\}$, $T = \{(1,1),(4,4)\}$, $EN(0) = \{\bar{1}, \bar{4}\}$ and $EN(n) = \phi$ for $n \geq 1$. So by Theorem 6, the number of end points of K is 2. The conditions 2(a),2(b) and 2(c) of Theorem 3 are satisfied. So K is homeomorphic to the unit interval.

Example 4. [$\sharp(EP) = 2$ and $\sharp(F_1) = 2$]

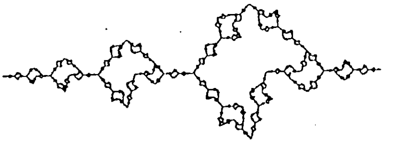


Put $\theta = \frac{\pi}{4}$ and $r = \frac{\sqrt{2}}{4}$.

Let $f_1(z) = \frac{1}{4}z$, $f_2(z) = r * e^{\theta i}z + \frac{1}{2}$, $f_3(z) = r * e^{-\theta i}z + \frac{1}{2}$, $f_4(z) = r * e^{(\pi-\theta)i}z + \frac{3}{2}$, $f_5(z) = r * e^{(\pi+\theta)i}z + \frac{3}{2}$, $f_6(z) = \frac{1}{4}z + \frac{3}{2}$.

Then $E = \{1,2,3,4,5,6\}$, $F_1 = \{1,6\}$, $A = \{1\bar{6}, 2\bar{1}, 2\bar{6}, 3\bar{1}, 3\bar{6}, 4\bar{1}, 4\bar{6}, 5\bar{1}, 5\bar{6}, 6\bar{1}\}$, $T = \{(1,1),(6,6)\}$, $EN(0) = \{\bar{1}, \bar{6}\}$ and $EN(n) = \phi$ for $n \geq 1$. So by Theorem 6, the number of end points of K is 2. The conditions 2(a) and 2(c) of Theorem 3 are satisfied, but for $a = 1\bar{6} \in A$, $Qa = \{1\bar{6}, 2\bar{1}, 3\bar{1}\}$ does not satisfy 2(b) of Theorem 3. So K is not homeomorphic to the unit interval.

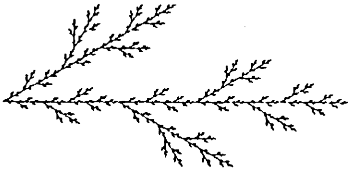
Example 5. [$\sharp(EP) = 2$ and $\sharp(F_1) = 1$]



Put $w_1 = 0, r_1 = \frac{1}{2}, w_2 = \frac{5}{18}\pi, r_2 = \frac{3}{20\cos w_2}$, $w_3 = -w_2$, $r_3 = r_2$, $w_4 = \pi - \tan^{-1}(\frac{9}{13}\tan w_2)$, $r_4 = |\frac{3}{20\cos w_4}|$, $w_5 = \tan^{-1}(\frac{39}{11}\tan w_2)$, $r_5 = \frac{11}{260\cos w_5}$, $w_6 = \pi$, $r_6 = \frac{4}{13}$,

$p_1 = q_1 = q_2 = q_3 = q_6 = 0$, $p_2 = p_3 = 1$, $p_4 = 1.6$, $q_4 = \frac{6}{65}\tan w_2$, $p_5 = 1.3$, $q_5 = -\frac{3}{10}\tan w_2$ and $p_6 = 2$. Let $f_j(z) = r_j * e^{w_j i}z + p_j + q_j i$ for $j = 1, 2, \ldots, 6$.
Then $E = \{1,2,3,4,5,6\}, F_1 = \{1\}, A = \{16\bar{1}, 2\bar{1}, 26\bar{1}, 3\bar{1}, 36\bar{1}, 4\bar{1}, 46\bar{1}, 5\bar{1}, 56\bar{1}, 626\bar{1}, 646\bar{1}, 66\bar{1}\}$, $T = \{(1,1)\}$, $EN(0) = \{\bar{1}\}$, $EN(1) = \{6\bar{1}\}$ and $EN(n) = \phi$ for $n \geq 2$. So by Theorem 6, the number of end points of K is 2.

Example 6. [$\sharp(EP) = \infty$]



Put $\theta = \frac{\pi}{6}$ and $r = \frac{1}{\sqrt{3}}$.

Let $f_1(z) = r * e^{\theta i}\bar{z}$, $f_2(z) = \frac{2}{3}\bar{z} + \frac{1}{3}$.

Then $E = \{1,2\}$, $F_1 = \{1,2\}$, $A = \{11\bar{2}, 2\bar{1}\}$, $T = \{(2,2)\}$, $EN(0) = \{\bar{2}\}$, $EN(1) = \{1\bar{2}\}$, $EN(2) = \{21\bar{2}\}$ and $EN(2) \neq \phi$. So by Proposition 3 and Theorem 6, the number of end points of K is ∞.

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