On the Connectivity of Julia Sets of Transcendental Entire Functions
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Abstract
We have two main purposes in this paper. One is to give some sufficient conditions for the Julia set of a transcendental entire function $f$ to be connected or to be disconnected as a subset of the complex plane $\mathbb{C}$. The other is to investigate the boundary of an unbounded periodic Fatou component $U$, which is known to be simply connected. These are related as follows: let $\varphi : \mathbb{D} \rightarrow U$ be a Riemann map of $U$ from a unit disk $\mathbb{D}$, then under some mild conditions we show the set $\Theta_\infty$ of all angles where $\varphi$ admits the radial limit $\infty$ are dense in $\partial \mathbb{D}$ if $U$ is an attracting basin, a parabolic basin or a Siegel disk. If $U$ is a Baker domain on which $f$ is not univalent, then $\Theta_\infty$ is dense in $\partial \mathbb{D}$ or at least its closure $\overline{\Theta}_\infty$ contains a certain perfect set, which means the boundary $\partial U$ has a very complicated structure. In all cases, this result leads to the disconnectivity of the Julia set $J_f$ in $\mathbb{C}$. We also consider the connectivity of the set $J_f \cup \{\infty\}$ in the Riemann sphere $\hat{\mathbb{C}}$ and show that $J_f \cup \{\infty\}$ is connected if and only if $f$ has no multiply-connected wandering domains.

1 Definitions and Results
Let $f$ be a transcendental entire function and $f^n$ denote the $n$-th iterate of $f$. Recall that the Fatou set $F_f$ and the Julia set $J_f$ of $f$ are defined as follows:

$$F_f := \{ z \in \mathbb{C} \mid \{f^n\}_{n=1}^{\infty} \text{ is a normal family in a neighborhood of } z \},$$

$$J_f := \mathbb{C} \setminus F_f.$$ 

It is possible to consider the Julia set to be a subset of the Riemann sphere $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ by adding the point of infinity $\infty$ to it. This definition is mainly adopted in the case of meromorphic functions (for example, see [Ber]) and also there are some researches on convergence phenomena of Julia sets as subsets of $\hat{\mathbb{C}}$ ([Ki], [Kr], [KrK]). In this setting, $J_f$ is compact in $\hat{\mathbb{C}}$ and hence $J_f$ is rather easy to handle. But for a transcendental entire function the suitable phase space as a dynamical system is the complex plane $\mathbb{C}$, not the Riemann sphere $\hat{\mathbb{C}}$, because $\infty$ is an essential singularity of $f$ and there seems to be no
reasonable way to define the value at \( \infty \). So it is more natural to regard \( J_f \) as a subset of \( \hat{\mathbb{C}} \) rather than of \( \mathbb{C} \) and hence we define \( J_f \) as above and write \( J_f \cup \{ \infty \} \) when we consider \( J_f \) to be a subset of \( \hat{\mathbb{C}} \).

A connected component \( U \) of \( F_f \) is called a Fatou component. A Fatou component is called a wandering domain if \( f^m(U) \cap f^n(U) = \emptyset \) for every \( m, n \in \mathbb{N} \) (\( m \neq n \)). If there exists an \( n_0 \in \mathbb{N} \) with \( f^{n_0}(U) \subseteq U \), \( U \) is called a periodic component and it is well known that there are following four possibilities:

1. There exists a point \( z_0 \in U \) with \( f^{n_0}(z_0) = z_0 \) and \( |(f^{n_0})'(z_0)| < 1 \) and every point \( z \in U \) satisfies \( f^{n_0k}(z) \to z_0 \) as \( k \to \infty \). The point \( z_0 \) is called an attracting periodic point and the domain \( U \) is called an attracting basin.

2. There exists a point \( z_0 \in \partial U \) with \( f^{n_0}(z_0) = z_0 \) and \( (f^{n_0})'(z_0) = e^{2\pi i\theta} \) (\( \theta \in \mathbb{Q} \)) and every point \( z \in U \) satisfies \( f^{n_0k}(z) \to z_0 \) as \( k \to \infty \). The point \( z_0 \) is called a parabolic periodic point and the domain \( U \) is called a parabolic basin.

3. There exists a point \( z_0 \in U \) with \( f^{n_0}(z_0) = z_0 \) and \( (f^{n_0})'(z_0) = e^{2\pi i\theta} \) (\( \theta \in \mathbb{R} \setminus \mathbb{Q} \)) and \( f^{n_0k}(U) \) is conjugate to an irrational rotation of a unit disk. The domain \( U \) is called a Siegel disk.

4. For every \( z \in U \), \( f^{n_0k}(z) \to \infty \) as \( k \to \infty \). The domain \( U \) is called a Baker domain.

In particular, if \( n_0 = 1 \), \( U \) is called an invariant component. \( U \) is called completely invariant if \( U \) satisfies \( f^{-1}(U) \subseteq U \). \( U \) is called a preperiodic component if \( f^m(U) \) is a periodic component for an \( m \geq 1 \). \( U \) is called eventually periodic if \( U \) is periodic or preperiodic.

It is known that eventually periodic components of a transcendental entire function are simply connected ([Ber], [EL1]) while a wandering domain can be multiply-connected ([Ba1], [Ba2], [Ba5]).

The boundary of unbounded periodic Fatou component can be extremely complicated. For example, consider the exponential family \( E_\lambda(z) := \lambda e^z \). If \( \lambda \) satisfies \( 0 < \lambda < \frac{1}{e} \), \( E_\lambda(z) \) has a unique attracting fixed point \( p_\lambda \) with an unbounded simply connected connected invariant basin \( \Omega(p_\lambda) \) and the Fatou set \( F_{E_\lambda} \) is equal to this basin ([DG]). Let \( \varphi : \mathbb{D} \to \Omega(p_\lambda) \) be a Riemann map of \( \Omega(p_\lambda) \) from a unit disk \( \mathbb{D} \), then the radial limit \( \lim_{r \to 1} \varphi(re^{i\theta}) \) exists for all \( e^{i\theta} \in \partial \mathbb{D} \) and moreover the set

\[
\Theta_\infty := \{ e^{i\theta} \mid \varphi(e^{i\theta}) := \lim_{r \to 1} \varphi(re^{i\theta}) = \infty \}
\]

is dense in \( \partial \mathbb{D} \) ([DG]). This implies that the Riemann map is highly discontinuous and hence the boundary of \( \Omega(p_\lambda) \), which is equal to \( J_{E_\lambda} \), is extremely complicated. From this fact, it follows that \( J_{E_\lambda} \) is disconnected in \( \mathbb{C} \), since \( \varphi \) is conformal the set

\[
\varphi(\{ re^{i\theta_1} \mid 0 \leq r < 1 \} \cup \{ re^{i\theta_2} \mid 0 \leq r < 1 \}) \subseteq U \quad (\theta_1, \theta_2 \in \Theta_\infty, \theta_1 \neq \theta_2)
\]
is a Jordan arc in $\mathbb{C}$ and this separates $J_{E_{\lambda}}$ into two disjoint relatively open subsets.

Taking these facts into account, we shall investigate the set $\Theta_{\infty}$ for a general unbounded periodic component $U$ and also consider the following problem:

**Problem**: When is the Julia set of a transcendental entire function $f$ connected or disconnected as a subset of $\mathbb{C}$?

If $f$ is a polynomial, the following criterion is well known. (For example, see [Bea] or [Mj]).

**Proposition A**: Let $f$ be a polynomial of degree $d \geq 2$. Then the Julia set $J_f$ is connected if and only if no finite critical values of $f$ tend to $\infty$ by the iterates of $f$.

Here, a critical value is a point $p := f(c)$ for a point $c$ with $f'(c) = 0$. This is a singularity of $f^{-1}$. For polynomials we have only to consider this type of singularities but there can be another type of singularities called an asymptotic value for the transcendental case. A point $p$ is called an asymptotic value if there exists a continuous curve $L(t)$ ($0 \leq t < 1$) called an asymptotic path with

$$\lim_{t \to 1} L(t) = \infty \quad \text{and} \quad \lim_{t \to 1} f(L(t)) = p.$$  

A point $p$ is called a singular value if it is either a critical or an asymptotic value and we denote the set of all singular values as $\text{sing}(f^{-1})$.

If $f$ is transcendental, however, the above criterion does not hold. For example, let us consider the exponential family $E_\lambda(z) := \lambda e^z$ again. If $\lambda$ satisfies $0 < \lambda < \frac{1}{e}$, the unique singular value $z = 0$ (this is an asymptotic value) is attracted to the fixed point $p_1$, and hence does not tend to $\infty$ but the Julia set $J_{E_{\lambda}}$ is disconnected as we mentioned above.

For other values of $\lambda$, for example $\lambda > \frac{1}{e}$, the singular value $z = 0$ may tend to $\infty$. If $f$ is a polynomial all of whose critical values tend to $\infty$, then $J_f$ is a Cantor set and especially disconnected. But on the other hand in this case $J_f$ is equal to the entire plain $\mathbb{C}$ ([D]) and hence connected.

Before considering the connectivity of $J_f$ in $\mathbb{C}$, we investigate the connectivity of $J_f \cup \{\infty\}$ in $\hat{\mathbb{C}}$. In this situation compactness of $J_f \cup \{\infty\}$ in $\hat{\mathbb{C}}$ makes the problem easier. Actually we can prove the following:

**Theorem 1**: Let $f$ be a transcendental entire function. Then the set $J_f \cup \{\infty\}$ in $\hat{\mathbb{C}}$ is connected if and only if $F_f$ has no multiply-connected wandering domains.

**Corollary 1**: Under one of the following conditions, $J_f \cup \{\infty\}$ in $\hat{\mathbb{C}}$ is connected.

1. $f \in B := \{f \mid \text{sing}(f^{-1}) \text{ is bounded}\}$.
2. $F_f$ has an unbounded component.
3. There exists a curve $\Gamma(t)$ ($0 \leq t < 1$) with $\lim_{t \to 1} \Gamma(t) = \infty$ such that $f|\Gamma$ is bounded. Especially $f$ has a finite asymptotic value.
Then how about \( J_f \) in \( \mathbb{C} \) itself? The results depend on whether \( F_f \) admits an unbounded component or not. In the case when \( F_f \) admits no unbounded components, we obtain the following:

**Theorem 2** Let \( f \) be a transcendental entire function. If all the components of \( F_f \) are bounded and simply connected, then \( J_f \) is connected.

The following is an easy consequence from Theorem 1 and 2.

**Corollary 2** Let \( f \) be a transcendental entire function. If all the components of \( F_f \) are bounded, then \( J_f \) is connected in \( \mathbb{C} \) if and only if \( J_f \cup \{\infty\} \) is connected in \( \bar{\mathbb{C}} \).

As we mentioned before, for the unbounded component \( \Omega(p_A) \) of \( F_{E_A} \) the set of all angles where the Riemann map \( \varphi : \mathbb{D} \to \Omega(p_A) \) admits the radial limit \( \infty \) is dense in \( \partial \mathbb{D} \) and this leads to the disconnectivity of \( J_{E_A} \). The main result of this paper is the generalization of this fact. Under some conditions this result holds for various kinds of unbounded periodic Fatou components. Here, a point \( p \in \partial U \) is accessible if there exists a continuous curve \( L(t) \) \((0 \leq t < 1)\) in \( U \) with \( \lim_{t \to 1} L(t) = p \).

**Main Theorem** Let \( U \) be an unbounded periodic Fatou component of a transcendental entire function \( f, \varphi : \mathbb{D} \to U \) be a Riemann map of \( U \) from a unit disk \( \mathbb{D} \), and

\[
P_{f^{m_0}} := \bigcup_{n=0}^{\infty} (f^{m_0})^n(\text{sing}((f^{m_0})^{-1})).
\]

We assume one of the following four conditions:

1. \( U \) is an attracting basin of period \( n_0 \) and \( \infty \in \partial U \) is accessible. There exists a finite point \( q \in \partial U \) with \( q \notin P_{f^{m_0}}, m_0 \in \mathbb{N} \) and a continuous curve \( C(t) \subset U \) \((0 \leq t \leq 1)\) with \( C(1) = q \) and satisfies \( f^{m_0}(C) \supset C \).

2. \( U \) is a parabolic basin of period \( n_0 \) and \( \infty \in \partial U \) is accessible. There exists a finite point \( q \in \partial U \) with \( q \notin P_{f^{m_0}}, m_0 \in \mathbb{N} \) and a continuous curve \( C(t) \subset U \) \((0 \leq t \leq 1)\) with \( C(1) = q \) and satisfies \( f^{m_0}(C) \supset C \).

3. \( U \) is a Siegel disk of period \( n_0 \) and \( \infty \in \partial U \) is accessible.

4. \( U \) is a Baker domain of period \( n_0 \) and \( f^{m_0} \) is not univalent. There exists a finite point \( q \in \partial U \) with \( q \notin P_{f^{m_0}}, m_0 \in \mathbb{N} \) and a continuous curve \( C(t) \subset U \) \((0 \leq t \leq 1)\) with \( C(1) = q \) and satisfies \( f^{m_0}(C) \supset C \).

Then the set

\[
\Theta_\infty := \{e^{i\theta} \mid \varphi(e^{i\theta}) := \lim_{r \to 1} \varphi(re^{i\theta}) = \infty\}
\]

is dense in \( \partial \mathbb{D} \) in the case of (1), (2) or (3). In the case of (4), the closure \( \overline{\Theta_\infty} \) contains a certain perfect set in \( \partial \mathbb{D} \). In particular, \( J_f \) is disconnected in all cases.

In the case of the exponential family, Devaney and Goldberg ([DG]) obtained the explicit
expression
\[ \varphi^{-1} \circ E_\lambda \circ \varphi(z) = \exp\left( \frac{\mu + \bar{\mu} z}{1 + z} \right), \quad \mu \in \{ z \mid \Im z > 0 \} \]
for a suitable Riemann map \( \varphi \) which was crucial to show the density of \( \Theta_\infty \) in \( \partial \mathbb{D} \). In general, of course, we cannot obtain the explicit form of \( \varphi^{-1} \circ f^{n_0} \circ \varphi(z) \) so instead of it we take advantage of a property of inner functions. In general analytic function \( g : \mathbb{D} \to \mathbb{D} \) is called an inner function if the radial limit \( g(e^{i\theta}) := \lim_{r \to 1} g(re^{i\theta}) \) exists for almost every \( e^{i\theta} \in \partial \mathbb{D} \) and satisfies \( |g(e^{i\theta})| = 1 \). It is easy to see that \( \varphi^{-1} \circ f^{n_0} \circ \varphi \) is an inner function. It is known that an inner function \( g \) has a unique fixed point \( p \in \mathbb{D} \) called a Denjoy-Wolff point and \( g^n(z) \) tends to \( p \) locally uniformly on \( \mathbb{D} \) ([DM]). The following is an important lemma for the proof of the Main Theorem.

**Lemma 1** Let \( g : \mathbb{D} \to \mathbb{D} \) be an inner function which is not a Möbius transformation and \( p \) its Denjoy-Wolff point.

1. If \( p \in \mathbb{D} \), then \( \bigcup_{n=1}^{\infty} g^{-n}(z_0) \supset \partial \mathbb{D} \) holds for every \( z_0 \in \mathbb{D} \setminus E \) where \( E \) is a certain exceptional set of logarithmic capacity zero.
2. If \( p \in \partial \mathbb{D} \), then \( \bigcup_{n=1}^{\infty} g^{-n}(z_0) \supset K \) holds for every \( z_0 \in \mathbb{D} \setminus E \) where \( E \) is a certain exceptional set of logarithmic capacity zero and \( K \) is a certain perfect set in \( \partial \mathbb{D} \).

If \( U \) is either an attracting basin or a parabolic basin and \( g = \varphi^{-1} \circ f^{n_0} \circ \varphi \), we can say more about the set \( \bigcup_{n=1}^{\infty} g^{-n}(z_0) \).

**Lemma 2** Let \( V \) be either an attracting basin or a parabolic basin (not necessarily unbounded) and \( g = \varphi^{-1} \circ f^{n_0} \circ \varphi \). Then there exists a set \( E \subset \mathbb{D} \) of logarithmic capacity zero such that
\[ \frac{\sigma_n(z_0, A)}{\sigma_n(z_0, \partial \mathbb{D})} \to \frac{\text{meas} A}{2\pi} \quad (n \to \infty) \]
holds for every \( z_0 \in \mathbb{D} \setminus E \) and every arc \( A \) in \( \partial \mathbb{D} \), where \( \sigma_n(z_0, A) = \sum_{\zeta}(1 - |\zeta|^2) \) and sum is taken over all \( \zeta = |\zeta|e^{i\theta} \) with \( g^n(\zeta) = z_0 \) and \( e^{i\theta} \in A \).

We omit the proofs of Lemma 1 and Lemma 2.

In §2 we prove Theorem 1 and Corollary 1. §3 consists of two subsections. In §3.1 we prove Theorem 2 and make some remarks on the sufficient conditions for \( f \) to admit no unbounded Fatou components. In §3.2 we prove the Main Theorem by using Lemma 1 and Lemma 2.

2 Connectivity of \( J_f \cup \{ \infty \} \) in \( \overline{\mathbb{C}} \)

**(Proof of Theorem 1):** The following criterion is well known. (See for example [Bea], p.81, Proposition 5.1.5).
Proposition B Let $K$ be a compact subset in $\mathbb{C}$. Then $K$ is connected if and only if each component of the complement $K^c$ is simply connected.

Since $J_f \cup \{\infty\}$ is compact in $\mathbb{C}$, we can apply Proposition B. As we mentioned in §1, eventually periodic components are simply connected. So if a Fatou component $U$ is not simply connected, then $U$ is necessarily a wandering domain which is not simply connected. This completes the proof. \qed

(Proof of Corollary 1): Under the condition (1), $f^n$ cannot tend to $\infty$ through $F_f$ ([EL2]). On the other hand, $f^n$ tends to $\infty$ on any multiply-connected wandering domains ([Ba4], [EL1]). So all the Fatou components are simply connected in this case. Under the condition (2) or (3), it is known that all the Fatou components must be simply connected ([Ba4], [EL1], p.620 Corollary 1, 2). \qed

Remark 1 (1) Let $S := \{f \mid \# \text{sing}(f^{-1}) < \infty\} \subset B$. Then there is even no wandering domain in $F_f$ for $f \in S$ ([GK]). For $f \in B$, $F_f$ may admit a wandering domain $U$ but $U$ must be simply connected as we mentioned above. Under an additional condition $J_f \cap (\text{derived set of } \bigcup_{n=0}^{\infty} f^n(\text{sing}(f^{-1}))) = \emptyset, \quad f \in B$ has also no wandering domain ([BHKMT]).

(2) We can conclude that in general if $J_f \cup \{\infty\}$ is disconnected, all the Fatou components are bounded and some of which are multiply-connected wandering domains.

3 Connectivity of $J_f$ in $\mathbb{C}$

3.1 The case when all the Fatou components are bounded

Suppose that a closed connected subset $K$ in $\mathbb{C}$ is bounded. Then all the components of the complement $K^c$ other than the unique unbounded component $V$ are simply connected. (Of course, $V \cup \{\infty\} \subset \mathbb{C}$ is simply connected). If $K$ is unbounded, then all the components of $K^c$ are simply connected, but the converse is false as the example $J_{E_{\lambda}}(0 < \lambda < \frac{1}{2})$ shows. (Compare with the Proposition B). But note that $J_{E_{\lambda}} \cup \{\infty\}$ is connected in $\mathbb{C}$. For the connectivity of a closed subset in $\mathbb{C}$, the following criterion holds.

Proposition 1 Let $K$ be a closed subset of $\mathbb{C}$. Then $K$ is connected if and only if the boundary of each component $U$ of the complement $K^c$ is connected.

(Proof): For the 'only if' part, see [New]. Suppose that $K$ is disconnected. Then there exist two closed sets $K_1$ and $K_2$ with $K = K_1 \cup K_2$ and $K_1 \cap K_2 = \emptyset$. Take a point $z_0$ with $d(z_0, K_1) = d(z_0, K_2)$ where $d$ denotes the Euclid distance in $\mathbb{C}$. Then $z_0 \in K^c$ and so let $U_0$
be the connected component of $K^2$ containing $z_0$. Since $\partial U_0$ is connected by the assumption, either $\partial U_0 \subset K_1$ or $\partial U_0 \subset K_2$. Without loss of generality we can assume $\partial U_0 \subset K_1$. On the other hand denote $r_0 := d(z_0, K_1) = d(z_0, K_2)$ and let $D_{r_0}(z_0) := \{z \mid |z - z_0| < r_0\}$. Then $D_{r_0}(z_0) \subset \overline{U_0}$ and there exists a point $w \in K_2$ with $w \in \overline{U_0}$. Since $w \in K_2 \subset K$, we have $w \in \partial U_0$ but this is a contradiction since $\partial U_0 \subset K_1$ and $K_1 \cap K_2 = \emptyset$. This completes the proof.

(Proof of Theorem 2): By Proposition 1, it is sufficient to to show that the boundary $\partial U$ is connected for each Fatou component $U$. Since $U$ is bounded, the boundary of $U$ as a subset of $\mathbb{C}$ and the one as the subset of $\hat{\mathbb{C}}$ coincide. Hence $U$ is simply connected if and only if $\partial U$ is connected ([Bea], p.81, Proposition 5.1.4). This completes the proof.

Remark 2 (1) Since a non-simply connected Fatou component is necessarily a wandering domain, the assumption of Theorem 2 is equivalent to that all the components of $F_f$ are bounded and $F_f$ admits no multiply-connected wandering domains.

(2) Several sufficient conditions are known for a transcendental entire function $f$ to admit no unbounded Fatou components as follows:

(i) ([Ba3]) $\log M(r) = O((\log r)^p)$ (as $r \to \infty$) where $M(r) = \sup |f(z)|$ and $1 < p < 3$.

(ii) ([S]) There exists $c \in (0, 1)$ such that $\log \log M(r) < (\log r)^c$ for large $r$.

(iii) ([S]) The order of $f$ is less than $\frac{1}{2}$ and $\frac{\log M(2r)}{\log M(r)} \to c$ (finite constant) as $r \to \infty$.

Note that the condition (ii) includes the condition (i).

3.2 In the case when $F_f$ admits an unbounded component

(Proof of Main Theorem): In what follows we assume that $n_0 = 1$ (that is, $U$ is an invariant component) and $m_0 = 1$ for simplicity. This causes no loss of generality, because we have only to consider $f^{m_0}$ instead of $f$ in general cases.

Case (1) Since $\infty$ is accessible, there exists a continuous curve $L(t)$ ($0 \leq t < 1$) in $U$ with $\lim_{t \to 1} L(t) = \infty$. By deforming $L(t)$ slightly, we construct a new curve $L(t)$ satisfying the following condition.

Lemma 3 There exists a curve $L(t)$ ($0 \leq t < 1$) with $\lim_{t \to 1} L(t) = \infty$ such that every branch of $f^{-n}$ can be analytically continued along it for every $n \in \mathbb{N}$.

(Proof): We may assume that $L(0) \notin P_f$, since $q \notin P_f$ we have $U \subset P_f$. Let $p_0 := L(0), p_1, p_2, \ldots$ be points on $L$ such that all the piecewise linear line segments connecting $p_0, p_1, p_2, \ldots$ lie in $U$. Let $F_n^{(1)}, F_n^{(2)}, \ldots, F_n^{(m)}, \ldots$ be all the branches of $f^{-n}$ which take values on $U$. The range of the suffix $m$ may be finite or infinite. Define

$$E_n^{(m)}(p_0) := \{e^{i\theta} \mid F_n^{(m)} \text{can be analytically continued along the ray}$$

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Then by the next Gross's Star Theorem ([Nev]), it follows that \( \Theta_p^{(m)}(p_0) \) has full measure in \( \partial \mathbb{D} \).

**Lemma C (Gross's Star Theorem)** Let \( f \) be an entire function and \( F \) a branch of \( f^{-1} \) defined in the neighborhood of \( p_0 \in \mathbb{C} \). Then \( F \) can be analytically continued along almost all rays from \( p_0 \) in the direction \( \theta \).

Then the set

\[
\Theta(p_0) := \bigcap_{n \geq 1, m \geq 1} \Theta_p^{(m)}(p_0)
\]

has also full measure in \( \partial \mathbb{D} \). Hence by changing \( p_1 \) slightly to a point \( p_1' \), the segments \( \overline{p_0p_1} \) and \( \overline{p_1'p_2} \) lie in \( U \) and all the branches \( F^{(m)}(n \geq 1, m \geq 1) \) can be analytically continued along \( \overline{p_0p_1} \). By the same method, we can find a point \( p_2' \) close to \( p_2 \) such that the segment \( \overline{p_1'p_2'} \) lies in \( U \) and has the same property as above. By repeating this argument, we can prove the Lemma 3.

Let \( I_n^{(m)}(t) := F^{(m)}(\mathcal{L}(t)) \) then we have \( \lim_{t \to -1} I_n^{(m)}(t) = \infty \). For suppose this is false, then there exist an increasing sequence of parameter values \( t_1 < t_2 < \cdots < t_k < \cdots \) and a finite point \( \alpha \) with \( \lim_{k \to \infty} I_n^{(m)}(t_k) = \alpha \not= \infty \). Then it follows that \( \lim_{k \to \infty} \mathcal{L}(t_k) = f^n(\alpha) \not= \infty \) and this contradicts the fact \( \lim_{k \to \infty} \mathcal{L}(t_k) = \infty \).

Let \( \varphi : \mathbb{D} \to U \) be a Riemann map of \( U \). Then

\[
\Gamma(t) := \varphi^{-1}(\mathcal{L}(t)) \quad \text{and} \quad I_n^{(m)}(t) := \varphi^{-1}(I_n^{(m)}(t))
\]

are curves in \( \mathbb{D} \) landing at a point in \( \partial \mathbb{D} \). This fact is not so trivial but follows from the proposition in [P] (p.29, Proposition 2.14). We may assume that \( \Gamma(t) \) lands at \( z = 1 \in \partial \mathbb{D} \) for simplicity. If \( \lim_{t \to -1} \gamma_n^{(m)}(t) = e^{i\theta_0} \), then since \( \lim_{t \to -1} \varphi(\gamma_n^{(m)}(t)) = \lim_{t \to -1} I_n^{(m)}(t) = \infty \), it follows that there exists the radial limit \( \lim_{t \to -1} \varphi(re^{i\theta}) \) and this is equal to \( \infty \). This fact follows from the theorem in [P] (p.34, Theorem 2.16). Therefore it is sufficient to show that the set of all the landing points of \( I_n^{(m)}(t) \) \((n \geq 1, m \geq 1)\) is dense in \( \partial \mathbb{D} \).

Let \( g := \varphi^{-1} \circ f \circ \varphi : \mathbb{D} \to \mathbb{D} \). Then by Fatou's theorem \( \varphi \) has radial limit \( \varphi(e^{i\theta}) = \lim_{r \to 1} \varphi(re^{i\theta}) \) is \( \partial U \) and non-constant for almost every \( e^{i\theta} \in \partial \mathbb{D} \). Hence \( f \circ \varphi(re^{i\theta}) \) is a curve landing at a point in \( \partial U \setminus \{\infty\} \) for almost every \( e^{i\theta} \in \partial \mathbb{D} \). Therefore it follows that \( \lim_{r \to 1} \varphi^{-1} \circ f \circ \varphi(re^{i\theta}) \in \partial \mathbb{D} \) a.e. and thus \( g \) is an inner function. Let \( \bar{C} := \varphi^{-1}(C) \) then by the same reason for \( \Gamma(t) \), \( \bar{C} \) is a curve in \( \mathbb{D} \) with an end point \( \bar{q} \in \partial U \) satisfying \( g(\bar{C}) \supset \bar{C} \). From the dynamics of \( g : \mathbb{D} \to \mathbb{D} \), it follows that the set

\[
\bigcup_{n=0}^{\infty} g^n(\bar{C}) \cup \{\bar{p}, \bar{q}\}
\]

is compact in \( \overline{\mathbb{D}} \) where \( \bar{p} = \varphi^{-1}(p) \) and \( \bar{p} \) is an attracting fixed point of \( g \) and the distance between this set and \( z = 1 \) is positive. Hence there exists \( \varepsilon_0 > 0 \) such that

\[
U_{\varepsilon_0}(1) \cap \left\{ \bigcup_{n=0}^{\infty} g^n(\bar{C}) \cup \{\bar{p}, \bar{q}\} \right\} = \emptyset
\]

(1)
Since \( \Gamma(t) \) lands at \( z = 1 \), there exists \( t_0 \in [0,1) \) such that \( \Gamma(t_0,1) \subset U_{\varepsilon_0}(1) \). So by rewriting \( \Gamma \) to \( \Gamma(t) \) \((0 \le t < 1)\) we may assume that \( \Gamma(t) \subset U_{\varepsilon_0}(1) \) for \( 0 \le t < 1 \).

Let \( K := \{ \xi \mid |\xi| \le 1 - \varepsilon_0 \} \) then since every point in \( \mathbb{D} \) tends to \( \bar{\mathbb{D}} \) under \( g^n \) and \( K \) is compact, there exists \( n_1 \in \mathbb{N} \) such that for every \( N \ge n_1 \) we have \( g^N(K) \subset U_{\varepsilon_0}(1) \). Then we have \( \gamma_n^{(m)}(t) \subset K^c \) for every \( N \ge n_1 \). For suppose that \( \gamma_n^{(m)}(t) \cap K \neq \emptyset \), then by operating \( f^N \) we have \( \Gamma(t) \cap K \neq \emptyset \) which contradicts \( \Gamma(t) \subset U_{\varepsilon_0}(1) \).

Now suppose that the conclusion does not hold. Then there exists

\[
(\theta_1, \theta_2) := \{ e^{i\theta} \mid \theta_1 < \theta < \theta_2 \} \subset \partial \mathbb{D} \quad \text{with} \quad \Theta_\infty \cap (\theta_1, \theta_2) = \emptyset.
\]

By changing the starting point \( \gamma(0) \) slightly, if necessary, we may assume that the points \( \gamma_n^{(m)}(0) \) \((n, m = 1, 2, \ldots)\) accumulate to all over \( \partial \mathbb{D} \) by Lemma 1 (1) while the end points \( \gamma_n^{(m)}(1) := \lim_{t \to 1} \gamma_n^{(m)}(t) \) \((n, m = 1, 2, \ldots)\) are not in \((\theta_1, \theta_2)\). Therefore there exists \( \gamma_n^{(m)}(t) \) such that \( \gamma_n^{(m)}(t) \subset K^c \) and \( \gamma_n^{(m)}(1) = \partial \mathbb{D} \setminus (\theta_1, \theta_2) \).

On the other hand there exist inverse images \( g^{-n}(\bar{C}) \) which have limit points on \((\theta_1, \theta_2)\) densely. The reason is as follows: Since \( q \notin P \), there exists a neighborhood \( V \) of \( q \) such that all the branches \( F_n^{(1)}, F_n^{(2)}, \ldots, F_n^{(m)}, \ldots \) can be defined. Let \( V_0 \subset V \) be a neighborhood of \( q \) with \( V_0 \subset V \). We may assume that \( C \subset V_0 \). Define

\[
c_n^{(m)}(t) := F_n^{(m)}(C(t)), \quad \bar{c}_n^{(m)}(t) := \varphi^{-1}(c_n^{(m)}(t)).
\]

Then \( c_n^{(m)}(t) \) is a curve in \( U \) landing at a point in \( \partial U \) and \( \bar{c}_n^{(m)}(t) \) is a curve in \( \bar{D} \) landing at a point in \( \partial \bar{D} \) by the same reason as before. Let \( (\theta_3, \theta_4) \subset (\theta_1, \theta_2) \) be any subarc of \((\theta_1, \theta_2)\). By changing the starting point \( C(0) \) slightly, if necessary, we may assume that the points \( c_n^{(m)}(0) \) \((n, m = 1, 2, \ldots)\) accumulate to \((\theta_3, \theta_4)\) by Lemma 1 (1). Since radial limits of \( \varphi \) exist and non-constant almost everywhere, by changing \( \theta_3 \) and \( \theta_4 \) slightly if necessary, we may assume that there exist the finite values \( \varphi(e^{i\theta_3}) \) and \( \varphi(e^{i\theta_4}) \) with \( \varphi(e^{i\theta_3}) \neq \varphi(e^{i\theta_4}) \). Then \( c_n^{(m)}(0) \) accumulate on \( \partial U \cap \varphi\{r e^{i\theta} \mid \theta_3 < \theta < \theta_4, \quad 0 \le r \le 1 \} \). In general the family of single-valued analytic branch of \( F^{(m)} \) \((n = 1, 2, \ldots)\) on a domain \( U_0 \) is normal and furthermore if \( U_0 \cap J \neq \emptyset \), any local uniform limit of a subsequence in the family is constant ([Bea], p.193, Theorem 9.2.1, Lemma 9.2.2). So the family \( \{ F_n^{(m)} | V_0 \} \) is normal and all its limit functions are constant and hence for a suitable subsequence the diameter of \( c_n^{(m)}(t) \) tends to zero, that is, \( c_n^{(m)}(t) \) must land in a point in \( (\theta_3, \theta_4) \). If the constant limit is \( \infty \), for large enough \( n \) the curves \( c_n^{(m)} \) cannot intersect both \( \{ \varphi(r e^{i\theta_3}) | 0 \le r \le 1 \} \) and \( \{ \varphi(r e^{i\theta_4}) | 0 \le r \le 1 \} \) which are bounded set, since the convergence is uniform on \( V_0 \). Hence again we can conclude that \( c_n^{(m)}(t) \) must land at a point in \( \partial U \cap \varphi\{r e^{i\theta} \mid \theta_3 < \theta < \theta_4, \quad 0 \le r \le 1 \} \) and therefore \( c_n^{(m)}(t) \) must land at a point in \( (\theta_3, \theta_4) \). This proves the assertion.

Then there exists \( \gamma_n^{(M_1)} \) such that \( \gamma_n^{(m)}(1) \cap \gamma_n^{(M_1)} \neq \emptyset \). We may assume that \( n_1 > N_1 \). Let \( u \in \gamma_n^{(m)} \cap \gamma_n^{(M_1)} \) then since \( u \in \gamma_n^{(m)} \), we have \( g^{n_1}(u) \in U_{\varepsilon_0}(1) \). On the other hand since \( u \in \gamma_n^{(M_1)} \) and \( n_1 > N_1 \), we have \( g^{n_1}(u) \in \bigcup_{n=0}^{\infty} g^n(C) \) which contradicts (1). Therefore

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$\Theta_\infty$ is dense in $\partial D$. Disconnectivity of $J_f$ easily follows by the same argument as in the case of $E_\lambda$ in §1. This completes the proof in the case of (1).

Case (2) The proof is quite parallel to the case (1). Note that by Lemma 2, $\bigcup_{n=1}^\infty g^{-n}(z_0) \supset \partial D (z_0 \in D \setminus E)$ holds for $g = \varphi^{-1} \circ f \circ \varphi$ in this case.

Case (3) Since $g(z) = e^{2\pi i \theta}$ with $\theta \in \mathbb{R} \setminus \mathbb{Q}$, the inverse image of $\Gamma(t)$ by $g^{-n}$ is unique and denote it by $\gamma_n(t)$. Then it is obvious that the end points of $\gamma_n(t)$ are dense in $\partial D$ and $\varphi$ attains radial limit $\infty$ there, since $g(z)$ is an irrational rotation and $\lim_{t \to 1} \varphi(\gamma_n(t)) = \lim_{t \to 1} f^{-1}(\varphi(\Gamma(t))) = \infty$.

Case (4) In this case we need not assume the accessibility of $\infty$, because this condition is automatically satisfied ([Ba6]). The set $\bigcup_{n=0}^\infty f^n(C)$ is a curve which may have self-intersections and tends to $\infty$. It is not difficult to take $L$ satisfying $L \cap (\bigcup_{n=0}^\infty f^n(C)) = \emptyset$. Hence we have $L \cap (\bigcup_{n=0}^\infty f^n(C)) = \emptyset$. The rest of the proof is quite parallel to the case (1) if the conclusion of Lemma 2 (1) holds for $g$. If we have only the conclusion of Lemma 2 (2), then we can prove that for every arc $A \subset \partial D$ with $A \cap K = \emptyset$, $A \cap \Theta_\infty = \emptyset$ holds by the similar argument.

References


[Kr] B. Krauskopf, *Convergence of Julia sets in the approximation of $\lambda e^z$ by $\lambda \left(1 + \frac{z^d}{d}\right)$*, Internat. J. Bif. & Chaos, 3 (1993), 257–270.


