On the Connectivity of Julia Sets of Transcendental Entire Functions

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Abstract

We have two main purposes in this paper. One is to give some sufficient conditions for the Julia set of a transcendental entire function f to be connected or to be disconnected as a subset of the complex plane $\mathbb{C}$. The other is to investigate the boundary of an unbounded periodic Fatou component U, which is known to be simply connected. These are related as follows: let $\varphi : \mathbb{D} \longrightarrow U$ be a Riemann map of U from a unit disk $\mathbb{D}$, then under some mild conditions we show the set Θ_∞ of all angles where φ admits the radial limit ∞ are dense in $\partial\mathbb{D}$ if U is an attracting basin, a parabolic basin or a Siegel disk. If U is a Baker domain on which f is not univalent, then Θ_∞ is dense in $\partial\mathbb{D}$ or at least its closure $\overline{\Theta_\infty}$ contains a certain perfect set, which means the boundary ∂U has a very complicated structure. In all cases, this result leads to the disconnectivity of the Julia set J_f in $\mathbb{C}$. We also consider the connectivity of the set $J_f \cup \{\infty\}$ in the Riemann sphere $\widehat{\mathbb{C}}$ and show that $J_f \cup \{\infty\}$ is connected if and only if f has no multiply-connected wandering domains.

1 Definitions and Results

Let f be a transcendental entire function and f^n denote the n-th iterate of f. Recall that *the Fatou set* F_f and *the Julia set* J_f of f are defined as follows:

$$\begin{aligned} F_f &:= \{z \in \mathbb{C} \mid \{f^n\}_{n=1}^{\infty} \text{ is a normal family in a neighborhood of } z\}, \\ J_f &:= \mathbb{C} \setminus F_f. \end{aligned}$$

It is possible to consider the Julia set to be a subset of the Riemann sphere $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ by adding the point of infinity ∞ to it. This definition is mainly adopted in the case of meromorphic functions (for example, see [**Ber**]) and also there are some researches on convergence phenomena of Julia sets as subsets of $\widehat{\mathbb{C}}$ ([**Ki**], [**Kr**], [**KrK**]). In this setting, J_f is compact in $\widehat{\mathbb{C}}$ and hence J_f is rather easy to handle. But for a transcendental entire function the suitable phase space as a dynamical system is the complex plane $\mathbb{C}$, not the Riemann sphere $\widehat{\mathbb{C}}$, because ∞ is an essential singularity of f and there seems to be no reasonable way to define the value at ∞. So it is more natural to regard J_f as a subset of $\mathbb{C}$ rather than of $\widehat{\mathbb{C}}$ and hence we define J_f as above and write $J_f \cup \{\infty\}$ when we consider J_f to be a subset of $\widehat{\mathbb{C}}$.

A connected component U of F_f is called *a Fatou component.* A Fatou component is called *a wandering domain* if $f^m(U) \cap f^n(U) = \emptyset$ for every $m, n \in \mathbb{N}$ $(m \neq n)$. If there exists an $n_0 \in \mathbb{N}$ with $f^{n_0}(U) \subseteq U$, U is called *a periodic component* and it is well known that there are following four possibilities:

1. There exists a point $z_0 \in U$ with $f^{n_0}(z_0) = z_0$ and $|(f^{n_0})'(z_0)| < 1$ and every point $z \in U$ satisfies $f^{n_0 k}(z) \to z_0$ as $k \to \infty$. The point z_0 is called *an attracting periodic point* and the domain U is called *an attracting basin.*

2. There exists a point $z_0 \in \partial U$ with $f^{n_0}(z_0) = z_0$ and $(f^{n_0})'(z_0) = e^{2\pi i\theta}$ $(\theta \in \mathbb{Q})$ and every point $z \in U$ satisfies $f^{n_0 k}(z) \to z_0$ as $k \to \infty$. The point z_0 is called *a parabolic periodic point* and the domain U is called *a parabolic basin.*

3. There exists a point $z_0 \in U$ with $f^{n_0}(z_0) = z_0$ and $(f^{n_0})'(z_0) = e^{2\pi i\theta}$ $(\theta \in \mathbb{R} \setminus \mathbb{Q})$ and $f^{n_0}|U$ is conjugate to an irrational rotation of a unit disk. The domain U is called *a Siegel disk.*

4. For every $z \in U$, $f^{n_0 k}(z) \to \infty$ as $k \to \infty$. The domain U is called *a Baker domain.*

In particular, if $n_0 = 1$, U is called *an invariant component.* U is called *completely invariant* if U satisfies $f^{-1}(U) \subseteq U$. U is called *a preperiodic component* if $f^m(U)$ is a periodic component for an $m \geq 1$. U is called *eventually periodic* if U is periodic or preperiodic. It is known that eventually periodic components of a transcendental entire function are simply connected ([**Ber**], [**EL1**]) while a wandering domain can be multiply-connected ([**Ba1**], [**Ba2**], [**Ba5**]).

The boundary of unbounded periodic Fatou component can be extremely complicated. For example, consider the exponential family $E_\lambda(z) := \lambda e^z$. If λ satisfies $0 < \lambda < \frac{1}{e}$, $E_\lambda(z)$ has a unique attracting fixed point p_λ with an unbounded simply connected completely invariant basin $\Omega(p_\lambda)$ and the Fatou set F_{E_λ} is equal to this basin ([**DG**]). Let $\varphi : \mathbb{D} \longrightarrow \Omega(p_\lambda)$ be a Riemann map of $\Omega(p_\lambda)$ from a unit disk $\mathbb{D}$, then the *radial limit* $\lim_{r \nearrow 1} \varphi(re^{i\theta})$ exists for all $e^{i\theta} \in \partial\mathbb{D}$ and moreover the set

$$\Theta_\infty := \{e^{i\theta} \mid \varphi(e^{i\theta}) := \lim_{r \nearrow 1} \varphi(re^{i\theta}) = \infty\}$$

is dense in $\partial\mathbb{D}$ ([**DG**]). This implies that the Riemann map is highly discontinuous and hence the boundary of $\Omega(p_\lambda)$, which is equal to J_{E_λ}, is extremely complicated. From this fact, it follows that J_{E_λ} is disconnected in $\mathbb{C}$, since φ is conformal the set

$$\varphi(\{re^{i\theta_1} \mid 0 \leq r < 1\} \cup \{re^{i\theta_2} \mid 0 \leq r < 1\}) \subset U \quad (\theta_1, \theta_2 \in \Theta_\infty,\ \theta_1 \neq \theta_2)$$

is a Jordan arc in $\mathbb{C}$ and this separates J_{E_λ} into two disjoint relatively open subsets.

Taking these facts into account, we shall investigate the set Θ_∞ for a genetal unbounded periodic component U and also consider the following problem

Problem : When is the Julia set of a transcendental entire function f connected or disconnected as a subset of $\mathbb{C}$?

If f is a polynomial, the following criterion is well known. (For example, see [Bea] or [M]).

Proposition A *Let f be a polynomial of degree $d \geq 2$. Then the Julia set J_f is connected if and only if no finite critical values of f tend to ∞ by the iterates of f.*

Here, *a critical value* is a point $p := f(c)$ for a point c with $f'(c) = 0$. This is a singularity of f^{-1}. For polynomials we have only to consider this type of singularities but there can be another type of singularities called *an asymptotic value* for the transcendental case. A point p is called an asymptotic value if there exists a continuous curve $L(t)$ $(0 \leq t < 1)$ called *an asymptotic path* with

$$\lim_{t\to 1} L(t) = \infty \quad \text{and} \quad \lim_{t\to 1} f(L(t)) = p.$$

A point p is called *a singular value* if it is either a critical or an asymptotic value and we denote the set of all singular values as $\operatorname{sing}(f^{-1})$.

If f is transcendental, however, the above criterion does not hold. For example, let us consider the exponential family $E_\lambda(z) := \lambda e^z$ again. If λ satisfies $0 < \lambda < \frac{1}{e}$, the unique singular value $z = 0$ (this is an asymptotic value) is attracted to the fixed point p_λ and hence does not tend to ∞ but the Julia set J_{E_λ} is disconnected as we mentioned above.

For other values of λ, for example $\lambda > \frac{1}{e}$, the singular value $z = 0$ may tend to ∞. If f is a polynomial all of whose critical values tend to ∞, then J_f is a Cantor set and especially disconnected. But on the other hand in this case J_f is equal to the entire plain $\mathbb{C}$ ([D]) and hence connected.

Before considering the connectivity of J_f in $\mathbb{C}$, we investigate the connectivity of $J_f \cup \{\infty\}$ in $\widehat{\mathbb{C}}$. In this situation compactness of $J_f \cup \{\infty\}$ in $\widehat{\mathbb{C}}$ makes the problem easier. Actually we can prove the following:

Theorem 1 *Let f be a transcendental entire function. Then the set $J_f \cup \{\infty\}$ in $\widehat{\mathbb{C}}$ is connected if and only if F_f has no multiply-connected wandering domains.*

Corollary 1 *Under one of the following conditions, $J_f \cup \{\infty\}$ in $\widehat{\mathbb{C}}$ is connected.*

(1) $f \in B := \{f \mid \operatorname{sing}(f^{-1}) \text{ is bounded}\}$.

(2) F_f has an unbounded component.

(3) There exists a curve $\Gamma(t)$ $(0 \leq t < 1)$ with $\lim_{t\to 1} \Gamma(t) = \infty$ such that $f|\Gamma$ is bounded. Especially f has a finite asymptotic value.

Then how about J_f in $\mathbb{C}$ itself? The results depend on whether F_f admits an unbounded component or not. In the case when F_f admits no unbounded components, we obtain the following:

Theorem 2 *Let f be a transcendental entire function. If all the components of F_f are bounded and simply connected, then J_f is connected.*

The following is an easy consequence from Theorem 1 and 2.

Corollary 2 *Let f be a transcendental entire function. If all the components of F_f are bounded, then J_f is connected in $\mathbb{C}$ if and only if $J_f \cup \{\infty\}$ is connected in $\widehat{\mathbb{C}}$.*

As we mentioned before, for the unbounded component $\Omega(p_\lambda)$ of F_{E_λ} the set of all angles where the Riemann map $\varphi : \mathbb{D} \longrightarrow \Omega(p_\lambda)$ admits the radial limit ∞ is dense in $\partial\mathbb{D}$ and this leads to the disconnectivity of J_{E_λ}. The Main result of this paper is the generalization of this fact. Under some conditions this result holds for various kinds of unbounded periodic Fatou components. Here, a point $p \in \partial U$ is *accessible* if there exists a continuous curve $L(t)$ $(0 \leq t < 1)$ in U with $\lim_{t \to 1} L(t) = p$.

Main Theorem *Let U be an unbounded periodic Fatou component of a transcendental entire function f, $\varphi : \mathbb{D} \longrightarrow U$ be a Riemann map of U from a unit disk $\mathbb{D}$, and*

$$P_{f^{n_0}} := \overline{\bigcup_{n=0}^{\infty} (f^{n_0})^n (\mathrm{sing}((f^{n_0})^{-1}))}.$$

We assume one of the following four conditions:

(1) U is an attracting basin of period n_0 and $\infty \in \partial U$ is accessible. There exists a finite point $q \in \partial U$ with $q \notin P_{f^{n_0}}$, $m_0 \in \mathbb{N}$ and a continuous curve $C(t) \subset U$ $(0 \leq t \leq 1)$ with $C(1) = q$ and satisfies $f^{m_0}(C) \supset C$.

(2) U is a parabolic basin of period n_0 and $\infty \in \partial U$ is accessible. There exists a finite point $q \in \partial U$ with $q \notin P_{f^{n_0}}$, $m_0 \in \mathbb{N}$ and a continuous curve $C(t) \subset U$ $(0 \leq t \leq 1)$ with $C(1) = q$ and satisfies $f^{m_0}(C) \supset C$.

(3) U is a Siegel disk of period n_0 and $\infty \in \partial U$ is accessible.

(4) U is a Baker domain of period n_0 and $f^{n_0}|U$ is not univalent. There exists a finite point $q \in \partial U$ with $q \notin P_{f^{n_0}}$, $m_0 \in \mathbb{N}$ and a continuous curve $C(t) \subset U$ $(0 \leq t \leq 1)$ with $C(1) = q$ and satisfies $f^{m_0}(C) \supset C$.

Then the set

$$\Theta_\infty := \{e^{i\theta} \mid \varphi(e^{i\theta}) := \lim_{r \nearrow 1} \varphi(re^{i\theta}) = \infty\}$$

is dense in $\partial\mathbb{D}$ in the case of (1), (2) or (3). In the case of (4), the closure $\overline{\Theta_\infty}$ contains a certain perfect set in $\partial\mathbb{D}$. In particular, J_f is disconnected in all cases.

In the case of the exponential family, Devaney and Goldberg ([DG]) obtained the explicit expression

$$\varphi^{-1}\circ E_\lambda\circ\varphi(z)=\exp i\Big(\frac{\mu+\bar\mu z}{1+z}\Big),\quad \mu\in\{z\mid \mathrm{Im}\ z>0\}$$

for a suitable Riemann map φ which was crucial to show the density of Θ_∞ in $\partial\mathbb{D}$. In general, of course, we cannot obtain the explicit form of $\varphi^{-1}\circ f^{n_0}\circ\varphi(z)$ so instead of it we take advantage of a property of inner functions. In general analytic function $g:\mathbb{D}\longrightarrow\mathbb{D}$ is called *an inner function* if the radial limit $g(e^{i\theta}):=\lim_{r\nearrow 1}g(re^{i\theta})$ exists for almost every $e^{i\theta}\in\partial\mathbb{D}$ and satisfies $|g(e^{i\theta})|=1$. It is easy to see that $\varphi^{-1}\circ f^{n_0}\circ\varphi$ is an inner function. It is known that an inner function g has a unique fixed point $p\in\overline{\mathbb{D}}$ called *a Denjoy-Wolff point* and $g^n(z)$ tends to p locally uniformly on $\mathbb{D}$ ([DM]). The following is an important lemma for the proof of the Main Theorem.

Lemma 1 *Let $g:\mathbb{D}\longrightarrow\mathbb{D}$ be an inner function which is not a Möbius transformation and p its Denjoy-Wolff point.*
(1) If $p\in\mathbb{D}$, then $\overline{\bigcup_{n=1}^\infty g^{-n}(z_0)}\supset\partial\mathbb{D}$ holds for every $z_0\in\mathbb{D}\setminus E$ where E is a certain exceptional set of logarithmic capacity zero.
(2) If $p\in\partial\mathbb{D}$, then $\overline{\bigcup_{n=1}^\infty g^{-n}(z_0)}\supset K$ holds for every $z_0\in\mathbb{D}\setminus E$ where E is a certain exceptional set of logarithmic capacity zero and K is a certain perfect set in $\partial\mathbb{D}$.

If U is either an attracting basin or a parabolic basin and $g=\varphi^{-1}\circ f^{n_0}\circ\varphi$, we can say more about the set $\bigcup_{n=1}^\infty g^{-n}(z_0)$.

Lemma 2 *Let U be either an attracting basin or a parabolic basin (not necessarily unbounded) and $g=\varphi^{-1}\circ f^{n_0}\circ\varphi$. Then there exists a set $E\subset\mathbb{D}$ of logarithmic capacity zero such that*

$$\frac{\sigma_n(z_0,A)}{\sigma_n(z_0,\partial\mathbb{D})}\rightarrow\frac{meas A}{2\pi}\quad (n\to\infty)$$

holds for every $z_0\in\mathbb{D}\setminus E$ and every arc A in $\partial\mathbb{D}$, where $\sigma_n(z_0,A)=\sum_\zeta(1-|\zeta|^2)$ and sum is taken over all $\zeta=|\zeta|e^{i\theta}$ with $g^n(\zeta)=z_0$ and $e^{i\theta}\in A$.

We omit the proofs of Lemma 1 and Lemma 2.

In §2 we prove Theorem 1 and Corollary 1. §3 consists of two subsections. In §3.1 we prove Theorem 2 and make some remarks on the sufficient conditions for f to admit no unbounded Fatou components. In §3.2 we prove the Main Theorem by using Lemma 1 and Lemma 2.

2 Connectivity of $J_f\cup\{\infty\}$ in $\widehat{\mathbb{C}}$

(Proof of Theorem 1): The following criterion is well known. (See for example [Bea], p.81, Proposition 5.1.5).

Proposition B *Let K be a compact subset in $\widehat{\mathbb{C}}$. Then K is connected if and only if each component of the complement K^c is simply connected.*

Since $J_f \cup \{\infty\}$ is compact in $\widehat{\mathbb{C}}$, we can apply Proposition B. As we mentioned in §1, eventually periodic components are simply connected. So if a Fatou component U is not simply connected, then U is necessarily a wandering domain which is not simply connected. This completes the proof. □

(Proof of Corollary 1): Under the condition (1), f^n cannot tend to ∞ through F_f ([EL2]). On the other hand, f^n tends to ∞ on any multiply-connected wandering domains ([Ba4], [EL1]). So all the Fatou components are simply connected in this case. Under the condition (2) or (3), it is known that all the Fatou components must be simply connected ([Ba4], [EL1], p.620 Corollary 1, 2). □

Remark 1 (1) Let $S := \{f \mid {}^{\#}\mathrm{sing}(f^{-1}) < \infty\} \subset B$. Then there is even no wandering domain in F_f for $f \in S$ ([GK]). For $f \in B$, F_f may admit a wandering domain U but U must be simply connected as we mentioned above. Under an additional condition

$$J_f \cap \left(\text{derived set of } \bigcup_{n=0}^{\infty} f^n(\mathrm{sing}(f^{-1}))\right) = \emptyset,$$

$f \in B$ has also no wandering domain ([BHKMT]).

(2) We can conclude that in general if $J_f \cup \{\infty\}$ is disconnected, all the Fatou components are bounded and some of which are multiply-connected wandering domains.

3 Connectivity of J_f in $\mathbb{C}$

3.1 The case when all the Fatou components are bounded

Suppose that a closed connected subset K in $\mathbb{C}$ is bounded. Then all the components of the complement K^c other than the unique unbounded component V are simply conneceted. (Of course, $V \cup \{\infty\} \subset \widehat{\mathbb{C}}$ is simply connected). If K is unbounded, then all the components of K^c are simply connected, but the converse is false as the example $J_{E_\lambda}(0 < \lambda < \frac{1}{e})$ shows. (Compare with the Proposition B). But note that $J_{E_\lambda} \cup \{\infty\}$ is connected in $\widehat{\mathbb{C}}$. For the connectivity of a closed subset in $\mathbb{C}$, the following criterion holds.

Proposition 1 *Let K be a closed subset of $\mathbb{C}$. Then K is connected if and only if the boundary of each component U of the complement K^c is connected.*

(Proof): For the 'only if' part, see [New]. Suppose that K is disconnected. Then there exist two closed sets K_1 and K_2 with $K = K_1 \cup K_2$ and $K_1 \cap K_2 = \emptyset$. Take a point z_0 with $d(z_0, K_1) = d(z_0, K_2)$ where d denotes the Euclid distance in $\mathbb{C}$. Then $z_0 \in K^c$ and so let U_0

be the connected component of K^c containing z_0. Since ∂U_0 is connected by the assumption, either $\partial U_0 \subset K_1$ or $\partial U_0 \subset K_2$. Without loss of generality we can assume $\partial U_0 \subset K_1$. On the other hand denote $r_0 := d(z_0, K_1) = d(z_0, K_2)$ and let $D_{r_0}(z_0) := \{z \mid |z - z_0| < r_0\}$. Then $\overline{D_{r_0}(z_0)} \subset \overline{U_0}$ and there exists a point $w \in K_2$ with $w \in \overline{U_0}$. Since $w \in K_2 \subset K$, we have $w \in \partial U_0$ but this is a contradiction since $\partial U_0 \subset K_1$ and $K_1 \cap K_2 = \emptyset$. This completes the proof. □

(Proof of Theorem 2): By Proposition 1, it is sufficient to to show that the boundary ∂U is connected for each Fatou component U. Since U is bounded, the boundary of U as a subset of $\mathbb{C}$ and the one as the subset of $\widehat{\mathbb{C}}$ coincide. Hence U is simply connected if and only if ∂U is connected ([**Bea**], p.81, Proposition 5.1.4). This completes the proof. □

Remark 2 (1) Since a non-simply connected Fatou component is necessarily a wandering domain, the assumption of Theorem 2 is equivalent to that all the components of F_f are bounded and F_f admits no multiply-connected wandering domains.

(2) Several sufficient conditions are known for a transcendental entire function f to admit no unbounded Fatou components as follows:

(i) ([**Ba3**]) $\log M(r) = O((\log r)^p)$ (as $r \to \infty$) where $M(r) = \sup_{|z|=r} |f(z)|$ and $1 < p < 3$.

(ii) ([S]) There exists $\varepsilon \in (0,1)$ such that $\log\log M(r) < \dfrac{(\log r)^{\frac{1}{2}}}{(\log\log r)^{\varepsilon}}$ for large r.

(iii) ([S]) The order of f is less than $\dfrac{1}{2}$ and $\dfrac{\log M(2r)}{\log M(r)} \to c$ (finite constant) as $r \to \infty$.

Note that the condition (ii) includes the condition (i).

3.2 In the case when F_f admits an unbounded component

(Proof of Main Theorem): In what follows we assume that $n_0 = 1$ (that is, U is an invariant component) and $m_0 = 1$ for simplicity. This causes no loss of generality, because we have only to consider f^{m_0} instead of f in general cases.

Case (1) Since ∞ is accessible, there exists a continuous curve $L(t)$ $(0 \le t < 1)$ in U with $\lim_{t \to 1} L(t) = \infty$. By deforming $L(t)$ slightly, we construct a new curve $\mathcal{L}(t)$ satisfying the following condition.

Lemma 3 *There exists a curve $\mathcal{L}(t)$ $(0 \le t < 1)$ with $\lim_{t \to 1} \mathcal{L}(t) = \infty$ such that every branch of f^{-n} can be analytically continued along it for every $n \in \mathbb{N}$.*

(Proof): We may assume that $L(0) \notin P_f$, since $q \notin P_f$ we have $U \not\subset P_f$. Let $p_0 := L(0), p_1, p_2, \ldots$ be points on L such that all the piecewise linear line segments connecting $p_0, p_1, p_2, \ldots$ lie in U. Let $F_n^{(1)}, F_n^{(2)}, \ldots, F_n^{(m)}, \ldots$ be all the branches of f^{-n} which take values on U. The range of the suffix m may be finite or infinite. Define

$$\Theta_n^{(m)}(p_0) := \{e^{i\theta} \mid F_n^{(m)} \text{can be analytically continued along the ray}$$

$\mid$ from p_0 in the direction $\theta\}$ $(n = 1, 2, \ldots)$.

Then by the next Gross's Star Theorem ([**Nev**]), it follows that $\Theta_n^{(m)}(p_0)$ has full measure in $\partial\mathbb{D}$.

Lemma C (**Gross's Star Theorem**) *Let f be an entire function and F a branch of f^{-1} defined in the neighborhood of $p_0 \in \mathbb{C}$. Then F can be analytically continued along almost all rays from p_0 in the direction θ.*

Then the set

$$\Theta(p_0) := \bigcap_{n\geq 1, m\geq 1} \Theta_n^{(m)}(p_0)$$

has also full measure in $\partial\mathbb{D}$. Hence by changing p_1 slightly to a point p_1', the segments $\overline{p_0p_1'}$ and $\overline{p_1'p_2}$ lie in U and all the branches $F_n^{(m)}$ $(n \geq 1,\ m \geq 1)$ can be analytically continued along $\overline{p_0p_1'}$. By the same method, we can find a point p_2' close to p_2 such that the segment $\overline{p_1'p_2'}$ lies in U and has the same property as above. By repeating this argument, we can prove the Lemma 3. □

Let $l_n^{(m)}(t) := F_n^{(m)}(\mathcal{L}(t))$ then we have $\lim_{t\to 1} l_n^{(m)}(t) = \infty$. For suppose this is false, then there exist an increasing sequence of parameter values $t_1 < t_2 < \cdots < t_k < \cdots$ and a finite point α with $\lim_{k\to\infty} l_n^{(m)}(t_k) = \alpha \neq \infty$. Then it follows that $\lim_{k\to\infty} \mathcal{L}(t_k) = f^n(\alpha) \neq \infty$ and this contradicts the fact $\lim_{k\to\infty} \mathcal{L}(t_k) = \infty$.

Let $\varphi : \mathbb{D} \longmapsto U$ be a Riemann map of U. Then

$$\Gamma(t) := \varphi^{-1}(\mathcal{L}(t)) \quad \text{and} \quad \gamma_n^{(m)}(t) := \varphi^{-1}(l_n^{(m)}(t))$$

are curves in $\mathbb{D}$ landing at a point in $\partial\mathbb{D}$. This fact is not so trivial but follows from the proposition in [**P**](p.29, Proposition 2.14). We may assume that $\Gamma(t)$ lands at $z = 1 \in \partial\mathbb{D}$ for simplicity. If $\lim_{t\to 1} \gamma_{n_0}^{(m_0)}(t) = e^{i\theta_0}$, then since $\lim_{t\to 1} \varphi(\gamma_{n_0}^{(m_0)}(t)) = \lim_{t\to 1} l_{n_0}^{(m_0)}(t) = \infty$, it follows that there exists the radial limit $\lim_{r\to 1} \varphi(re^{i\theta_0})$ and this is equal to ∞. This fact follows from the theorem in [**P**] (p.34, Theorem 2.16). Therefore it is sufficient to show that the set of all the landing points of $\gamma_n^{(m)}(t)$ $(n \geq 1, m \geq 1)$ is dense in $\partial\mathbb{D}$.

Let $g := \varphi^{-1} \circ f \circ \varphi : \mathbb{D} \longrightarrow \mathbb{D}$. Then by Fatou's theorem φ has radial limit $\varphi(e^{i\theta}) = \lim_{r\nearrow 1} \varphi(re^{i\theta}) \in \partial U$ and non-constant for almost every $e^{i\theta} \in \partial\mathbb{D}$. Hence $f \circ \varphi(re^{i\theta})$ is a curve landing at a point in $\partial U \setminus \{\infty\}$ for almost every $e^{i\theta} \in \partial\mathbb{D}$. Therefore it follows that $\lim_{r\nearrow 1} \varphi^{-1} \circ f \circ \varphi(re^{i\theta}) \in \partial\mathbb{D}$ a.e. and thus g is an inner function. Let $\tilde{C} := \varphi^{-1}(C)$ then by the same reason for $\Gamma(t)$, $\tilde{C}$ is a curve in $\mathbb{D}$ with an end point $\tilde{q} \in \partial U$ satisfying $g(\tilde{C}) \supset \tilde{C}$. From the dynamics of $g : \mathbb{D} \to \mathbb{D}$, it follows that the set $\bigcup_{n=0}^{\infty} g^n(\tilde{C}) \bigcup \{\tilde{p},\ \tilde{q}\}$ is compact in $\overline{\mathbb{D}}$ where $\tilde{p} = \varphi^{-1}(p)$ and $\tilde{p}$ is an attracting fixed point of g and the distance between this set and $z = 1$ is positive. Hence there exists $\varepsilon_0 > 0$ such that

$$U_{\varepsilon_0}(1) \bigcap \left\{ \bigcup_{n=0}^{\infty} g^n(\tilde{C}) \bigcup \{\tilde{p},\ \tilde{q}\} \right\} = \emptyset \tag{1}$$

Since $\Gamma(t)$ lands at $z = 1$, there exists $t_0 \in [0,1)$ such that $\Gamma|[t_0,1) \subset U_{\varepsilon_0}(1)$. So by rewriting $\Gamma|[t_0,1)$ to $\Gamma(t)$ $(0 \le t < 1)$ we may assume that $\Gamma(t) \subset U_{\varepsilon_0}(1)$ for $0 \le t < 1$. Let $K := \{z \mid |z| \le 1-\varepsilon_0\}$ then since every point in $\mathbb{D}$ tends to $\tilde{p}$ under g^n and K is compact, there exists $n_1 \in \mathbb{N}$ such that for every $N \ge n_1$ we have $g^N(K) \subset U_\varepsilon(\tilde{p})$. Then we have $\gamma_N^{(m)}(t) \subset K^c$ for every $N \ge n_1$. For suppose that $\gamma_N^{(m)}(t) \cap K \ne \emptyset$, then by operating f^N we have $\Gamma(t) \cap K \ne \emptyset$ which contradicts $\Gamma(t) \subset U_{\varepsilon_0}(1)$.

Now suppose that the conclusion does not hold. Then there exists

$$(\theta_1,\theta_2) := \{e^{i\theta} \mid \theta_1 < \theta < \theta_2\} \subset \partial\mathbb{D} \quad \text{with} \quad \Theta_\infty \cap (\theta_1,\theta_2) = \emptyset.$$

By changing the starting point $\Gamma(0)$ slightly, if necessary, we may assume that the points $\gamma_n^{(m)}(0)$ $(n,m = 1,2,\cdots)$ accumulate to all over $\partial\mathbb{D}$ by Lemma 1 (1) while the end points $\gamma_n^{(m)}(1) := \lim_{t\to1}\gamma_n^{(m)}(t)$ $(n,m=1,2,\cdots)$ are not in (θ_1,θ_2). Therefore there exists $\gamma_{n_1}^{(m_1)}(t)$ such that $\gamma_{n_1}^{(m_1)}(t) \subset K^c$ and $\gamma_{n_1}^{(m_1)}(1) \in \partial\mathbb{D} \setminus (\theta_1,\theta_2)$

On the other hand there exist inverse images $g^{-n}(\tilde{C})$ which have limit points on (θ_1,θ_2) densely. The reason is as follows: Since $q \notin P_f$, there exists a neighborhood V of q such that all the branches $F_n^{(1)}, F_n^{(2)}, \ldots, F_n^{(m)}, \ldots$ can be defined. Let $V_0 \subset V$ is a neighborhood of q with $\overline{V_0} \subset V$. We may assume that $C \subset V_0$. Define

$$c_n^{(m)}(t) := F_n^{(m)}(C(t)), \quad \tilde{c}_n^{(m)}(t) := \varphi^{-1}(c_n^{(m)}(t)).$$

Then $c_n^{(m)}(t)$ is a curve in U landing at a point in ∂U and $\tilde{c}_n^{(m)}(t)$ is a curve in $\mathbb{D}$ landing at a point in $\partial\mathbb{D}$ by the same reason as before. Let $(\theta_3,\theta_4) \subset (\theta_1,\theta_2)$ be any subarc of (θ_1,θ_2). By changing the starting point $\tilde{C}(0)$ slightly, if necessary, we may assume that the points $\tilde{c}_n^{(m)}(0)$ $(n,m=1,2,\cdots)$ accumulate to (θ_3,θ_4) by Lemma 1 (1). Since radial limits of φ exist and non-constant almost everywhere, by changing θ_3 and θ_4 slightly if necessary, we may assume that there exist the finite values $\varphi(e^{i\theta_3})$ and $\varphi(e^{i\theta_4})$ with $\varphi(e^{i\theta_3}) \ne \varphi(e^{i\theta_4})$. Then $c_n^{(m)}(0)$ accumulate on $\partial U \cap \varphi(\{re^{i\theta} \mid \theta_3 < \theta < \theta_4,\ 0 \le r \le 1\})$. In general the family of single-valued analytic branch of f^{-n} $(n = 1,2,\ldots)$ on a domain U_0 is normal and furthermore if $U_0 \cap J_f \ne \emptyset$, any local uniform limit of a subsequence in the family is constant ([**Bea**], p.193, Theorem 9.2.1, Lemma 9.2.2). So the family $\{F_n^{(m)}|V_0\}$ is normal and all its limit functions are constant and hence for a suitable subsequence the diameter of $c_{n_k}^{(m_k)}(t)$ tends to zero, that is, $c_{n_k}^{(m_k)}(t)$ must land at a point in $\partial U \cap \varphi(\{re^{i\theta} \mid \theta_3 < \theta < \theta_4,\ 0 \le r \le 1\})$ if the constant limit is finite. Therefore $\tilde{c}_{n_k}^{(m_k)}(t)$ must land at a point in (θ_3,θ_4). If the constant limit is ∞, for large enough n_k the curves $c_{n_k}^{(m_k)}$ cannot intersect both $\{\varphi(re^{i\theta_3}) \mid 0 \le r \le 1\}$ and $\{\varphi(re^{i\theta_4}) \mid 0 \le r \le 1\}$ which are bounded set, since the convergence is uniform on V_0. Hence again we can conclude that $c_{n_k}^{(m_k)}(t)$ must land at a point in $\partial U \cap \varphi(\{re^{i\theta} \mid \theta_3 < \theta < \theta_4,\ 0 \le r \le 1\})$ and therefore $\tilde{c}_{n_k}^{(m_k)}(t)$ must land at a point in (θ_3,θ_4). This proves the assertion.

Then there exists $\tilde{c}_{N_1}^{(M_1)}$ such that $\gamma_{n_1}^{(m_1)} \cap \tilde{c}_{N_1}^{(M_1)} \ne \emptyset$. We may assume that $n_1 > N_1$. Let $u \in \gamma_{n_1}^{(m_1)} \cap \tilde{c}_{N_1}^{(M_1)}$ then since $u \in \gamma_{n_1}^{(m_1)}$, we have $g^{n_1}(u) \in U_{\varepsilon_0}(1)$. On the other hand since $u \in \tilde{c}_{N_1}^{(M_1)}$ and $n_1 > N_1$, we have $g^{n_1}(u) \in \bigcup_{n=0}^{\infty} g^n(\tilde{C})$ which contradicts (1). Therefore

Θ_∞ is dense in $\partial\mathbb{D}$. Disconnectivity of J_f easily follows by the same argument as in the case of E_λ in §1. This completes the proof in the case of (1). □

Case (2) The proof is quite parallel to the case (1). Note that by Lemma 2, $\overline{\bigcup_{n=1}^{\infty} g^{-n}(z_0)} \supset \partial\mathbb{D}$ $(z_0 \in \mathbb{D} \setminus E)$ holds for $g = \varphi^{-1} \circ f \circ \varphi$ in this case. □

Case (3) Since $g(z) = e^{2\pi i\theta_0}$ with $\theta_0 \in \mathbb{R} \setminus \mathbb{Q}$, the inverse image of $\Gamma(t)$ by g^{-n} is unique and denote it by $\gamma_n(t)$. Then it is obvious that the end points of $\gamma_n(t)$ are dense in $\partial\mathbb{D}$ and φ attains radial limit ∞ there, since $g(z)$ is an irrational rotation and

$$\lim_{t\to 1} \varphi(\gamma_n(t)) = \lim_{t\to 1} f^{-1}(\varphi(\Gamma(t))) = \infty. \qquad \square$$

Case (4) In this case we need not assume the accessibility of ∞, because this condition is automatically satisfied ([**Ba6**]). The set $\cup_{n=0}^{\infty} f^n(C)$ is a curve which may have self-intersections and tends to ∞. It is not difficult to take L satisfying $L \cap (\cup_{n=0}^{\infty} f^n(C)) = \emptyset$. Hence we have $\mathcal{L} \cap (\cup_{n=0}^{\infty} f^n(C)) = \emptyset$. The rest of the proof is quite parallel to the case (1) if the conclusion of Lemma 2 (1) holds for g. If we have only the conclusion of Lemma 2 (2), then we can prove that for every arc $A \subset \partial\mathbb{D}$ with $A \cap K \neq \emptyset$, $A \cap \Theta_\infty \neq \emptyset$ holds by the similar argument. □

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