Bifurcation along Arcs in Antiholomorphic Dynamics

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Abstract

In this note, a bifurcation phenomenon along arcs in the dynamics of a family of antiholomorphic maps is considered. By explicitly calculating the multipliers of 2-periodic points, we can describe the exact subarcs of the boundary of the hyperbolic component of period one of the the multicorn, along which bifurcation occurs.

1 Introduction

In this paper, we consider the dynamics of a family of antiholomorphic polynomials of degree $d \geq 2$ of the form : $f_c(z) = \overline{z}^d + c,\ \ c \in \mathbb{C}$. Though f_c itself is not holomorphic, its second iterate $f_c^{\circ 2}(z) = (z^d + \overline{c})^d + c$ is holomorphic. So, we can define its filled-in Julia set $K_c = K(f_c)$ and Julia set $J_c = J(f_c)$ analogously as in the polynomial case :

$$\begin{aligned} K_c &= \{z \in \mathbb{C}; \text{its forward orbit } \{f_c^{\circ n}(z)\}_{n=0}^{\infty} \text{ is bounded}\}, \\ J_c &= \partial K_c. \end{aligned}$$

We are mainly concerned with the connectedness locus of this family :

$$M_d^* = \{c \in \mathbb{C}; J_c \text{ is connected}\}.$$

We call it the *multicorn*. The case $d = 2$ has been investigated by several authors. Milnor [Mil] called M_2^* the *tricorn* and Rippon et.al. [Rip] called it the *Mandelbar set*. It is characterized also by

$$M_d^* = \{c \in \mathbb{C}; 0 \in K_c\}.$$

Note that 0 is the unique critical point of this family.

It can be regarded as an analogy with the (generalized) Mandelbrot set M_d for the polynomial family : $P_c(z) = z^d + c, \ c \in \mathbb{C}$. In fact, they share same properties to a certain extent. For example, Nakane [Nak1] showed that the tricorn M_2^* is connected. His proof works also for any $d \geq 2$ and we can say M_d^* is connected for any d.

But there are some differences between M_d and M_d^*. In [Rip] they showed, for $d = 2$, that a bifurcation of attracting fixed points into attracting 2-cycles occurs across an arc and not at a single point. This never happens for the quadratic family $\{P_c\}$ since it depends holomorphically on the parameter c. The boundaries of two hyperbolic components must meet only at a single point.

This paper is an attempt to investigate such a bifurcation on the boundary of any hyperbolic component of odd period k. Especially in case $k = 1$, we give a complete description of the bifurcation by explicitly calculating the multiplier of the 2-periodic point.

Let W be a hyperbolic component of odd period k of M_d^* and ∂W be its boundary. Suppose $c \in \partial W$. Then, there exists an indifferent k-periodic point z_c of f_c satisfying $(\partial f_c^{\circ 2k}/\partial z)(z_c) = 1$. Note that, for any periodic point z_c of f_c of odd period k,

$$(\partial f_c^{\circ 2k}/\partial z)(z_c) = |(\partial f_c^{\circ k}/\partial \bar{z})(z_c)|^2 \geq 0.$$

Hence, an indifferent k-periodic point of f_c must always be a rationally indifferent k-periodic point of $f_c^{\circ 2}$ with multiplier 1.

Thus ∂W is a real algebraic set and is real-analytically parametrized at least locally, which we denote by $c = c(t)$. Let z_t be a rationally indifferent k-periodic point of $f_{c(t)}$. Actually, in Nakane and Schleicher [NS], we show that each connected component of $\partial W -$ {cusp points} is real analytically parametrized by the *Ecalle height* of the critical value.

Lemma 1.1 *If* $(\partial^2 f_{c(t)}^{\circ 2k}/\partial z^2)(z_t) = 0$, *then* $(\partial^3 f_{c(t)}^{\circ 2k}/\partial z^3)(z_t) \neq 0$.

PROOF. Since the combinatorial rotation number is one at z_c and the number of critical orbits of $f_c^{\circ 2}$ is two, the multiplicity of z_c as a k-periodic point of $f_c^{\circ 2}$ is at most three. From the assumption, it is not two. So it is three. This completes the proof. □

Definition 1.2 (Parabolic arcs and cusps) *We call* $c = c(t) \in \partial W$ *a* cusp *point if* $(\partial^2 f_{c(t)}^{\circ 2k}/\partial z^2)(z_t) = 0$. *Otherwise, we call it a* non-cusp *point. We call each connected component of* $\partial W - \{\text{cusp points}\}$ *a* parabolic arc.

For example, in case $k = 1$, its boundary ∂W is parametrized by

$$c(t) = z_t - \bar{z}_t^{\,d}, z_t = d^{-1/(d-1)} e^{it}, \ \ 0 \le t \le 2\pi. \tag{1}$$

Here is our main theorem of this paper.

Theorem 1.3 *In case* $k = 1$, *bifurcation occurs on* $c = c(t)$ *in (1) with*

$$\frac{4j+1}{2(d+1)}\pi < t < \frac{4j+3}{2(d+1)}\pi, \quad 0 \le j \le d.$$

REMARK. The value $t/2\pi$ above corresponds to the internal angle of the hyperbolic component of period one. The above theorem implies that bifurcation occurs just on one half of the boundary in the sense of internal angles. Cusp points correspond to $t = \frac{2j+1}{d+1}\pi, \ 0 \le j \le d$.

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2 Multipliers of 2k-periodic points

In this section, we investigate the bifurcation along arcs by a precise estimate of the multipliers of the bifurcating $2k$-cycles.

Though in [Rip], they have shown such a bifurcation only in a small neighborhood of a cusp point of the hyperbolic component of period one, we get a general result for any period. Especially, in case of period one, we get an exact subarc of the boundary across which bifurcation occurs.

Let W be a hyperbolic component of odd period k of M_d^*. We use the same notation as in the introduction. First, we will show the following.

Lemma 2.1 *Suppose k is odd. Let z_0 be a $2k$-periodic point of f_c and put $w_0 = f_c^{\circ k}(z_0)$. Then*

$$(\partial f_c^{\circ 2k}/\partial z)(w_0) = \overline{(\partial f_c^{\circ 2k}/\partial z)(z_0)}.$$

Furthermore, if z_0 is a k-periodic point of f_c,

$$(\partial f_c^{\circ 2k}/\partial z)(z_0) = |(\partial f_c^{\circ k}/\partial \overline{z})(z_0)|^2 \geq 0.$$

PROOF. Let $z_j = f_c^{\circ j}(z_0)$. Then $w_j = z_{k+j}$. By the chain rule, we have

$$\begin{aligned}(\partial f_c^{\circ 2k}/\partial z)(w_0) &= (\partial f_c^{\circ k}/\partial \overline{z})(f_c^{\circ k}(w_0))\overline{(\partial f_c^{\circ k}/\partial \overline{z})(w_0)} \\ &= (\partial f_c^{\circ k}/\partial \overline{z})(z_0)\overline{(\partial f_c^{\circ k}/\partial \overline{z})(w_0)} \\ &= \overline{(\partial f_c^{\circ 2k}/\partial z)(z_0)}.\end{aligned}$$

Especially, if z_0 is k-periodic, then $w_0 = z_0$. Hence

$$(\partial f_c^{\circ 2k}/\partial z)(z_0) = |(\partial f_c^{\circ k}/\partial \overline{z})(z_0)|^2 \geq 0.$$

This completes the proof. □

We will estimate the multiplier of the $2k$-cycle of f_c near ∂W. To do so, we calculate its $2k$-periodic point asymptotically. Though the main theorem states only for the case $k = 1$, we can calculate them in a general form for any odd k.

We use the following notations :

$$\begin{aligned}F(\overline{z}, c, \overline{c}) &= f_c^{\circ k}(z), \\ G(z, c, \overline{c}) &= f_c^{\circ 2k}(z), \\ x &= \varphi_t(z) \equiv z - z_t, \\ c &= c(t) + s, \quad s = r\omega, \quad r > 0, \quad \omega = e^{i\theta}, \\ H(x) &= G(x + z_t, c(t) + s, \overline{c(t)} + \overline{s}) - z_t.\end{aligned}$$

Then, we have a Taylor expansion of H with respect to x:

$$H(x) = G_c s + G_{\overline{c}}\overline{s} + (G_z + G_{zc}s + G_{z\overline{c}}\overline{s})x + G_{zz}x^2/2 + G_{zzz}x^3/6 + \ldots$$

Here $G_c = G_c(z_t, c(t), \overline{c(t)})$ etc. are partial derivatives of G. Note that $G_z \equiv 1$.

We find a Puiseux expansion of the fixed point x of H (which corresponds to the $2k$-periodic point of $f_{c(t)}$) with respect to r of the form :

$$x = ar^{1/2} + br + O(r^{3/2}).$$

By subordinating this in H, we have

$$\begin{aligned} 0 &= H(x) - x \\ &= (G_c\omega + G_{\bar{c}}\overline{\omega})r + (G_{zc}\omega + G_{z\bar{c}}\overline{\omega})r(ar^{1/2} + br + \dots) \\ &\quad + G_{zz}(a^2r + 2abr^{3/2} + \dots)/2 + G_{zzz}a^3r^{3/2}/6 + \dots \end{aligned}$$

Comparing the coefficients of r and $r^{3/2}$, it follows

$$\begin{aligned} G_c\omega + G_{\bar{c}}\overline{\omega} + G_{zz}a^2/2 &= 0, \\ (G_{zc}\omega + G_{z\bar{c}}\overline{\omega})a + G_{zz}ab + G_{zzz}a^3/6 &= 0. \end{aligned}$$

Hence we have,

$$\begin{aligned} a &= a(t) = \pm\sqrt{-2(G_c\omega + G_{\bar{c}}\overline{\omega})/G_{zz}}, \\ b &= b(t) = -(G_{zc}\omega + G_{z\bar{c}}\overline{\omega} + G_{zzz}a^2/6)/G_{zz}. \end{aligned}$$

Let $\rho = \rho(t)$ be the multiplier of the fixed point x of H, i.e., the multiplier of the $2k$-periodic point of $f_{c(t)}$.

Lemma 2.2 *We have*

$$\rho = 1 + iAr^{1/2} + Br + O(r^{3/2}), \tag{2}$$

where

$$\begin{aligned} A &= A(t) = \pm\sqrt{2G_{zz}(G_c\omega + G_{\bar{c}}\overline{\omega})}, \\ B &= B(t) = -2G_{zzz}(G_c\omega + G_{\bar{c}}\overline{\omega})/3G_{zz}. \end{aligned}$$

PROOF. By a direct calculation, it follows

$$\begin{aligned} \rho &= H'(x) \\ &= 1 + (G_{zc}\omega + G_{z\bar{c}}\overline{\omega})r + G_{zz}x + G_{zzz}x^2/2 + \dots \\ &= 1 + (G_{zc}\omega + G_{z\bar{c}}\overline{\omega})r + G_{zz}(ar^{1/2} + br) + G_{zzz}a^2r/2 + \dots \\ &= 1 + G_{zz}ar^{1/2} + (G_{zc}\omega + G_{z\bar{c}}\overline{\omega} + G_{zz}b + G_{zzz}a^2/2)r + \dots \\ &= 1 + iAr^{1/2} + Br + O(r^{3/2}). \end{aligned}$$

This completes the proof. □

Proposition 2.3 *$A = A(t)$ and $B = B(t)$ are real-valued outside $\partial\overline{W}$ for any odd k.*

PROOF. We calculate A and B. Since $G = F(\overline{F}, c, \overline{c})$, we have

$$\begin{aligned} G_z &= F_{\bar z}(\overline{F})\overline{F_{\bar z}}, \\ G_{zz} &= F_{\bar z\bar z}(\overline{F})\overline{F_{\bar z}}^2 + F_{\bar z}(\overline{F})\overline{F_{\bar z\bar z}}, \\ G_{zzz} &= F_{\bar z\bar z\bar z}(\overline{F})\overline{F_{\bar z}}^3 + 3F_{\bar z\bar z}(\overline{F})\overline{F_{\bar z\bar z}F_{\bar z}} + F_{\bar z}\overline{F_{\bar z\bar z\bar z}}. \end{aligned}$$

By putting $z = z_t, c = c(t)$, we have

$$\begin{aligned} G_z &= |F_{\bar z}|^2 \equiv 1, \\ G_{zz} &= \sqrt{\overline{F_{\bar z}}}2\mathrm{Re}(F_{\bar z\bar z}\overline{F_{\bar z}}^{3/2}), \\ G_{zzz} &= \overline{F_{\bar z}}(3|F_{\bar z\bar z}|^2 + 2\mathrm{Re}(F_{\bar z\bar z\bar z}\overline{F_{\bar z}}^2)), \\ G_c &= F_c + F_{\bar z}\overline{F_{\bar c}}, \\ G_{\bar c} &= F_{\bar c} + F_{\bar z}\overline{F_c} = F_{\bar z}\overline{G_c}. \end{aligned}$$

Hence we can compute A and B as follows :

$$\begin{aligned} A &= \pm 2\sqrt{2\mathrm{Re}(F_{\bar z\bar z}\overline{F_{\bar z}}^{3/2})\mathrm{Re}((F_c\omega + F_{\bar c}\overline{\omega})\sqrt{\overline{F_{\bar z}}})}, \\ B &= -\frac{2\mathrm{Re}((F_c\omega + F_{\bar c}\overline{\omega})\sqrt{\overline{F_{\bar z}}})}{3\mathrm{Re}(F_{\bar z\bar z}\overline{F_{\bar z}}^{3/2})}(3|F_{\bar z\bar z}|^2 + 2\mathrm{Re}(F_{\bar z\bar z\bar z}\overline{F_{\bar z}}^2)). \end{aligned}$$

Hence, we have shown that B is real. As for A, this calculation shows that A is real or pure imaginary. By Lemma 2.1 and the following Lemma 2.4, A must be real outside $\partial\overline{W}$. This completes the proof of Proposition 2.3. □

Lemma 2.4 *On ∂W - {cusp points}, an attracting and repelling k-cycles collapse and bifurcate into a $2k$-cycle outside $\overline{W}$.*

We omit the proof. The case $d = 2, k = 1$ was shown in [Rip].

REMARK. Proposition 2.3 assures that, if $A \neq 0$, we can define and make use of the Ecalle cylinder in a neighborhood of ∂W. See Lavaurs [Lav] or Shishikura [Shi2]. In fact, $A \neq 0$ implies a rotation around the bifurcating

$2k$-periodic point, which suggests the existence of an eggbeater dynamics. Ecalle cylinder, the fundamental domain of this eggbeater dynamics, is thus assured to exist. The following calculation implies that, in case $k = 1$, $A \neq 0$ is equivalent to the fact that $c = c(t)$ is not a cusp point.

Then, the absolute value of ρ is expressed by :

$$\begin{aligned} |\rho|^2 &= (1 + Br)^2 + A^2 r + O(r^{3/2}) \\ &= 1 + (2B + A^2)r + O(r^{3/2}). \end{aligned}$$

Hence, for sufficiently small $r > 0$, x is attracting if and only if $2B + A^2 < 0$. Here,

$$2B + A^2 = \frac{2}{3} \frac{G_c \omega + G_{\bar{c}} \overline{\omega}}{G_{zz}} (3G_{zz}^2 - 2G_{zzz}).$$

Thus we have obtained a preliminary version of Theorem 1.3.

Theorem 2.5 *For general odd k, suppose $A \neq 0$. Then bifurcation occurs across the subarcs of ∂W satisfying*

$$\frac{2B + A^2}{A^2} = 1 - \frac{2G_{zzz}}{3G_{zz}^2} < 0.$$

REMARK. We can state the above theorem in terms of the holomorphic index. Let z_0 be a fixed point of a holomorphic function f. The holomorphic index $i(f, z_0)$ of f at z_0 is defined by

$$i(f, z_0) = \frac{1}{2\pi i} \oint_{|z - z_0| = \varepsilon} \frac{dz}{z - f(z)}.$$

For example, $i(f, z_0) = 1/(1 - f'(z_0))$ if z_0 is a repelling fixed point, and $i(f, z_0) = 2f'''(z_0)/3f''(z_0)^2$ if z_0 is a parabolic fixed point with multiplier one. Thus we have $i(f_{c(t)}^{\circ 2k}, z_t) = 2G_{zzz}/3G_{zz}^2$. This was suggested by M. Shishikura. A detailed discussion will be published elsewhere.

For general k, we cannot calculate A and B. But it is possible for $k = 1$, in which case,

$$\begin{aligned} G &= (z^d + \bar{c})^d + c, \\ G_c &= 1, \\ G_{\bar{c}} &= d(z^d + \bar{c})^{d-1}, \end{aligned}$$

$$\begin{aligned}
G_z &= d^2 z^{d-1}(z^d+\overline{c})^{d-1},\\
G_{zz} &= d^2(d-1)z^{d-2}(z^d+\overline{c})^{d-1} + d^3(d-1)z^{2d-2}(z^d+\overline{c})^{d-2}\\
G_{zzz} &= d^2(d-1)[(d-2)z^{d-3}(z^d+\overline{c})^{d-1} + 3d(d-1)z^{2d-3}(z^d+\overline{c})^{d-2}\\
&\quad + d^2(d-2)z^{3d-3}(z^d+\overline{c})^{d-3}].
\end{aligned}$$

Hence, by putting $z = z_t = d^{-1/(d-1)}e^{it}, c = c(t) = z_t - \overline{z}_t^d$, and using $f_{c(t)}(z_t) = z_t$, we have

$$\begin{aligned}
G_{\overline{c}} &= d\overline{z}_t^{d-1} = e^{-i(d-1)t},\\
G_{zz} &= d^2(d-1)(z_t^{d-2}\overline{z}_t^{d-1} + dz_t^{2d-2}\overline{z}_t^{d-2}) = (d-1)d^{1/(d-1)}(e^{-it}+e^{idt}),\\
G_{zzz} &= d^2(d-1)[(d-2)z_t^{d-3}\overline{z}_t^{d-1} + 3d(d-1)z_t^{2d-3}\overline{z}_t^{d-2} + d^2(d-2)z_t^{3d-3}\overline{z}_t^{d-3}]\\
&= (d-1)d^{2/(d-1)}[(d-2)e^{-2it} + 3(d-1)e^{i(d-1)t} + (d-2)e^{2idt}].
\end{aligned}$$

Then it follows

$$A = \pm 2\sqrt{(d-1)d^{1/(d-1)}\cos\left(\frac{(d-1)t}{2}+\theta\right)\cos\frac{(d+1)t}{2}} \neq 0,$$

for $t \in I \equiv (-\pi/(d+1), \pi/(d+1))$ if we take $\theta = 0$, and

$$\begin{aligned}
1 - \frac{2G_{zzz}}{3G_{zz}^2} &= 1 - \frac{2(d-1)d^{2/(d-1)}\{(d-2)(e^{-2it}+e^{2idt}) + 3(d-1)e^{i(d-1)t}\}}{3\{(d-1)d^{1/(d-1)}(e^{-it}+e^{idt})\}^2}\\
&= 1 - \frac{2e^{i(d-1)t}\{3(d-1) + (d-2)(e^{i(d+1)t}+e^{-i(d+1)t})\}}{3(d-1)(e^{-it}+e^{idt})^2}\\
&= 1 - \frac{2\{3(d-1) + 2(d-2)\cos(d+1)t\}}{3(d-1)(2\cos((d+1)t/2))^2}\\
&= \frac{(d+1)\cos(d+1)t}{3(d-1)(\cos(d+1)t+1)}.
\end{aligned}$$

From this and the symmetry of M_d^*, we get the conclusion of Theorem 1.3.

References

[Rip] W.D. Crowe, R. Hasson, P.J. Rippon, P.E.D. Strain-Clark: On the structure of the Mandelbar set. Nonlinearity 2 (1989), pp. 541–553.

[DH1] A. Douady and J. Hubbard: Étude dynamique des polynômes complexes. Publ. Math. d'Orsay, 1er partie, 84-02;2me partie, 85-04.

[HNS] J.H.Hubbard, S.Nakane and D.Schleicher: Non-pathwise connectivity of the multicorn. in preparation.

[J] H. Jeroulli: Indice holomorphe et multiplicateur: Application a l'etude d'explosion des cylindres. Thèse de doctrat de l'Universitè de Paris-Sud, Orsay, France, 1994.

[Lav] P. Lavaurs: Systemes dynamiques holomorphes: explosion de points paraboliques . These de doctrat de l'Universite de Paris-Sud, Orsay, France, 1989.

[M2] J. Milnor: *Dynamics in one complex variable: introductory lectures.* Stony Brook Preprint #1990/5.

[Mil] J. Milnor: Remarks on iterated cubic maps. Experimental Math. 1 (1992), pp. 5–24.

[Nak1] S. Nakane: Connectedness of the tricorn. Erg. Th. Dyn. Sys. 13 (1993), pp. 349–356.

[Nak2] S. Nakane: On quasiconformal equivalence on the boundary of the tricorn. in "Structure and Bifurcation of Dynamical Systems," World Sci. Publ., 1993, pp. 154–167.

[NS] S. Nakane and D. Schleicher: Hyperbolic Components of Multicorns. in preparation.

[Shi1] M. Shishikura: On the parabolic bifurcation of holomorphic maps. in "Dynamical Systems and Chaos," World Sci. Publ., 1990, pp. 478–486.

[Shi2] M. Shishikura: The Hausdorff dimension of the boundary of the Mandelbrot set and Julia sets. To appear in Annals of Math.

[Win] R. Winters: Bifurcation in families of antiholomorphic and biquadratic maps. Thesis at Boston Univ., 1990.