COMPLETIONS OF STRATIFIED ENDS

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1. Introduction

1.1. A famous result of L. Siebenmann characterizes those topological manifolds which are the interiors of compact manifolds with boundary. Elsewhere we have recently shown that his theorem generalizes to the context of *stratified spaces*. Our purpose here is to explain the main results of our work briefly. See [1] for the full account.

2. Definitions

Let X^n, $n \geq 6$ be a tame ended topological n-manifold. Siebenmann proves that there is a single obstruction $\sigma(X)$, in $\tilde{K}_0(\mathbb{Z}\pi)$, with the property that $\sigma(X) = 0$ if and only if X^n is the interior of a compact manifold with boundary. By the work of Freedman and Quinn [3] one can also allow $n \geq 5$, if π is not too complicated. The group π denotes the fundamental group of the end of X, which can be described as the $Holink(\widehat{X}, \infty)$; here $\widehat{X}$ denotes the one-point compactification of X. The space, $Holink(X, A)$, (the "homotopy link" of A in X), is defined for any subspace A of a topological space X as:

$$Holink(X, A) = \{\sigma \in \mathrm{Map}([0,1], X) \mid \sigma^{-1}(A) = 0\}.$$

It is given the compact-open topology. It comes with two maps,

$$A \xleftarrow{p_X} Holink(X, A) \xrightarrow{j_X} X - A : \quad j_X(\sigma) = \sigma(1), p_X(\sigma) = \sigma(0).$$

It is used by Quinn [6] as a homotopical analogue for the normal sphere bundle of A in X.

F. Quinn generalizes Siebenmann's result greatly. For any locally compact pair (X, A), where A is closed and *tame* in X, $X - A$ is an n-manifold ($n \geq 6$) and A is an ANR, Quinn [4],[5],[6] defines an obstruction, $q_0(X, A) \in \tilde{K}_0^{lf}(A, p_X)$, which vanishes if and only if A has a mapping cylinder neighborhood in X. Here the map $p_X : Holink(X, A) \to A$ is the projection. This concept of tameness is discussed by many others at this conference. The foundational concepts surrounding controlled K-theory have recently been greatly clarified by the eminently readable paper of Ranicki and Yamasaki [7].

2.1. Quinn's obstruction, $q_0(X, A)$ can be localized in the following way: let A be a closed and tame subset of X, and X' an open subset of X. Then $A' := X' \cap A$ is tame in X' and $i^* q_0(X, A) = q_0(X', A')$ where $i^* : \tilde{K}_0^{lf}(A, p_X) \to \tilde{K}_0^{lf}(A', p_{X'})$ is the restriction map. Using these maps one can define, for every subset $B \subset A$,

$$K_0^{lf}((A, p_X)_{(B)}) = \varinjlim K_0^{lf}(A', p_{X'} | A')$$

First author partially supported by NSF Grant DMS 95-04264.

(the direct limit is over the $X-$ neighborhoods, X', of B). Then the image of any one of the obstructions, $q_0(X', A')$ in $K_0^{lf}((A, p_X)_{(B)})$ is independent of the X-neighborhood, X', chosen. We will write this image $q_0((X, A)_{(B)})$.

2.2. In geometric topology, the stratified analogue of a topological manifold is a *stratified space*. This concept was introduced by Quinn under the name "manifold homotopy stratified set"; our terminology is due to Hughes and Weinberger.

A *stratified space* is a locally compact space, finitely filtered by closed subsets $X = X^n \supset X^{n-1} \supset \cdots \supset X^{-1} = \emptyset$. Each *stratum* $X_i := X^i - X^{i-1}$ must be a manifold, and the *boundary of X,* defined by the rule, $\partial X := \cup_i(\partial X_i)$, must be closed in X. (It is customary, to arrange the indexing so that $dim(X_i) = i$). It is also required that X_i must be tame in $X_i \cup X_j$ for each $j > i$. The projection, $Holink(X_i \cup X_j, X_i) \xrightarrow{p} X_i$ must be a fibration, and the inclusion $Holink(\partial X_i \cup \partial X_j, \partial X_i) \to Holink(X_i \cup X_j, X_i)|_{\partial X_i}$ must be a fiber homotopy equivalence over ∂X_i.

3. Main results

Let X be a stratified space with empty boundary. We seek a *completion* of X, i.e. a compact stratified space $\bar{X}$ such that $X = \bar{X} - \partial\bar{X}$, and $\partial\bar{X}$ has a collar neighborhood in $\bar{X}$. It is easy to see that a necessary condition for X having a completion is to be *tame ended*. This means that the one point compactification of X, $\widehat{X} := X \cup \infty$, is again a stratified space. The stratification of the one-point compactification is the following;

$$\widehat{X}_0 = X_0 \cup \infty; \quad \widehat{X}_j = X_j, \, \forall j > 0.$$

An equivalent formulation is that $\{\infty\}$ is tame in $X_j \cup \{\infty\}$ for each j. Notice that, by reverse tameness each X^j can have only finitely many ends.

A completion may not always exist; a weaker requirement would be an *exhaustion* of X. This is defined to be an increasing sequence of compact stratified subspaces of X, with bicollared boundaries in X, whose union is X. An exhaustion is also obstructed in the category of stratified spaces.

Our main results, 3.2, 3.3, and 3.7, say that a completion (or an exhaustion) exists if a single obstruction vanishes.

3.1. *The End Obstruction.* Let X be a tame ended stratified space. For each integer $m > 0$ we define, (as in 2.1, above),

$$\gamma_m(X) = q_0((\widehat{X}^m, \widehat{X}^{m-1})_{(\infty)}) \in K_0^{lf}((\widehat{X}^{m-1}, p_{\widehat{X}^m})_{(\infty)})$$

As before, the map $p_{\widehat{X}^m} : Holink(\widehat{X}^m, \widehat{X}^{m-1}) \to \widehat{X}^{m-1}$ denotes the Holink projection. Set also:

$$\gamma_*(X) = \bigoplus_m \gamma_m(X) \in \bigoplus_m K_0^{lf}((\widehat{X}^{m-1}, p_{\widehat{X}^m})_{(\infty)}).$$

3.2. Theorem. *Suppose X is a stratified space, with empty boundary, which admits a completion. Then $\gamma_*(X) = 0$.*

3.3. Theorem. *Let X be a tame ended stratified space with empty boundary. Let A be any closed pure subset of X, containing X^5, such that A admits a completion, $\bar{A}$. Suppose $\gamma_*(X) = 0$. Then X admits a completion $\bar{X}$ such that $Cl_{\bar{X}}(A) = \bar{A}$.*

(A *pure* subset is one which is the union of components of strata.)

3.4. Note *This result reduces to Siebenmann's theorem when X has only one stratum.*

3.5. Note Following Weinberger, we say that a finite group action on a manifold, (M, G) is a *stratified G manifold* if the fixed set of each subgroup, M^H, is a manifold, and M^H is locally flat in M^K for each $K \subset H$. By (1.4, 1.5 and 1.6 of [6]), this is equivalent to saying that $X = M/G$ is a stratified space when it is stratified by its orbit type components. A corollary of our main theorem is an end-completion result for G-manifolds:

Corollary 3.6. *Let (M, G) be a stratified G-manifold with $\partial M = \emptyset$. Then (M, G) is the interior of a compact stratified G-manifold with collared boundary iff $X = M/G$ is tame ended, $\gamma_*(X) = 0$, and X^5 has a completion.*

The obstruction to finding an exhaustion for the stratified space X turns out to have the form $\partial\gamma_*(X)$, where ∂ is a map we will not define here in complete generality. Instead we will give the definition of $\partial\gamma_n(X)$ in the special case when X^{n-1} admits a completion. In this case an ∞- neighborhood in X^{n-1} has the form $B \times [0, \infty)$, for some stratified space B. Then the *open cone* of B, OB which can be thought of as $B \times (0, \infty]/B \times \infty$ is a neighborhood of ∞ in $\widehat{X}^{n-1}$; moreover ∞ has a cofinal sequence of such neighborhoods, $B \times (k, \infty]/B \times \infty$, $k = 0, 1, 2, \ldots$. The restriction maps connecting the K-theory of these are isomorphisms. This implies that the obstruction $\gamma_*(X)$ reduces to $\gamma_n(X)$, and moreover, that $\tilde{K}_0^{lf}((\widehat{X}^{n-1}, p_{\widehat{X}})_{(\infty)}))$ can be identified to $\tilde{K}_0^{lf}(OB, p_{\widehat{X}}|_{OB})$, where $Holink(\widehat{X}, \widehat{X}^{n-1})|_{OB} \xrightarrow{p_{\widehat{X}}|_{OB}} OB$ is the projection map. The inclusion map induces a restriction map:

$$\tilde{K}_0^{lf}(\ OB, p) \longrightarrow \tilde{K}_0^{lf}(B \times (0, \infty), p|_{B\times(0,\infty)})$$

which amounts then to a map:

$$\partial_n : K_0^{lf}((\widehat{X}^{n-1}, p_{\widehat{X}})_{(\infty)}) \to K_{-1}(B, p_B)$$

where p_B denotes the restriction of the holink projection over B.

This is the map we seek. It turns out that $\partial_n\gamma_n(X) \in K_{-1}(B, p_B)$ is the obstruction to finding an exhaustion of X:

Theorem 3.7 (Exhaustibility Theorem)**.** *Let X be a tame ended n-dimensional stratified space with empty boundary for which X^{n-1} admits a completion. Assume that $\partial\gamma_n(X) = 0$. Then X admits an exhaustion.*

Conversely, if X admits an exhaustion, and all the fundamental groups of the fibers of the map, $Holink(X, X^{n-1}) \xrightarrow{p_X} X^{n-1}$ are good*, then $\partial\gamma_n(X) = 0$.*

3.8. We say a group G is *good* if $K_i(\mathbb{Z}[G]) = 0$ for $i \leq -2$.
No example of a group which is not good is known. Moreover, a recent theorem of Farrell and Jones [2] shows that any subgroup of a uniform discrete subgroup of a virtually connected Lie group must be good.

3.9. There are stratified G-manifolds which are not exhaustable, but are tame ended. In fact, there is a semifree action of $G = \mathbb{Z}/6\mathbb{Z}$ on $M_1' = \mathbb{R}^{2n+1} - \{0\}$, $n \geq 2$ with fixed set $\mathbb{R}^1 - \{0\}$, for which $\partial\gamma_{2n+1}(M/G) \neq 0$ in $K_{-1}(\mathbb{Z}G) \oplus K_{-1}(\mathbb{Z}G) \oplus K_{-1}(\mathbb{Z}G) \oplus K_{-1}(\mathbb{Z}G)$. Furthermore if $M_1 = S^{2n} \times S^1$ and $M_1' \xrightarrow{\pi} M_1$ is the usual covering map, then π is equivariant with respect to a stratified G-action on M_1. This G-manifold, (M_1, G) is h-cobordant (stratified and equivariant) to some (M_0, G), whose infinite cyclic cover (M_0', G) has the form $(V - \{0\}, G)$, where (V, G) is a linear representation of G. This example and the more general question of realizability of the obstruction $\gamma_*(X)$ are thoroughly analyzed in the 1996 Ph.D. thesis of B. Vajiac.

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