THE ASYMPTOTIC METHOD IN THE NOVIKOV CONJECTURE

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The famous Hirzebruch signature theorem asserts that the signature of a closed oriented manifold is equal to the integral of the so called L-genus. An immediate corollary of this is the homotopy invariance of $< L(M), [M] >$. The L-genus is a characteristic class of tangent bundles, so the above remark is a non-trivial fact. The problem of higher signatures is a generalization of the above consideration. Namely we investigate whether the higher signatures are homotopy invariants or not. The problem is called the Novikov conjecture. The characteristic numbers are closely related to the fundamental groups of manifolds.

There are at least two proofs of the signature theorem. One is to use the cobordism ring. The other is to use the Atiyah-Singer index theorem. Recall that the signature is equal to the index of the signature operators. The higher signatures are formulated as homotopy invariants of bordism groups of $B\Gamma$. The problem was solved using the Atiyah-Singer index theorem in many partial solutions. Here we have the index-theoretic approach in mind when considering the higher signatures. Roughly speaking, a higher signature is an index for a signature operator with some coefficients. To interpret the number as a generalized signature, one considers homology groups with rational group ring coefficients. By doing surgery on the homology groups, we obtain non degenerate symmetric form $\sigma \in L(\Gamma)$ over the group ring. It is called the Mishchenko-Ranicki symmetric signature. This element is a homotopy invariant of manifolds. Mishchenko introduced Fredholm representations, obtaining a number $\sigma(\mathbb{F})$ from a Fredholm representation $\mathbb{F}$ and σ. On the other hand, one can construct a virtual bundle over $K(\Gamma, 1)$ from a Fredholm representation. By pulling back the bundle through maps from the base manifolds to $K(\Gamma, 1)$, we can make a signature operator with coefficients. Mishchenko discovered the generalized signature theorem which asserts the coincidence of the index of the operators and $\sigma(\mathbb{F})$. Thus a higher signature coming from a Fredholm representation is an oriented homotopy invariant.

In [CGM] the authors showed that all higher signatures come from Fredholm representations for large class of discrete groups, including word hyperbolic groups. They formulated the notion of a proper Lipschitz cohomology class in group cohomology. It corresponds to a Fredholm representation in K-theory. In fact for many discrete groups, any class of group cohomology can be represented by a proper Lipschitz cohomology class. Their method depends on the existence of finite dimensional spaces of $\mathbb{Q}$ type $K(\Gamma, 1)$.

On the other hand for larger classes of discrete groups, we cannot expect existence of such good spaces. In [G], Gromov introduced a very large class of discrete groups, the quasi geodesic bicombing groups. This class is characterized by convexity of the Cayley graph. Hyperbolic groups are contained in the class. For the class we cannot ensure the existence of good spaces as above. Moreover it is unknown whether $H^n(\Gamma;\mathbb{Q})$ is zero for sufficiently large n. To overcome this difficulty, the following is shown in [K]. We realize $K(\Gamma,1)$ by infinite dimensional space and approximate it by a family of finite dimensional spaces. By applying the method of [CGM] for finite dimensional spaces iteratively, it turns out that any cohomology class comes from a Fredholm representation asymptotically. It suffices for the Novikov conjecture because of finite dimensionality of manifolds.

§1 Geometric interpretation

To indicate the geometric features of higher signatures, let us consider signatures of submanifolds(see[G2]). Let Γ be a discrete group, M be a closed manifold and $f: M \to K(\Gamma,1)$ be a smooth map. Let us assume that $K(\Gamma,1)$ is realized by a closed manifold V ($\dim M \geq \dim V$). Then for a regular value $m \in V$, the cobordism class of $W = f^{-1}(m)$ is defined uniquely up to homotopy class of f. Moreover the Poincaré dual class of $[W] \in H_*(M)$ is $f^*([V])$ where $[V] \in H^{\dim V}(V)$ is the fundamental cohomology class. Notice that the normal bundle of W is trivial. Thus

$$\sigma(W) \ = < L(W),[W] > = < L(M),[W] > = < L(M)f^*([V]),[M] > .$$

$\sigma(W)$ is a higher signature of M which we now define as follows.

Definition 1-1. Let M be a closed manifold and Γ be a discrete group. Then a higher signature of M is a characteristic number

$$< L(M)f^*(x),[M] >$$

where $f: M \to K(\Gamma,1)$ is a continuous map and $x \in H^*(\Gamma;\mathbb{Q})$.

It is conjectured that these characteristic numbers are all homotopy invariants.

Let us see another geometric interpretation. Let $F \to X \to M$ be a smooth fiber bundle over M and assume F is $4k$ dimensional. Then the flat bundle induced from the fibration $H \to M$ has a natural involution $*$. Thus H splits as $H = H_+ \oplus H_-$ and by the index theorem for families, it follows

$$\sigma(X) \ = < L(M)ch(H_+ - H_-),[M] > .$$

As a corollary, we see that the right hand side is a homotopy invariant of fiber bundles over M(see[At]).

It is not necessary to construct a fiber bundle corresponding to each higher signature. To induce the homotopy invariance, we only need a flat bundle and an involution over M. From the point of view, Lusztig succeeded in verifying Novikov conjecture for free abelian groups by the analytic method ([L]). Let Y be a compact topological space and X be $2k$ dimensional compact manifold. Let $\rho : Y \times \pi_1(X) \to U(p,q)$ be a family of $U(p,q)$ representations of the fundamental group of X. Then one can construct a vector bundle E over $Y \times X$ which is flat in the X direction. E is admitted a non degenerate hermitian form $<,>$ and E splits as $E = E_+ \oplus E_-$. Using the splitting, we obtain a family of quadratic forms

$$\sigma_y : H^k(X;E) \times H^k(X;E) \to \mathbb{C}.$$

Naturally there corresponds $\sigma(X,\rho) \in K(Y)$ which is homotopy invariant of X. Lusztig discovered the index theorem as follows. Let $\pi : Y \times X \to Y$ be the projection. Then

$$\pi_*(L(X)ch(E_+ - E_-)) \;=\; ch(\sigma(X,\rho)).$$

In particular we can take as Y the representation space of the fundamental group of X. In the special case of the free abelian group $\mathbb{Z}^n$, $U(1)$ the representation space is the dual torus which is topologically isomorphic to the torus T^n. In the case of a single $U(1)$ representation, one can only obtain the signature. However Lusztig found the following. There exist bases $\{a_i\}$ and $\{b_i\}$ of $H^{2*}(T^n;\mathbb{Z})$ and $H^{2n}(\hat{T}^n;\mathbb{Z})$ such that

$$ch(\sigma(X,L)) \;=\; \sum_i < L(X)f^*(a_i),[X] > b_i$$

where $f : X \to T^n$ induces an isomorphism of the fundamental groups. This is enough to verify the Novikov conjecture for free abelian groups. In the case of general noncommutative discrete groups, the representation space will be too complicated and it will be very difficult to apply this method to general noncommutative discrete groups.

§2 Fredholm representation

Mishchenko discovered the infinite dimensional version of the method of flat vector bundles.

Definition 2-1. Let Γ be a discrete group. Then a Fredholm representation of Γ is a set $(H_1, H_2, \rho_1, \rho_2, F)$ where
(1) H_1, H_2 are Hilbert spaces,
(2) $F : H_1 \to H_2$ is a Fredholm map,
(3) $\rho_i : \Gamma \to U(H_i, H_i)$ is a unitary representation such that $\rho_2(\gamma)F - F\rho_1(\gamma)$ is a compact operator for any $\gamma \in \Gamma$.

Using a Fredholm representation, we can construct a virtual bundle over $K(\Gamma,1)$ as follows. From the condition (3), we can construct an equivariant continuous map $f : E\Gamma \to B(H_1, H_2)$ which satisfies
(1) for some point $x \in E\Gamma$, $f(x) = F$,
(2) for any points $x, y \in E\Gamma$, $f(x) - f(y)$ is a compact operator.

Notice that f is unique up to homotopy. Then the virtual bundle is $(f : E\Gamma \times_\Gamma H_1 \to E\Gamma \times_\Gamma H_2)$ and we write

$$\mu : \{ \text{ Fredholm representations } \}/ \text{ homotopy} \to \text{ Virtual bundles over } B\Gamma.$$

Theorem 2-2 (Mishchenko). *Let $f : M \to B\Gamma$ be a continuous map. Then*

$$< L(M)f^*(ch(\mu(F))), [M] >$$

is an oriented homotopy invariant of M.

Let us interpret this theorem as an infinite version of the one of Lusztig. By doing surgery on the homology groups with local coefficient, we have the resulting homology only on the middle dimension. Poincaré duality on the homology gives a symmetric form σ. This is an element of the Wall L-group $L(\Gamma)$ of the fundamental group Γ, represented by a group ring valued nondegenerate symmetric matrix. If there is a unitary representation of Γ, then the matrix can be regarded as an invertible self adjoint operator on an infinite dimensional Hilbert space. The Fredholm operator F of a Fredholm representation decomposes into an operator valued 2 by 2 matrix $\{F_{i,j}\}_{i,j=1,2}$ corresponding to the decomposition of the Hilbert space into positive and negative parts of the self adjoint operators. It turns out that the diagonal parts F_{11} and F_{22} are also Fredholm operators and F_{12}, F_{21} are compact operators. This follows essentially from the almost commutativity of the unitary representations and the Fredholm operator in the definition of Fredholm representation. Thus we obtain a number $\mathrm{index}F_{11} - \mathrm{index}F_{22}$. Mishchenko discovered the generalized signature theorem which asserts the coincidence of this number and the characteristic number of the above theorem. The process is parallel to the signature theorem in the case of the simply connected spaces.

§3 Novikov conjecture for word hyperbolic groups

It is natural to ask how large $ch^*(\mu($ Fredholm representations $))$ is in $H^{2*}(\Gamma;\mathbb{Q})$. By a celebrated work by A.Connes, M.Gromov and H.Moscovici, it is shown that if Γ is hyperbolic, then they occupy in $H^{2*}(\Gamma;\mathbb{Q})$.

In some cases of discrete groups, Eilenberg-Maclane spaces are realized by (compact) smooth manifolds. In particular compact negatively curved manifolds themselves are Eilenberg-Maclane spaces. Hyperbolic groups are introduced by Gromov. The class is characterized by the essential properties which are possessed by the fundamental groups of compact negatively curved manifolds. Though the class is very large, they have reasonable classifying spaces which are enough to work instead of Eilenberg-Maclane spaces, at least for the Novikov conjecture. The spaces are called Rips complexes.

Fact 3-1. Let Γ be a discrete group. Then there exists a family of finite dimensional simplicial complexes $\{P_n(\Gamma)\}_{1\leq n}$. They satisfy the following:
(1) Γ acts on each $P_n(\Gamma)$ proper discontinuously with compact quotient,
(2) if Γ is torsion free, then the action is also free,
(3) $P_1(\Gamma) \subset \cdots \subset P_n(\Gamma) \subset P_{n+1}(\Gamma) \ldots$,
(4) if Γ is hyperbolic, then $P_n(\Gamma)$ is contractible for sufficiently large n.

In particular, torsion free hyperbolic groups have $B\Gamma$ represented by finite dimensional simplicial complexes. In the following, we shall write $\tilde{P}_n/\Gamma$ as a tubular neighborhood in an embedding $P_n(\Gamma)/\Gamma \to \mathbb{R}^N$. $\tilde{P}_n/\Gamma$ is an open manifold with the induced metric from $\mathbb{R}^N$. In the following, Γ is a hyperbolic group.

Kasparov KK-groups.

Before explaining the method of [CGM], we shall quickly review Kasparov's KK-theory. The KK-groups are used effectively to prove Novikov conjecture. KK is a bifunctor from a pair of distinct spaces (X,Y) to abelian groups which is covariant on X and contravariant on Y. The KK-groups include both K-cohomology and K-homology.

Roughly speaking K-homology consists of the set of Dirac operators on spaces. Precisely an element of $K_0(X)$ is represented by (M,E,φ) where

(1) M is an even dimensional spin^c manifold which need not be compact or connected,

(2) E is a complex vector bundle over M,

(3) φ is a proper map from M to X.

$K_0(X)$ is the set of the above triples quotiented by a certain equivalence relation. It is dual to K-cohomology and the pairing is to take the index on twisted vector bundles. Let S be the spin^c vector bundle over M and $D_E : S_+ \otimes E \to S_- \otimes E$ be the Dirac operator on M. Then the pairing of K theory is $< F,(M,E,\varphi) >=$ index$D_{E\otimes F}$.

Fact 3-2. There exists a Chern character isomorphism,

$$ch_* : K_0(X) \otimes \mathbb{Q} = H_{2*}^{\inf}(X;\mathbb{Q})$$

by $\varphi_*(ch^*(E) \cup td(M) \cap [M])$ where $H_*^{\inf}$ is the homology with locally finite infinite support.

Roughly speaking $KK(X,Y)$ is the set of sections over a family of elements of $K_0(X)$ over Y. Thus if Y is a point,

$$KK_*(X,\mathrm{pt}) = K_*(X).$$

There is an analytical interpretation of topological K-homology. Let $C_0(X)$ be the set of the continuous functions on X vanishing at infinity. $C_0(X)$ is C^* algebra whose C^* norm is to take pointwise supremum. The analytical K-homology $\hat{K}(X)$ is the set $(H_0 \oplus H_1, \rho_0, \rho_1, T)$ quotiented by an equivalence relation, where

(1) H_i is a Hilbert space,

(2) $\rho_i : C_0(X) \to B(H_i)$ is a $*-$homomorphism,

(3) $T : H_0 \to H_1$ is a bounded operator such that $I - T^*T$, $I - TT^*$, $\rho_1(a)T - T\rho_0(a)$, are all compact operators.

The explicit map $K_0(X) \to \hat{K}_0(X)$ is to take L^2 sections of twisted spin^c vector bundles, $(L^2(M, S\hat{\otimes}E), D_E, \varphi)$. Though D_E is an unbounded operator, by making pseudo differential calculus, we can construct a bounded operator. If M is compact, then it is $D_E(I + D_E^* D_E)^{-\frac{1}{2}}$. As φ is a proper map, it pulls back $C_0(X)$ to $C_0(M)$ and the $*-$homomorphism is the multiplication by $\varphi^*(a)$, $a \in C_0(X)$. If X is a point, then an element of $\hat{K}_0(X)$ is represented by a Fredholm operator over Hilbert spaces. $K_0($ pt $)$ is naturally isomorphic to $\mathbb{Z}$ by taking Fredholm indices. $KK($ pt $, Y)$ is a family of Fredholm operators over Y. Thus

$$KK_*(\mathrm{pt}, Y) = K^*(Y).$$

Now let us define the KK-groups. First, let us recall the definition of the analytical K homology (1), (2), (3) and consider the family version. (1) The set of sections over the family of Hilbert spaces over Y admits a natural $C_0(Y)$-module structure. (2) As the $*-$homomorphism ρ_i action is fiberwise, it commutes with that of $C_0(Y)$. (3) A family of compact operators will be formulated as an element of a norm closure of finite rank projections in the set of endomorphisms of the $C_0(Y)$-module. Soon we define this precisely.

Let us consider the triple (E, ϕ, F) where

(1) E is a $\mathbb{Z}_2$-graded right $C_0(Y)$-module with a $C_0(Y)$ valued inner product. It is complete with respect to C^* norm of $C_0(Y)$. E is called a Hilbert module over $C_0(Y)$.

(2) ϕ is a degree 0 $*-$homomorphism from $C_0(X)$ to $B(E)$ where $B(E)$ is the set of $C_0(Y)$-module endomorphisms. $C_0(X)$ acts on E from the left.

(3) $F \in B(E)$ is of degree 1 such that $[F - F^*]\phi(a)$, $[\phi(a), F]$ and $(F^2 - 1)\phi(a)$ are all compact endomorphisms. A compact endomorphism is an element of $B(E)$ which lies in the closure of linear span of the rank one projections $\theta_{x,y} \in B(E)$, $\theta_{x,y}(z) = x < y, z >$. We denote the set of compact endomorphisms by $K(E)$

If $C_0(Y)$ itself is considered as $C_0(Y)$-module, then $B(C_0(Y))$ is the set of bounded continuous functions on Y. $K(C_0(Y))$ is also $C_0(Y)$.

Let us denote the set of the above triples (E, ϕ, F) by $E(X, Y)$. Notice that we can replace $C_0(X)$ and $C_0(Y)$ by any C^*-algebras A, B and write $E(A, B)$ for the set of triples which satisfy the above (1), (2), (3) replacing $C_0(X)$ by A and $C_0(Y)$ by B.

Now let us introduce a homotopy equivalence relation as follows. (E_1, ϕ_1, F_1) is equivalent to (E_2, ϕ_2, F_2) if there exists $(E, \phi, F) \in E(A, C([0, 1], B))$ such that $(E \hat{\otimes}_{f_i} B, f_i \circ \phi, (f_i)_* F)$ is isomorphic to (E_i, ϕ_i, F_i) where $f_i : C([0, 1], B) \to B$ is the evaluation maps.

Definition 3-3. $KK(X, Y) = E(X, Y)/$ homotopy .

It turns out that $KK(X, Y)$ is a group. $KK(A, B)$ is defined similarly.

Notice that $KK(\text{ pt }, \mathbb{R}^n)$ is isomorphic to the K-homology of $\mathbb{R}^n$ with compact support which is isomorphic to $\mathbb{Z}$. The generator of $KK(\text{ pt }, \mathbb{R}^n)$ is expressed using Clifford algebra. Let $n = 2k$ be even. Then by identifying $\mathbb{R}^n$ with $\mathbb{C}^k$, any vector in $\mathbb{R}^n$ acts on $\wedge \mathbb{C}^k$ by Clifford multiplication. Then the generator is

$$\{C_0(\mathbb{R}^n, \wedge \mathbb{C}^k), F(x) = \frac{x}{1 + |x|}\}$$

in $KK(\text{ pt }, \mathbb{R}^n)$.

There is also equivariant KK-theory. Let A and B admit automorphisms of Γ. If X and Y are Γ spaces, then $C_0(X)$ and $C_0(Y)$ have natural Γ actions. Let $E_\Gamma(A, B)$ be the set of triples $(E, \phi, F) \in E(A, B)$ such that there exists an action of Γ on E which satisfy

(1) $g(a\zeta b) = (ga)(g\zeta)(gb)$, $< g\zeta, g\zeta' >= g < \zeta, \zeta' >$

(2) $\phi(a)(gFg^{-1} - F)$ is a compact endomorphism of E.

Homotopy equivalence is defined similarly.

Definition 3-4. $KK_\Gamma(X,Y) = E_\Gamma(X,Y)/$ homotopy .

Notice that $KK_\Gamma(\mathrm{pt},\mathrm{pt})$ is the set of Fredholm representations quotiented by homotopy equivalence.

There is a very important operation in KK-theory, called the intersection product pairing (see [Bl])

$$KK_\Gamma(X,Y) \times KK_\Gamma(Y,Z) \to KK_\Gamma(X,Z).$$

Lipschitz geometry.

In the essential step, [CGM] constructs an element $\varphi \in KK_\Gamma(\mathrm{pt},\tilde{P}_n)$. Roughly speaking, the construction is as follows.

First of all, using the contractibility of $\tilde{P}_n$, one constructs a map which induces Poincaré duality,

$$\alpha : \tilde{P}_n \times_\Gamma \tilde{P}_n \to T\tilde{P}_n/\Gamma.$$

Namely for $\beta \in H^*(\tilde{P}_n/\Gamma;\mathbb{Q})$, $z \in H_*^{\inf}(\tilde{P}_n/\Gamma;\mathbb{Q})$,

$$\alpha\cap : H^{\inf}(\tilde{P}_n/\Gamma;\mathbb{Q}) \to H^*(P_n/\Gamma;\mathbb{Q})$$
$$\alpha \cap (z) = \int_{\tilde{z}\times_\Gamma\tilde{\beta}} \alpha.$$

Proposition 3-5[CGM]. *If α is fiberwise proper Lipschitz, then one can construct φ.*

Let $\alpha : \tilde{P}_n \times_\Gamma \tilde{P}_n \to T\tilde{P}_n/\Gamma = \tilde{P}_n/\Gamma \times \mathbb{R}^N$ be the fiberwise proper Lipschitz map which induces Poincaré duality. Let us take $e \in \tilde{P}_n$ and restrict α on $\tilde{P}_n \times e$. Then

$$\varphi = \{C_0(\tilde{P}_n, \wedge\mathbb{C}^K), \frac{\alpha(x)}{1+|\alpha(x)|}\}$$

in $KK_\Gamma($ pt $,\tilde{P}_n)$ is the desired one. If α is not fiberwise Lipschitz, then the above φ does not define an element of the equivariant KK-group. To ensure $\gamma F\gamma^{-1} - F$ is a compact endomorphism, it is enough to see that $|\gamma F(x)\gamma^{-1} - F(x)|$ goes to zero when x goes to infinity. This follows, by simple calculation, from the Lipschitzness of α.

A priori, we only have a fiberwise proper map which induces Poincaré duality. It is natural to try to deform the map so that it becomes fiberwise proper Lipschitz by a proper homotopy. To do so, first using the hyperbolicity, we have the following map.

Proposition 3-6. *Let us take a sufficiently large $n > 0$ and a sufficiently small constant $0 < \mu < 1$. The there exists a map*

$$F : \tilde{P}_n \times_\Gamma \tilde{P}_n \to \tilde{P}_n \times_\Gamma \tilde{P}_n$$

such that

(1) F is fiberwise μ Lipschitz,

(2) there exists a fiberwise proper homotopy F_t which connects F to the identity.

Remark 3-7. The existence of such F implies that $\tilde{P}_n$ must be contractible.

Let us take sufficiently large r and put $D = \{(x,y) \in \tilde{P}_n \times_\Gamma \tilde{P}_n; d(x,y) \leq r\}$. By modifying α slightly, we may assume that $\mu^{-1}\alpha \circ F|_{\partial F^{-1}(D)} = \alpha|_{\partial F^{-1}(D)}$.

Let us put

$$B_i = \{(x,y) \in \tilde{P}_n \times_\Gamma \tilde{P}_n; F^i(x,y) = F \circ F \ldots F(x,y) \in D\}$$

$$D_i = B_i - B_{i-1}.$$

Let us define

$$\alpha_\infty : \tilde{P}_n \times_\Gamma \tilde{P}_n \to T\tilde{P}_n$$

$$\alpha_\infty|_{D_i} = \mu^{-i+1}\alpha \circ F^{i-1}.$$

It is not difficult to see that α_∞ is fiberwise proper Lipschitz and it is fiberwise proper homotopic to α.

Using the Kasparov intersection product, we have a map

$$\varphi : KK_\Gamma(\tilde{P}_n, \text{ pt }) \to KK_\Gamma(\text{ pt }, \text{ pt })$$

$$\varphi(x) = \varphi \times x.$$

Theorem 3-8[CGM]. *There exists the following commutative diagram.*

$$\begin{array}{ccc}
KK_\Gamma(\tilde{P}_n(\pi), \mathbb{Q}) & \xrightarrow{\phi\circ\mu} & K^{2*}(B\Gamma) \\
\Big\downarrow{\scriptstyle ch_*} & & \Big\downarrow{\scriptstyle ch^*} \\
H_{2*}^{inf}(\tilde{P}_n(\Gamma)/\Gamma) & \xrightarrow{PD} & H^{2*}(B\Gamma)
\end{array}$$

where PD is the Poincare duality.

In the case of cohomology groups of odd degrees, we can reduce to the case of even ones by considering $\mathbb{Z} \times \Gamma$. Thus

Corollary 3-9. *Let Γ be a hyperbolic group and $f : M \to K(\Gamma, 1)$ be a continuous map. Then $< L(M)f^*(x), [M] >$ is an oriented homotopy invariant for any $x \in H^*(\Gamma; \mathbb{Q})$. Namely let $p : M_1 \to M_2$ be an oriented homotopy equivalence. Then*

$$< L(M_1)(p \circ f)^*(x), [M_1] >=< L(M_2)f^*(x), [M_2] > .$$

Notice that in the case of hyperbolic groups, we have used the fact that $K(\Gamma, 1)$ was realized by a finite dimensional simplicial complex over $\mathbb{Q}$. On the other hand, we cannot expect it on more large classes of discrete groups, in particular quasi geodesic bicombing groups which we shall treat in the next section. For the class, we cannot expect even that the ranks of cohomology over $\mathbb{Q}$ are finite.

§4 Quasi geodesic bicombing groups

In [ECHLPT], a very large class of discrete groups is defined. The elements of the class are called combing groups. It contains hyperbolic groups and quasi geodesic bicombing groups defined later.

Theorem 4-1[ECHLPT]. *If Γ is a combable group, then $K(\Gamma, 1)$ space can be realized by a CW complex such that the number of cells in each dimension is finite.*

As an immediate corollary of this, we can see that $\dim H^n(\Gamma; \mathbb{Q}) < \infty$ for each n. Using this fact, in the following construction, we shall make an analogy of the case of hyperbolic groups on spaces which approximates $K(\Gamma, 1)$.

A set of generators of a discrete group determines a 1 dimensional simplicial complex called Cayley graph $G(\Gamma)$. $G(\Gamma)$ has a natural metric. Notice that the universal covering spaces of non positively curved manifolds have the convex property. With this in mind, we shall define the following.

Definition 4-2[G1]. If Γ has the following properties, we call it a bicombing group. Let us fix a generating set of Γ. Then there exists a continuous and Γ equivariant map

$$S : \Gamma \times \Gamma \times [0, 1] \to G(\Gamma)$$

such that for some $k \geq 1, C \geq 0$, it satisfies

$$S(\gamma_1, \gamma_2, 0) = \gamma_1, \quad S(\gamma_1, \gamma_2, 1) = \gamma_2,$$
$$d(S_t(\gamma_1, \gamma_2), S_t(\gamma_1', \gamma_2')) \leq k(td(\gamma_2, \gamma_2') + (1 - t)d(\gamma_1, \gamma_1')) + C.$$

Though $S(\gamma_1, \gamma_2, \quad) : [0, 1] \to G(\pi)$ connects γ_1 and γ_2, we shall require balanced curves.

Definition 4-3[G1]. Let Γ be bicombing. We say that Γ is bounded if for some $k \geq 1, C \geq 0$, it satisfies

$$d(S_t(\gamma_1, \gamma_2), S_{t'}(\gamma_1, \gamma_2)) \leq k|t - t'|d(\gamma_1, \gamma_2) + C$$

Definition 4-4. Γ : bounded bicombing is quasi geodesic if for every γ, a sufficiently small ϵ and $0 \leq t < s < t + \epsilon \leq 1$, $S(e, \gamma, t) \neq S(e, \gamma, s)$. Moreover let us denote a unit speed path of $S(e, \gamma, \quad)$ by ω_γ.

$\omega_\gamma : [0, |S(e, \gamma, \quad)|] \to G(\Gamma)$. Then for $d(\gamma_1, \gamma_2) \leq 1$,

$$Ud(\omega_{\gamma_1}, \omega_{\gamma_2}) = sup_t d(\omega_{\gamma_1}(t), \omega_{\gamma_2}(t)) \leq C,$$
$$|S(e, \gamma, \quad)| \leq k|\gamma| + C$$

Furthermore, for some $A > 0, B \geq 0$, S_t satisfies

$$d(\gamma, S_t(\gamma, \gamma')) \geq Atd(\gamma, \gamma') - B.$$

Using S, it is easy to prove the following lemma.

Lemma 4-5[Al]. *If Γ is a quasi geodesic bicombing group, then each Rips complex $P_i(\Gamma)$ is contractible in $P_n(\Gamma)$ for large $n = n(i)$.*

From this, we see that for a torsion free quasi geodesic bicombing group Γ, $P_\infty(\Gamma) = lim_i P_i(\Gamma)$ is a realization of $E\Gamma$. Unlike to the case of hyperbolic groups, we cannot make $F : \tilde{P}_n \times_\Gamma \tilde{P}_n(\Gamma) \to \tilde{P}_n \times_\Gamma \tilde{P}_n(\Gamma)$ as before, because P_n is not contractible in itself. However we have the following family of maps.

Proposition 4-6. *Let us take an arbitrary family of small constants $\{\mu_i\}_{0\leq i}$, $1 >> \dots \mu_i >> \mu_{i-1} \dots >> \mu_0 > 0$. Then for some family of Rips complexes $\{P_{n(i)}\}_{0\leq i}$, there exists a family of maps*

$$F_i : \tilde{P}_{n(0)} \times_\pi \tilde{P}_{n(i)} \to \tilde{P}_{n(0)} \times_\pi \tilde{P}_{n(i+1)}$$

such that
(1) F_i is fiberwise μ_i Lipschitz,
(2) F_i is fiberwise proper homotopic to the inclusion.

Let $\alpha_0 : \tilde{P}_{n(0)} \times_\pi \tilde{P}_{n(0)} \to \mathbb{R}^N$ be a fiberwise proper map which induces Poincaré duality. Using the above family of maps, we can construct the following commutative diagram of maps.

$$\begin{array}{ccc}
\tilde{P}_0 \times_\Gamma \tilde{P}_0 & \xrightarrow{\alpha_0} & \mathbb{R}^{N_0} \\
{\scriptstyle incl}\downarrow & & \downarrow{\scriptstyle incl} \\
\tilde{P}_0 \times_\Gamma \tilde{P}_1 & \xrightarrow{\alpha_1} & \mathbb{R}^{N_1} \\
{\scriptstyle incl}\downarrow & & \downarrow{\scriptstyle incl} \\
\dots & \longrightarrow & \dots
\end{array}$$

To produce a proper Lipschitz map, we need to control growth of these maps at infinity. In this case, we can construct α_i which satisfy the following. There exist families of constants $\{C_i\}$, $\{a_i\}$ such that the Lipschitz constant of α_i on $N_i(r) = \{(x,y) | x \in \tilde{P}_0, y \in \tilde{P}_i, d_i(x,y) \leq r\}$, for sufficiently large r, is bounded by $C_i H(a_i r)$ where H is a Lipschitz function on $[0,\infty)$.

Proposition 4-7. *Using these maps, we have*

$$\alpha_\infty : \tilde{P}_{n(0)} \times_\pi \tilde{P}_{n(0)} \to \mathbb{R}^\infty$$

which is fiberwise proper homotopic to α. Moreover let

$$pr : \mathbb{R}^\infty \to \mathbb{R}^N, \qquad N = dim \tilde{P}_{n(0)}.$$

Then $pr \circ \alpha_\infty$ is fiberwise proper Lipschitz.

Let us recall that homology commutes with spaces under the direct limit operation. Thus $H_*(P_\infty/\Gamma) = lim_n H_*(P_n/\Gamma)$. With the fact that the rank of $H_N(P_n/\Gamma)$ is finite for every N, we have

Lemma 4-8. *Let us take any large N. Then for $* < N$, there exists n such that*

$$i_* : H_*(P_n/\Gamma; \mathbb{Q}) \to H_*(P_\infty/\Gamma; \mathbb{Q})$$

is surjective where $i : P_n(\pi)/\pi \to P_\infty(\pi)/\pi$ is the inclusion.

As before, we can construct $\varphi \in KK_\Gamma(\text{ pt }, \tilde{P}_n)$.

Theorem 4-9. *The following diagram commutes.*

$$\begin{array}{ccc} KK_\Gamma(\tilde{P}_n, \ pt\ ) & \xrightarrow{\mu\circ\psi} & K^{2*}(B\Gamma) \\ \Big\downarrow{\scriptstyle ch_*} & & \Big\downarrow{\scriptstyle i^*\circ ch^*} \\ H^{inf}_{2*}(\tilde{P}_n/\Gamma) & \xrightarrow{PD} & H^{2*}(P_n/\Gamma). \end{array}$$

We cannot construct the homotopy between $\mathrm{pr}\circ\alpha_\infty$ and α_0 through the map to $\mathbb{R}^N$. Let $2K = N$ and $Z = \tilde{P}_{n(0)} \times P_{n(0)}$. To show the commutativity of the diagram, we construct a homotopy between the following two elements in $KK(\text{ pt }, Z)$.

$$\alpha_0^*(\beta) = \{C_0(Z, \wedge\mathbb{C}^K), F_0(x) = \frac{\alpha_0(x)}{1 + |\alpha_0(x)|}\}$$
$$(\text{ pr } \circ\alpha_\infty)^*(\beta) = \{C_0(Z, \wedge\mathbb{C}^K), F_\infty(x) = \frac{\mathrm{pr}\circ\alpha_\infty(x)}{1 + |\mathrm{pr}\circ\alpha_\infty(x)|}\}.$$

Let $\Delta_N = \wedge\mathbb{C}^N$. Then there are natural inclusions $\Delta_N \subset \Delta_{N+1} \subset \dots$ which preserves the metrics. Let $[\wedge C^\infty]$ be the infinite dimensional Hilbert space which is the completion of the union. By adding degenerate elements, we can express

$$\alpha_0^*(\beta) = \{C_0(Z, [\wedge\mathbb{C}^\infty]), F_0 \oplus G_0\}$$
$$(\text{ pr } \alpha_\infty)^*(\beta) = \{C_0(Z, [\wedge\mathbb{C}^\infty]), F_\infty \oplus G_\infty\}$$

Using the proper homotopy between α_0 and $\mathrm{pr}\circ\alpha_\infty$ through maps to $\mathbb{R}^\infty$, we can construct the homotopy between the elements in $KK(\text{ pt }, Z)$.

Corollary 4-10. *Let Γ be a torsion free quasi geodesic bicombing group. Then for arbitrary large N and $x \in H^{2*}(B\Gamma; \mathbb{Q})$, there exists a Fredholm representation F such that $x - ch(\mu(F)) \in H^*(B\Gamma; \mathbb{Q})$, $* \geq N$.*

Corollary 4-11. *For torsion free quasi geodesic bicombing groups, the higher signatures are oriented homotopy invariants.*

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