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FACTORIZATION OF AUTOMORPHISMS OF A MODULE OVER A LOCAL RING

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Abstract

Let R be a commutative local ring with the identity 1 and the unique maximal ideal $\mathfrak{m}$, M be a free module of rank n over R, and σ be in $\mathrm{Aut}_R M$.

Then, we factorize M into a direct sum of m free submodules $M_1, M_2, ..., M_m$ such that each M_i is σ-invariant modulo $\mathfrak{m}$ for $i = 1, 2, ..., m$ and m is the number of the polynomials in the system of invariants of σ modulo $\mathfrak{m}$.

Further it is shown that there exists a basis X for M over R for which σ is factorized into a product of elements in $\mathrm{Aut}_R M$ in the form of

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$$\sigma = \rho_1\rho_2\cdots\rho_m\gamma_1\gamma_2\cdots\gamma_m,$$

where each ρ_i is a cyclic permutation on X, each γ_i is simple, i.e., fixes $n-1$ elements in X. If 2 is a unit in R, then $\rho_1\rho_2\cdots\rho_m$ can be replaced by a product of $n-m$ symmetries in $\mathrm{Aut}_R M$. As a result if 2 is a unit in R, then σ is a product of n or less than n simple elements.

1. Introduction

Throughout R is a commutative local ring with the identity 1 and the unique maximal ideal $\mathfrak{m}$, M is a free module of rank n over R, $\mathrm{Aut}_R M$ is the automorphism group on M and σ is in $\mathrm{Aut}_R M$.

We have two chanonical maps

$$\pi_R : R \to \overline{R} = R/\mathfrak{m} \quad \text{defined by} \quad a \mapsto \overline{a} = a + \mathfrak{m}$$

and

$$\pi_M : M \to \overline{M} = M/\mathfrak{m}M \quad \text{defined by} \quad x \mapsto \overline{x} = x + \mathfrak{m}M.$$

Since $\overline{R}$ is a field, we may regard $\overline{M}$ as a vector space over $\overline{R}$ of dimension n by the scalar multiplication $\overline{a}\,\overline{x} = \overline{ax}$ for $a \in R$ and $x \in M$. Clearly the ring homomorphism π_R is a surjective R-module homomorphism if we regard $\overline{R}$ as a module over R by a scalar multiplication $a\overline{b} = \overline{ab}$ for $a, b \in R$. Also π_M is a surjective R-module homomorphism, since $\overline{M}$ is an R-module by $a\overline{x} = \overline{ax}$ for $a \in R$ and $\overline{x} \in \overline{M}$.

Further, for $x \in M$ and $\sigma \in \mathrm{Aut}_R M$, if we define $\overline{\sigma}\overline{x} = \overline{\sigma x}$, we obtain an automorphism $\overline{\sigma}$ of $\overline{M}$, that is, $\overline{\sigma} \in \mathrm{Aut}_{\overline{R}}\overline{M}$. Thus, we have the third chanonical map

$$\pi_E : \mathrm{Aut}_R M \to \mathrm{Aut}_{\overline{R}}\overline{M} \quad \text{by} \quad \sigma \mapsto \overline{\sigma},$$

which is also a surjective group homomorphism.

As we have mentioned in the abstract our purpose is to present factorizations of the module M and the automorphism σ on M. However, the details of the theorems will be stated explicitly in the next section, and the proofs of them will be done in Section 3. In our proof of the theorems, the structure theorem of a finitely generated torsion module over PID will play a central role.

2. Statement of Theorem A and Theorem B

Let $R[t]$ be the polynomial ring in t over R and $\overline{R}[t]$ be the polynomial ring over the field $\overline{R}$ in t. For $\bar{f}(t) \in \overline{R}[t]$ and $\bar{x} \in \overline{M}$ if we define a scalar multiplication by

$$\bar{f}(t)\bar{x} = \bar{f}(\bar{\sigma})\bar{x},$$

$\overline{M}$ is endowed with the structure of $\overline{R}[t]$-module. Since $\overline{R}[t]$ is a PID, applying the structure theorem of a finitely generated torsion module over a PID, we have a set of monic polynomials

$$\mathcal{F} = \{f_1(t),\ f_2(t), \ldots,\ f_m(t)\} \text{ in } R[t] \tag{2.1}$$

such that

$$\overline{\mathcal{F}} = \{\bar{f}_1(t),\ \bar{f}_2(t), \ldots,\ \bar{f}_m(t)\} \text{ in } \overline{R}[t] \text{ with } \bar{f}_1 \mid \bar{f}_2 \mid \cdots \mid \bar{f}_m \tag{2.2}$$

is the system of invariants of $\bar{\sigma}$.

$\mathbb{N}$ denotes the set of natural numbers.

Theorem A. *Let R be a local ring with 1 and the unique maximal ideal $\mathfrak{m}$, M be a free module of rank n over R, and σ be in $\operatorname{Aut}_R M$. Then, there exist $x_1, x_2, \ldots, x_m$ in M and $n_1, n_2, \ldots, n_m$ in $\mathbb{N}$ such that*

$$M = M_1 \oplus M_2 \oplus \cdots \oplus M_m \tag{2.3}$$

and

$$M_i = Rx_i \oplus R\sigma x_i \oplus \cdots \oplus R\sigma^{n_i - 1}x_i \quad \text{for } i = 1, 2, \ldots, m, \tag{2.4}$$

where $\sigma^{n_i} x_i$ *is in* M_i *modulo* $\mathfrak{m}$, x_i*'s are unique for* σ *modulo* $\mathfrak{m}$ *and* n_i*'s are unique for* σ.

In particular M_i *contains a free submodule* L_i *such that*

$$\operatorname{rank} L_i = n_i - 1 \text{ and } \sigma L_i \subseteq M_i \text{ for } i = 1, 2, ..., m. \tag{2.5}$$

In Theorem B, we will try various factorizations of σ into a product of special automorphisms in $\operatorname{Aut}_R M$. We will now introduce such special automorphisms.

Let γ be in $\operatorname{Aut}_R M$. If there are two bases $X = \{x, x_1, x_2, ..., x_{n-1}\}$ and $X' = \{x', x_1, x_2, ..., x_{n-1}\}$ for M over R such that

$$\gamma = 1 \text{ on } \{x_1, x_2, ..., x_{n-1}\} \text{ and } \gamma x = x',$$

γ is called a *simple element* with axis $\{x_1, x_2, ..., x_{n-1}\}$, or simply γ is *simple* in X. In particular, γ is a *symmetry* if $x' = -x$ and a *transvection* if $x' = x + y$ for some y in $Rx_1 \oplus Rx_2 \oplus \cdots \oplus Rx_{n-1}$. Further let ρ be in $\operatorname{Aut}_R M$. If there is a basis $X = \{x_1, x_2, ..., x_n\}$ such that ρ is a permutation on X, then we call ρ a *permutation element* in $\operatorname{Aut}_R M$ with respect to X, or simply a *permutation* in X. In particular, we say that ρ is *cyclic* in X if it is a cyclic permutation on X. Also an element δ in $\operatorname{Aut}_R M$ is said to be *diagonal* in X if the matrix of δ relative to X is diagonal.

Theorem B. *Under the same assumption as Theorem* A *we have a basis* X *for* M *over* R *with a partition* $X = X_1 \cup X_2 \cup \cdots \cup X_n$ *for which* σ *can be factorized in* $\operatorname{Aut}_R M$ *in each way as* (I), (II) *and* (III) *following*:

(I) $\sigma = \rho\gamma_1\gamma_2 \cdots \gamma_m$ *with* $\rho = \rho_1\rho_2 \cdots \rho_m$,

where for $i = 1, 2, ..., m$

(i) ρ_i *is cyclic on* X_i *of length* n_i,

(ii) $n_i = |X_i| = \deg f_i$ *for an invariant system* $\{f_1, f_2, ..., f_m\}$ *of* σ *modulo* $\mathfrak{m}$, *and*

(iii) γ_i *is simple on* X_i, *i.e., fixes* $n_i - 1$ *elements in* X_i.

(II) $\sigma = \rho\delta\tau_1\tau_2 \cdots \tau_m$,

where

(i) ρ *is that of* (I),

(ii) *the matrix of* δ *is diagonal relative to* X,

and

(iii) $\tau_1, \tau_2, ..., \tau_m$ *are transvections in* X.

(III) *If* 2 *is a unit*,

$$\sigma = \sigma_1\sigma_2 \cdots \sigma_{n-m}\gamma_1\gamma_2 \cdots \gamma_m = \sigma_1\sigma_2 \cdots \sigma_{n-m}\delta\tau_1\tau_2 \cdots \tau_m,$$

where

(i) $\sigma_1, \sigma_2, ..., \sigma_{n-m}$ *are symmetries*,

(ii) $\gamma_1, \gamma_2, ..., \gamma_m$ *are those of* (I),

and

(iii) δ, $\tau_1, \tau_2, ..., \tau_m$ *are those of* (II).

3. Proof for the Theorems

For an ideal $\mathfrak{a}$ of R and a subset S of M we denote the submodule of M spanned by S over $\mathfrak{a}$ by $\langle S\rangle_{\mathfrak{a}}$, i.e.,

$$\langle S\rangle_{\mathfrak{a}} = \sum_{s\in S}\mathfrak{a}s.$$

We need the following lemma.

Lemma. *For a subset* $Y = \{y_1, y_2, ..., y_n\}$ *of* M, *the following statements* (a) *and* (b) *are equivalent*:

(a) *Y is a basis for M over R.*

(b) $\overline{Y}$ *is a basis for* $\overline{M}$ *over* $\overline{R}$.

Proof. Let $Z = \{z_1, z_2, \ldots, z_n\}$ be a basis for M over R. Then we have a matrix $P \in M_n(R)$ such that

$$^tY = P^tZ,$$

therefore

$$^t\overline{Y} = \overline{P}^t\overline{Z}.$$

Thus,

(a) $\leftrightarrows$ P is invertible $\leftrightarrows$ $|P| \notin \mathfrak{m}$ $\leftrightarrows$ $|\overline{P}| \neq \overline{0}$ $\leftrightarrows$ $\overline{P}$ is invertible $\leftrightarrows$ (b).

□

Since $\overline{\mathcal{F}} = \{\bar{f}_1(t), \bar{f}_2(t), \ldots, \bar{f}_m(t)\}$ is the system of the invariant of $\overline{\sigma}$ in $\mathrm{Aut}_{\overline{R}}\overline{M}$, by the structure theorem of $\overline{M}$ we have vectors

$$\{x_1', x_2', \ldots, x_n'\} \subseteq \overline{M}$$

such that if we set for $i = 1, 2, \ldots, m$

$$X_i' = \{x_i', \overline{\sigma}x_i', \ldots, \overline{\sigma}^{n_i-1}x_i'\}, \quad n_i = \deg \bar{f}_i,$$

then

$$X' = X_1' \cup X_2' \cup \cdots \cup X_m'$$

is a basis for $\overline{M}$ over $\overline{R}$.

Choose an arbitrary inverse image $x_i \in M$ of $x_i' \in \overline{M}$ by π_M, i.e., $\overline{x}_i = x_i'$ for $i = 1, 2, \ldots, m$. Set

$$X_i = \{x_i, \sigma x_i, \ldots, \sigma^{n_i-1}x_i\} \text{ for } i = 1, 2, \ldots, m$$

and

$$X = X_1 \cup X_2 \cup \cdots \cup X_m.$$

Then, since $\overline{X}_i = X_i$ for $i = 1, 2, \ldots, m$, we have $\overline{X} = X'$. Consequently X is a basis for M over R by the lemma.

Now we prove our theorems.

(a) Proof for Theorem A.

Let M_i be the submodule spanned by X_i for $i = 1, 2, ..., m$, and L_i be the submodule spanned by $X_i \backslash \{\sigma^{n_i - 1} x_i\}$ for $i = 1, 2, ..., m$. Then Theorem A is straightforward.

(b) Proof for Theorem B.

By (2.1) we have a set of polynomials $\mathcal{F} = \{f_1(t), f_2(t), ..., f_m(t)\}$ in $R[t]$. Before we begin the proof for Theorem B we want to observe that the constant term of the invariant $f_i(t)$ is a unit in R for $i = 1, 2, ..., m$. To do so we write for $i = 1, 2, ..., m$

$$f_i(t) = -a_0(i) - a_1(i)t - \cdots - a_{n_i - 1}(i)t^{n_i - 1} + t^{n_i}, \; a_h(i) \in R \tag{3.1}$$

for $h = 0, 1, ..., n_i - 1$. The companion matrix $C(f_i(t))$ of $f_i(t)$ is denoted by

$$C(f_i(t)) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \cdots & \\ 0 & 0 & 0 & \cdots & 1 \\ a_0(i) & a_1(i) & a_2(i) & \cdots & a_{n_i - 1}(i) \end{pmatrix} \tag{3.2}$$

and we set the block diagonal matrix A as

$$A = \begin{pmatrix} C(f_i(t)) & & & 0 \\ & C(f_2(t)) & & \\ & & \ddots & \\ 0 & & & C(f_m(t)) \end{pmatrix} \in M_n(R). \tag{3.3}$$

Then the matrix $\overline{A}$ of $\overline{\sigma}$ is

$$\overline{A} = \begin{pmatrix} C(\bar{f}_i(t)) & & & 0 \\ & C(\bar{f}_2(t)) & & \\ & & \ddots & \\ 0 & & & C(\bar{f}_m(t)) \end{pmatrix} \in M_n(\overline{R}), \tag{3.4}$$

where

$$C(\bar{f}_i(t)) = \begin{pmatrix} \bar{0} & \bar{1} & \bar{0} & \cdots & \bar{0} \\ \bar{0} & \bar{0} & \bar{1} & \cdots & \bar{0} \\ & & & \cdots & \\ \bar{0} & \bar{0} & \bar{0} & \cdots & \bar{1} \\ \bar{a}_0(i) & \bar{a}_1(i) & \bar{a}_2(i) & \cdots & \bar{a}_{n_i-1}(i) \end{pmatrix}. \tag{3.5}$$

This yields that

$$a_0(i) \in R^* \text{ (the unit group of } R) \tag{3.6}$$

for $i = 1, 2, ..., m,$ since $\bar{\sigma}$ is an automorphism and so $|\,\bar{A}\,| \neq 0.$

Now we return to our proof for Theorem B, first we treat (I) of the theorem. For $i = 1, 2, ..., m$ define $\rho_i \in \mathrm{Aut}_R M$ by

$$\rho_i = 1 \text{ on } X \setminus X_i$$

and on X_i by cyclic, i.e.,

$$\rho_i : x_i \to \sigma x_i \to \cdots \to \sigma^{n_i - 1} x_i \to x_i,$$

a cyclic permutation of length $n_i = \deg f_i$. Set $\rho = \rho_1 \rho_2 \cdots \rho_m$.

Then, for $i = 1, 2, ..., m$ we have

$$\rho^{-1}\sigma = 1 \text{ on } X_i \setminus \sigma^{n_i - 1} x_i \tag{3.7}$$

and by (3.4) and (3.5) it holds that

$$\begin{aligned} \sigma(\sigma^{n_i - 1} x_i) &= \sigma^{n_i} x_i \\ &= u_i + a_0(i) x_i + a_1(i) \sigma x_i + \cdots + a_{n_i - 2}(i) \sigma^{n_i - 2} x_i \\ &\quad + a_{n_i - 1}(i) \sigma^{n_i - 1} x_i + v_i, \end{aligned} \tag{3.8}$$

for $i = 1, 2, ..., m,$ where

$$u_i \in \langle X_1 \cup X_2 \cup \cdots \cup X_{i-1} \rangle_{\mathfrak{m}}$$

and

$$v_i \in \langle X_{i+1} \cup X_{i+2} \cup \cdots \cup X_m \rangle_{\mathfrak{m}}.$$

Hence,

$$\begin{aligned}\rho^{-1}\sigma(\sigma^{n_i-1}x_i) = \rho^{-1}\sigma^{n_i}x_i \\ &= \rho^{-1}u_i + a_1(i)x_i + a_2(i)\sigma x_i + \cdots + a_{n_i-1}(i)\sigma^{n_i-2}x_i \\ &\quad + a_0(i)\sigma^{n_i-1}x_i + \rho^{-1}v_i, \qquad (3.9)\end{aligned}$$

where

$$\rho^{-1}u_i \in \langle X_1 \cup X_2 \cup \cdots \cup X_{i-1} \rangle_{\mathfrak{m}},$$

$$\rho^{-1}v_i \in \langle X_{i+1} \cup X_{i+2} \cup \cdots \cup X_m \rangle_{\mathfrak{m}}$$

and

$a_0(i)$ is a unit of R by (3.6).

This implies that if we set

$$Y_i = \{x_i, \sigma x_i, ..., \sigma^{n_i-2}x_i, \rho^{-1}\sigma^{n_i}x_i\},$$

i.e., replacing the last element $\sigma^{n_i-1}x_i$ in X_i by $\rho^{-1}\sigma^{n_i}x_i,$ then we see that

$$S_j = Y_1 \cup Y_2 \cup \cdots \cup Y_{j-1} \cup X_j \cup \cdots \cup X_m$$

is also a basis for M over R for $j = 1, 2, ..., m,$ where we understand that

$$S_1 = X_1 \cup X_2 \cup \cdots \cup X_m, \text{ i.e., } Y_1 \cup Y_2 \cup \cdots \cup Y_{j-1} = \varnothing$$

if $j = 1.$

Define $\gamma_j \in \mathrm{Aut}_R M$ by

$$\gamma_j = 1 \text{ on } Y_1, \ldots, Y_{j-1} \cup X_j \setminus \{\sigma^{n_j - 1} x_j\} \cup X_{j+1} \cdots \cup X_m$$

and

$$\gamma_j \sigma^{n_j - 1} x_j = \rho^{-1} \sigma^{n_j} x_j.$$

Then γ_j is simple and

$$\gamma_1 : X = \{X_1, X_2, \ldots, X_m\} \to \{Y_1, X_2, \ldots, X_m\},$$

$$\gamma_2 : \{Y_1, X_2, \ldots, X_m\} \to \{Y_1, Y_2, X_3, \ldots, X_m\},$$

$$\cdots$$

$$\gamma_m : \{Y_1, Y_2, \ldots, Y_{m-1}, X_m\} \to \{Y_1, Y_2, \ldots, Y_{m-1}, Y_m\},$$

i.e.,

$$X = S_1 \overset{\gamma_1}{\to} S_2 \overset{\gamma_2}{\to} S_3 \cdots \overset{\gamma_m}{\to} S_m.$$

On the other hand, since

$$\rho^{-1}\sigma : \{X_1, X_2, \ldots, X_m\} \to \{Y_1, Y_2, \ldots, X_m\}, \text{ i.e., } X = S_1 \overset{\rho^{-1}\sigma}{\to} S_m,$$

we conclude that

$$\rho^{-1}\sigma = \gamma_m \cdots \gamma_2 \gamma_1$$

and so

$$\sigma = \rho \gamma_m \cdots \gamma_2 \gamma_1 = \rho_1 \rho_2 \cdots \rho_m \gamma_m \cdots \gamma_2 \gamma_1.$$

By a renumbering of $\gamma_1, \gamma_2, \ldots, \gamma_m$, we have (I).

Next we prove (II). Define $\delta \in \mathrm{End}_R M$ by D, a diagonal matrix in X such that

$$D = \mathrm{diag}(A(1), A(2), \ldots, A(m)) \text{ with } A(i) = \mathrm{diag}(1, 1, \ldots, 1, a_0(i))$$

for $i = 1, 2, \ldots, m$, where

(i) $A(i)$ is an $n_i \times n_i$ diagonal matrix,

(ii) $a_0(i)$ is that of (3.6) and

(iii) $a_0(i)$ is $(n(i), n(i))$-entry in D for $n(i) = n_1 + n_2 + \cdots + n_i$.

Since $a_0(i)$'s are all units in R by (3.6), we have δ in $\mathrm{Aut}_R M$ and by (3.7) and (3.9) we have

$$\delta^{-1}\rho^{-1}\sigma = 1 \text{ on } X_i \backslash \{\sigma^{n_i - 1}x_i\} \text{ for } i = 1, 2, \ldots, m, \tag{3.10}$$

and

$$\begin{aligned}\delta^{-1}\rho^{-1}\sigma(\sigma^{n_i-1}x_i) &= \delta^{-1}\rho^{-1}\sigma^{n_i}x_i \\ &= \delta^{-1}\rho^{-1}u_i + a_1(i)x_i + a_2(i)\sigma x_i + \cdots + a_{n_i-1}(i)\sigma^{n_i-2}x_i \\ &\quad + \sigma^{n_i-1}x_i + \delta^{-1}\rho^{-1}v_i,\end{aligned} \tag{3.11}$$

with

$$\rho^{-1}u_i \in \langle X_1 \cup X_2 \cup \cdots \cup X_{i-1}\rangle_{\mathfrak{m}},$$

$$\rho^{-1}v_i \in \langle X_{i+1} \cup X_{i+2} \cup \cdots \cup X_m\rangle_{\mathfrak{m}}.$$

Note that for $i = 1, 2, \ldots, m$ the coefficient $a_0(i)$ of $\sigma^{n_i-1}x_i$ is 1 in the above expression.

This enables us to define transvections $\tau_1, \tau_2, \ldots, \tau_m$ in $\mathrm{Aut}_R M$ by

$$\tau_j = 1 \text{ on } Y_1 \cup \cdots Y_{j-1} \cup X_j \backslash \{\sigma^{n_j-1}x_j\} \cup X_{j+1} \cup X_m$$

and

$$\tau_j(\sigma^{n_j-1}x_j) = \delta^{-1}\rho^{-1}\sigma^{n_j}x_j,$$

for $j = 1, 2, \ldots, n$ for which we have

$$\delta^{-1}\rho^{-1}\sigma = \tau_1\tau_2\cdots\tau_m.$$

Thus, we have obtained $\sigma = \rho\delta\tau_1\tau_2\cdots\tau_m$ as was to be shown for (II).

Finally we treat (III).

By (I) we have $\sigma = \rho\gamma_1\gamma_2\cdots\gamma_m$ with $\rho = \rho_1\rho_2\cdots\rho_m$. Since a cyclic permutation is a product of transpositions, we have on X_i

$$\rho_i = (x_i \ \sigma x_i \cdots \sigma^{n_i-1}x_i) = (x_i \ \sigma^{n_i-1}x_i)\cdots(x_i \ \sigma^2 x_i)(x_i \ \sigma x_i).$$

Further if 2 is a unit in R, then we have

$$Rx_i \oplus R\sigma^j x_i = Ru_{ij} \oplus Rv_{ij}$$

for

$$u_{ij} = x_i + \sigma^j x_i, \ \ v_{ij} = x_i - \sigma^j x_i$$

and the above transposition $(x_i \ \sigma^j x_i)$ acts

$$(x_i \ \sigma^j x_i) = 1 \text{ on } u_{ij} \text{ and } -1 \text{ on } v_{ij}$$

for $i = 1, 2, ..., m$ and $j = 1, 2, ..., n_i - 1$. Therefore, if we define symmetries $\sigma_{ij} \in \mathrm{Aut}_R M$ for $i = 1, 2, ..., m$ and $j = 1, 2, ..., m$ by

$$\sigma_{ij} = 1 \text{ on } \{u_{ij}\} \cup X \setminus \{x_i, \sigma^j x_i\}$$

and

$$\sigma_{ij} = -1 \text{ on } \{v_{ij}\},$$

we have

$$\sigma_{ij} = (x_i \ \sigma^j x_i) \text{ on } \{x_i, \sigma^j x_i\} \text{ and } \sigma_{ij} = 1 \text{ on } X \setminus \{x_i, \sigma^j x_i\}$$

which yields that

$$\begin{aligned}\rho &= \rho_1\rho_2\cdots\rho_m \\ &= (\sigma_{1(n_1-1)}\cdots\sigma_{12}\sigma_{11})(\sigma_{2(n_2-1)}\cdots\sigma_{22}\sigma_{21})\cdots(\sigma_{m(n_m-1)}\cdots\sigma_{m2}\sigma_{m1}).\end{aligned}$$

From this and by

$$(n_1 - 1) + (n_2 - 1) + \cdots (n_m - 1) = n_1 + n_2 + \cdots + n_m - m = n - m$$

we obtain (III), which completes our proof for Theorem B. □

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