On topological completeness of decorated exponential families

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Abstract

In this paper, we give two kind of proofs of topological completeness of decorated exponential families.

1 Introduction and main results

As stated in [5], the exponential family and the sine family are topological complete in the following sense.

Definition We say that a family of entire functions is *topologically complete* if any entire function topologically conjugate to an element of it is actually conformally conjugate to an element of it.

First, recall that topological completeness of the exponential family follows from the classical result as below.

Proposition 1 *Any holomorphic universal covering surface of* $\mathbf{C}_w - \{0\}$ *is conformally equivalent to the whole plane, say* $\mathbf{C}_\zeta$. *Moreover, the covering projection* π *is represented as*

$$w = \pi(\zeta) = e^{a\zeta+b}$$

with suitable a $(\neq 0)$ *and* b.

Similarly, since any element of the sine family is the projection of a holomorphic universal covering of the orbifold of $(0,3;2,2,\infty)$ type, the sine family is topologically complete.

But topological completeness seems to be known for few other families. Some exceptions are decorated exponential families. Such families has been getting more and more attentions. See [1], [4], [6] and [8].

In this paper, we give two proofs of the following

Theorem 2 *For any positive integers k and n, the doubly decorated exponential family*

$$\mathcal{F}_{k,n} = \{P(z)\exp Q(z)\},$$

where $P(z)$ and $Q(z)$ move over all polynomials of degree k and n, respectively, is topologically complete.

Remark The first proof of Theorem 2 in §2 is essentially due to Keen [3] and Stallard [7], and relies on quasiconformal maps. The second one in §3 is more elementary, and gives concrete information of the covering structure induced by an element of the family.

On the other hand, the additively decoreted exponential families such as considered in [8] are not topologically complete. We only state the following

Theorem 3 *The family*

$$\mathcal{F}_+ = \{z + a + e^z \mid a \in \mathbf{C}\}$$

is not topologically complete.

We prove Theorem 3 in §4 by giving an entire function quasiconformally conjugate to an element, but conformally conjugate to no one, of $\mathcal{F}_+$.

2 The first proof of Theorem 2

We begin with the following

Lemma 4 *Let g be topologically conjugate to some $f(z) = P(z)\exp Q(z) \in \mathcal{F}_{k,n}$. Then g is quasiconformally equivalent to f, namely, there are quasiconformal maps ψ_1 and ψ_2 of $\mathbf{C}$ such that*

$$\psi_2 \circ f = g \circ \psi_1.$$

Proof. Let $\varphi(z)$ be a homeomorphism of C onto itself such that $\varphi \circ f = g \circ \varphi$, and E_f be the set of all singular values of f. Since E_f is a finite set, there is an isotopy Φ relative to E_f which connects φ with a quasiconformal map ψ_2.

By the assumption, we can lift Φ to an isotopy $\tilde{\Phi}$ such that $g \circ \tilde{\Phi} = \Phi \circ f$. And $\tilde{\Phi}$ connects φ with a quasiconformal map ψ_1 satisfying $\psi_2 \circ f = g \circ \psi_1$. ∎

The first proof of Theorem 2. We may assume that $\varphi(0) = 0$. Then since $f^{-1}(0)$ and $g^{-1}(0)$ have the same number of points counting multiplicity, we find a polynomial $R(z)$ of degree k such that $g(z)/R(z)$ has no zeros. Hence we can write $g(z)$ as $R(z) \exp h(z)$ with an entire function $h(z)$.

Since quasiconformal maps are Hölder continuous, there are some $K > 1$ and $A > 1$ such that

$$A^{-1}|z|^{1/K} \leq |\psi_j(z)| \leq A|z|^K$$

for each j and every z with sufficeintly large $|z|$. Hence on $\{|z| = r\}$ with large r, we have

$$|g(z)| = |\psi_2 \circ f \circ \psi_1^{-1}(z)| \leq A|M(f, A^K r^K)|^K,$$

where $M(f,r) = \max_{|z|=r} |f(z)|$. Here since $f \in \mathcal{F}_{k,n}$, $\log |M(f, A^K r^K)|^K$ has a polynomial growth order with respect to r. Hence there are some C and N such that

$$|\mathrm{Re}\ h(z)| \leq Cr^N$$

for every z with sufficeintly large $|z|$, which implies that $h(z)$ is a polynomial.

Finally, let n' be the degree of $h(z)$. Then $g(z)$ has exaxtly $k + n' - 1$ critical points including multiplicity, which should equals to $k + n - 1$. Thus we have $n' = n$. ∎

3 The second proof of Theorem 2

We give here a more elementary proof of Theorem 2. For this purpose, fix an entire function g which is topologically equivalent to an element f in $\mathcal{F}_{k,n}$. Let $\varphi(z)$ be a homeomorphism of C onto itself such that $\varphi \circ f = g \circ \varphi$, and we again assume that $\varphi(0) = 0$. For the sake of simplicity, we explain only the case that $Q(z) = z$ and $f(z) = P(z)e^z$ has exactly k non-zero critical values, and let $\mathcal{F}_k$ be the set of all such $P(z)e^z$. Other cases can be treated similarly.

First, we apply a cut-paste surgery to f; namely, let $\{a_j\}_{j=1}^k$ be critical values of f, and take mutually disjoint smooth arcs $\{L_j\}_{j=1}^k$ on $\mathbf{C}-\{0\}$ starting from a_j and tending to ∞. Consider f as a (branched and incomplete) covering of the target $\mathbf{C}_w$ of f by the domain $\mathbf{C}_z$ of f. (So all a_j and L_j are on $\mathbf{C}_w$.) Cut $\mathbf{C}_z$ along all lifts of L_j with respect to this covering. Then $\mathbf{C}_z$ is decomposed into $k+1$ components, say $\{R_1, \cdots, R_k, S\}$. Each component can be re-pasted over L_j, and we have $k+1$ Riemann surfaces spreading holomorphically over $\mathbf{C}_w$. Here, k of them corresponding to $\{R_1, \cdots, R_k\}$ are holomorphic universal covering surfaces of $\mathbf{C}_w$, denoted by $\{\hat{R}_1, \cdots, \hat{R}_k\}$, while the remaining one, say $\hat{S}$, is a holomorphic universal covering surface of $\mathbf{C}_w - \{0\}$.

As noted above, f restricted on S considered as a subset of $\hat{S}$, which in turn we denote as $\mathbf{C}_\zeta$, can be expressed as

$$f|_S(\zeta) = e^{a\zeta+b}$$

with suitable a and b.

Definition We call S the *exponential part* of the covering surface $\mathbf{C}_z$, and each R_k a $\mathbf{C}$-*decoration.*

Conversely, such decoration structure characterizes the family $\mathcal{F}_k$. See Proposition 9.

We begin with the definitions. Let S^0 be a universal covering surface of $\mathbf{C}_w - \{0\}$. We decorate k copies R_j^0 of the whole plane inductively as follows: take mutually disjoint k arcs $\{L_j^0\}$ starting from a point in $\mathbf{C}_w - \{0\}$ and tending to ∞. Fix a lift of L_1^0 on S^0 and decorate R_1^0 by connecting crosswise along this lift. Then, we have a branched incomplete covering of $\mathbf{C}_w$. Next we fix a lift of L_2^0 to this new covering surface, and decorate R_2^0 along the lift. Repeating this process, we have a branched and incomplete covering surface D of the whole plane.

Definition We call D a *k-th decorated exponential covering surface*, and the projection π to $\mathbf{C}_w$ a *k-th decorated exponential function.*

The domain D of π is clearly simply connected. Moreover, the following fact is essentially well-known.

Lemma 5 *Every k-th decorated exponential function π can be regarded as an entire function.*

Proof. For the sake of convenience, we include a standard proof. Roughly speaking, on the neighborhood of the infinity, the covering structure is the logarithmic one, and also so does for the non-compact component of the covering surface restricted on a neighborhood of 0. Thue we can conctruct a ring domain with atrbitrary large modulus as follows.

First fix $A < (1 <) B$ so that $\{A < |w| < B\}$ contains all critical values. Fix a positive N so large that each crirical points are contained in some p-th sheet with $|p| \leq N$ (under a suitable choice of the 0-th sheet of S^0). Take a sufficiently large n $(> N)$. When $|p| \leq N$, then let W_p be the part of the p-th sheet of S^0 over

$$\{A/n < |w| < A\} \cup \{B < |w| < Bn\},$$

possibly with the part of R_j^0 decorated to the p-th sheet over $\{B < |w| < Bn\}$. When $|p| > N$, let W_p be the part of the p-th sheet over

$$\{A/n < |w| < Bn\}.$$

Then the union $W = \cup_{p=-n}^{n} W_p$ can be considered as a doubly conneted region on the domain D of π. Define the conformal density ρ on W as follows; on W_p with $|p| \leq N$, we set ρ as the pull-back of $|dw|/|w|$ by π, and on any other W_p, we set ρ as the pull-back of $|dw|/(p|w|)$. Then, we can see that the ρ-length of any arc connecting boundary components is greater than $C \log n$ with a positive C independent of n. Since the ρ-area of W is

$$4(2N + k + 1)\pi \log n + 4\pi \sum_{p=N+1}^{n} \frac{2\log n + \log(B/A)}{p^2},$$

we can conclude that the modulus of W is greater than $C' \log n$ with a positive C' independent of n, which shows that the domain D of π is conformally equivalent to the whole plane. ■

Now, the covering structure of π depends on the choice of the isotopy class of the arcs $\{L_j^0\}$ relative to end points, and the configuration tree induced from the order of the C-decorations.

Definition The *configuration tree* for a k-th decoreated exponential function consists of vertices $\{r_1, \cdots, r_k, n \in \mathbf{Z}\}$, where each r_j represents R_j^0 and each n represents the n-th sheet of S^0 and they are connected by sides which are given by the following injunctions;

1. Add one side connecting n with $n+1$ for every n.

2. Add one side connecting r_1 with 0 (, which determines the 0-th sheet of S^0).

3. Add one side connecting r_2 with r_1 if R_2^0 is connected to R_1^0. And if not, some sheet, say n_2-th sheet, of S^0 is connected with R_2^0. Then Add one side connecting r_2 with n_2.

4. Add one side connecting r_j with one of $\{r_1, \cdots, r_{j-1},\ n \in \mathbf{Z}\}$ by similar injunctions as in (3), for every $j > 2$.

The configuration tree determines a k-th decorated exponential function in the following sence.

Lemma 6 *Suppose that two k-th decorated exponential function h_1 and h_2 has the same singular values and the same configuration tree (with respect to the same cut arcs), then there is a conformal automorphism A of the whole plane such that*

$$h_1 = h_2 \circ A.$$

Proof. Let S^m be the exponential part and $\{R_j^m\}$ the $\mathbf{C}$-decorations for each $m = 1, 2$. Then there is a conformal map $A_0 : \hat{S}^1 \to \hat{S}^2$ such that the branch point for $\hat{R}_1^1$ is sent to that for $\hat{R}_1^2$ and $\hat{\pi}_2 \circ A_0 \circ \hat{\pi}_1^{-1}$ is the identitical automorphism of $\mathbf{C}^*$, where $\hat{\pi}_m$ is the natural projection of $\hat{S}^m$. Then A_0 can be extended to a biholomorphic maps from $\hat{S}^1$ decorated with $\hat{R}_1^1$ to $\hat{S}^2$ decorated with $\hat{R}_1^2$, which should send the branch point for $\hat{R}_2^1$ to that for $\hat{R}_2^2$. Repeating this, we have an affine map A as desired. ∎

On the other hand, every k-th decorated exponential covering surfaces can be realized by an element of $\mathcal{F}_k$. To show this, first we recall that critical values moves biholomorphically with respect to coefficients of monic polynomials in the following sense.

Lemma 7 *Let $P_0(z)e^z$ be an element of $\mathcal{F}_k$ with a monic polynomial*

$$P_0(z) = z^k + c_k^0 z^{k-1} + \cdots + c_1^0$$

and mutually different non-zero critical values $\mathbf{a}_0 = (a_1^0, \cdots, a_k^0)$. Then the map from $\mathbf{c} = (c_1, \cdots, c_k)$ to $\mathbf{a} = (a_1, \cdots, a_k)$ is biholomorphic in a neighborhood of $\mathbf{c}_0 = (c_1^0, \cdots, c_k^0)$.

Proof. First, since critical points $\mathbf{p} = (p_1, \cdots, p_k)$ of $P_0(z)e^z$ are zeros of $P(z) + P'(z)$, $\mathbf{p}$ is clearly a holomorphic function of $\mathbf{c}$, and

$$(P + P')'(p_j)\,\frac{\partial p_j}{\partial c_\ell} = -p_j^{\ell-1} - (\ell - 1)p_j^{\ell-2}.$$

Here by the assumption, $(P + P')'(p_j)$ is non-zero for every j, and hence the Jacobian is non-degenerate.

Next, since $\mathbf{a} = (P(p_1)e^{p_1}, \cdots, P(p_k)e^{p_k})$, we have

$$\frac{\partial a_j}{\partial c_\ell} = \left(P'(p_j)\frac{\partial p_j}{\partial c_\ell} + p_j^{\ell-1}\right) e^{p_j} + P(p_j)e^{p_j}\frac{\partial p_j}{\partial c_\ell} = p_j^{\ell-1}e^{p_j}.$$

Hence, we conclude the assertion. ■

Actually, for every point $\mathbf{p} = (p_1, \cdots, p_k)$ in the region

$$\Omega = \{\mathbf{p} = (p_1, \cdots, p_k) \in \mathbf{C}^k \mid \prod_{i \neq j}(p_i - p_j) \neq 0\},$$

it is easily to give an element of $\mathcal{F}_k$ with monic polynomial which has the critical points $\mathbf{p}$. And we have shown that the map Φ from Ω into $\mathbf{C}^k$, which sends $\mathbf{p}$ to critical values $\mathbf{a}$ of the corrsponding element, is locally biholomorphic.

So set

$$\Omega' = \{\mathbf{a} = (a_1, \cdots, a_k) \in \mathbf{C}^k \mid a_1 \cdots a_k \prod_{i \neq j}(a_i - a_j) \neq 0\},$$

and consider to take lifts of arcs in Ω' starting from $\mathbf{a}_0$ with respect to Φ, where $\mathbf{a}_0$ is as in Lemma 7. And we have the following

Lemma 8 *The germ of Φ^{-1} which sends $\mathbf{a}_0$ to $\mathbf{p}_0$ can be continued analytically along any arc in Ω' starting from $\mathbf{a}_0$, where $\mathbf{p}_0$ corresponds to $\mathbf{c}_0$ in Lemma 7.*

Proof. Suppose that there were an arc $\gamma : [0,1] \to \Omega'$ starting from $\mathbf{a}_0$ which can not be lifted into Ω from $\mathbf{p}_0$ with respect to Φ. Further, we may assume that $\gamma : [0,r] \to \Omega'$ has such a lift for every $r < 1$.

If the end of the lift of $\gamma : [0,1) \to \Omega'$ from $\mathbf{p}_0$ accumulates to a point of Ω, then by the fact noted before Lemma 8, we would have a lift of the whole given arc γ. So the end should diverge to the boundary. Here since γ is in Ω', the end can not tend to the relative boundary of Ω in $\mathbf{C}^k$. Hence it diverges to the infinity. But then, γ should end at $\{a_1 \cdots a_k = 0\}$ or diverge to the infinity, which is impossible. ∎

Thus perturbation of critical values can be realized by elements of $\mathcal{F}_k$ with monic polynomials, and hence we can deform a k-th decorated exponential function within the family $\mathcal{F}_k$ rather freely. Thus we have the following

Proposition 9 *The family $\mathcal{F}_k$ contains all k-th decorated exponential functions modulo conformal conjugation.*

Proof. Fix an element h_0 of $\mathcal{F}_k$ with a monic polynomial and a k-th decorated exponential function π_0 arbitrarily. First, by starting from h_0 and moving critical values of h_0 so that they are always mutually distinct, we obtain by Lemma 8 an element in $\mathcal{F}_k$ whose critical values is just the same as those of π_0 including the order.

When the configuration trees are not the same, we deform it by moving each critical value around another critical value or 0. Then again by Lemma 8, applying finite number of such deformations along suitable loops, we can get an element h of $\mathcal{F}_k$ which has the same configuration tree as that of π_0.

Then by Lemma 6, we can lift the identical map between the target planes of the exponential parts to a conformal map A of the domain of π_0 such that

$$\pi_0 \circ A = h.$$

This implies that π_0 is conformally conjugate to an element of $\mathcal{F}_k$. ∎

The second proof of Theorem 2. Now back to g in the beginning of this section, it is clear that g gives a k-th decorated exponential covering surface, for g is topologically conjugate to f. Hence Proposition 9 gives the assertion. ∎

4 A proof of Theorem 3

To show Theorem 3, we take a_0 so that each critical values of $f(z) = z+a_0+e^z$ belongs to mutually distinct attractive basins for fixed points of f. For instance, take $1/2$ as a_0.

The grand orbits of such attractive basins are mutually disjoint. The crutial fact is the following

Proposition 10 *For every $g \in \mathcal{F}_+$, the complex dynamics of g is invariant under conjugation by the translation $T(z) = z + 2\pi i$:*

$$g = T \circ g \circ T^{-1}.$$

Proof of Theorem 3. Fix the grand orbit of an attractive basin D of f, and denote it by $[D]$. (See [2] and [8] for terminologies and fundamental facts.) For an invariant Beltrami coefficeint μ supported on $[D]$, let ϕ_μ be a quasiconformal map of $\mathbf{C}$ with the Beltrami coefficient μ on $[D]$ and 0 on $\mathbf{C} - [D]$. Then $g = \phi \circ f \circ \phi^{-1}$ with $\phi = \phi_\mu$ is again an entire function and $\phi([D])$ is the grand orbit of an attractive basin of a fixed point of g. Since the Teichmüller space of $R = [D]/f$ is non-trivial, we can choose μ so that $\phi[D]$ gives a once-punctured torus not conformally equivalent to R.

Here the grand orbit of any other attractive basin of g still gives the same R. Thus we conclude that g does not belong to $\mathcal{F}_+$ by Proposition 10. ■

Remark Actually, we have shown that $\mathcal{F}_+$ is not quasiconformally complete.

Also note that, when we start with

$$f(z) = z + 1 + e^z,$$

then every critical point of f determines a superattractive basin. And we can not deform f quasiconformally on the grand orbits of such basins. The situation is similar when we start with

$$z + e^z,$$

which has infinitely many Baker domains (cf. [8]).

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