Dynamics and kneading sequences of skew tent maps

Keiko ICHIMURA
Department of Information Sciences, Ochanomizu University
2-1-1 Otsuka, Bunkyo-ku, Tokyo, 112 Japan

Abstract

We study the dynamics of the two–parameter family of skew tent maps, namely how the change of the behavior of the map is depending on parameters. So we decompose the parameter space by the nature of periodic points. Especially we prove that the form of attracting orbit is of stair type and give an explicit proof of the monotonicity of kneading sequence.

1 Introduction

Dynamics of the family of skew tent maps has been studied in, for example, [ITN79] and [MV91].

According to J. Milnor[Mil85], "attractor" is defined as the set which attracts almost all orbits in the neighborhood. To say precicely, the set A is called *attractor* of f when it satisfies the following four conditions:

(1) A is a closed set.

(2) $f(A) = A$.

(3) The set A has a neighborhood U such that $\lim_{n\to\infty} f^n(x) \in A$ for almost all $x \in U$.

(4) There exists a point $x \in A$ whose orbit is dense in A.

If attractor A is a set of k points, then each point in A is mapped to itself under f^k and called *attracting k–periodic point.*

Suppose that x is a k–periodic point of f and f is differentiable on the neighborhood of orbit of x. If $|(f^k)'(x)| < 1$, then x is attractive.

According to R. L. Devaney [Dev89], the dynamics of f defined on X is said to be *chaotic* if f satisfies the following:

(1) f has *sensitive dependance on the initial condition*; there exists $\delta > 0$ such that, for any $x \in X$ and any neighborhood U_x, there exists $y \in U_x$ and $n \geq 0$ such that $|f^n(x) - f^n(y)| > \delta$.

(2) f is topologically transitive; for any pair of open sets $U, V \subset X$ there exists $n \geq 0$ such that $f^n U \cap V \neq \phi$.

(3) periodic points are dense in X.

For two maps $f : V \to V$ and $g : U \to U$, if there exists a homeomorphism h such that $h \circ f = g \circ h$, then f and g are said to be topologically conjugate, denoted by $f \sim g$. The map h is called *topologicall conjugacy*.

For the next family of *skew tent maps* on **R**, a quite different bifurcation occurs, compared with the one which occurs for the logistic family $g_u(x) = ux(1-x)$ with $3 \leq u \leq 4$.

$$F_\mu(x) = \begin{cases} ax + \mu, & \text{for } x \leq 0 \\ -bx + \mu, & \text{for } x \geq 0 \end{cases}$$

where $0 < a < 1,\ b > 1$.

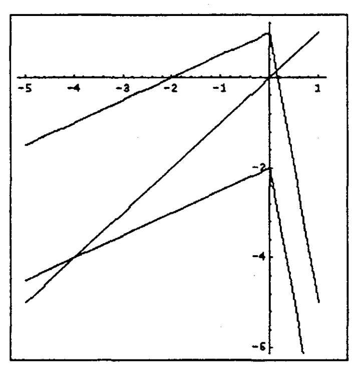

Figure 1: The graphs of F_μ for $\mu = 1,\ -2$, where $a = \frac{1}{2}$, $b = 6$.

First let us fix a, b and think μ as the parameter. Then a bifurcation occurs at $x = 0$ and $\mu = 0$ (see [NY95]). If we take $a = \frac{1}{2}$ and $b = 3.2$ (resp. $4.14, 4.42$ or 5.5), bifurcation diagrams are given in from Fig.2 to Fig. 5. In Fig.2, the family exibits a bifurcation from an attracting fixed point to attracting 3–periodic points. In each Fig.3, 4, 5, the family exhibits a bifurcation from an attracting fixed point to attracting intervals on which F_μ is chaotic. From these figures, we can see that there are various types with respect to what bifurcation occurs at $\mu = 0$. On the other hand, for any $\mu > 0$, there does not occur new bifurcation as long as the parameter pair (a, b) is fixed. It is because $F_\mu \sim F_1$ for all $\mu > 0$.

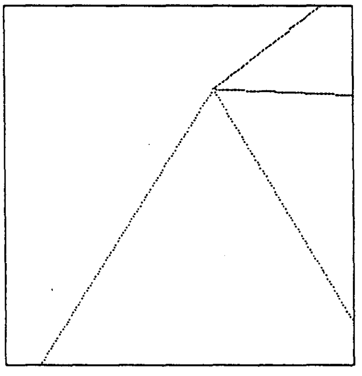

Figure 2: $a = \frac{1}{2}$, $b = 3.2$. F_μ exhibits bifurcation from an attracting fixed point to attracting 3–periodic points.

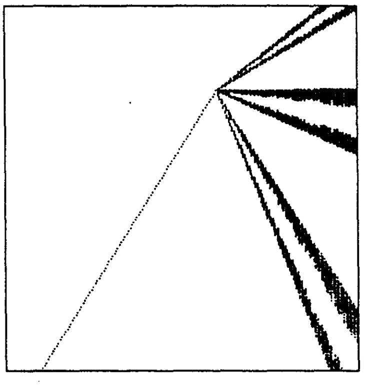

Figure 3: $a = \frac{1}{2}$, $b = 4.14$. F_μ exhibits bifurcation from an attracting fixed point to six attracting intervals. F_μ is chaotic on those intervals.

In this paper, we will study the characterization of an attracting periodic orbit of skew tent map. After this, for example, we will see when the bifurcation like as Fig. 2 occurs.

We remark that putting the results in the sequel paper [ITO] in this volume together, we can completely obtain the partition of D in terms of the characters of dynamics of $f_{a,b}$.

For our purpose, we have only to fix $\mu = 1$ and analyze the dynamics of $f_{a,b}$ with the following parameter plane D:

$$D = \{(a,b);\ a > 0,\ b > 1,\ a + b \geq ab\}.$$

$$f_{a,b}(x) = \begin{cases} ax + 1, & \text{for } x \leq 0 \\ -bx + 1, & \text{for } x \geq 0 \end{cases}, \text{where } (a,b) \in D.$$

We expand the range of the parameter a from $\{0 < a < 1\}$ to $\{a > 0\}$. If $a + b < ab$, almost all orbits tend to $-\infty$; if $a + b \geq ab$, then the orbit of x in $\mathbf{R} \setminus I_{a,b}$ tends to either the interval $[f_{a,b}^2(0),\ f_{a,b}(0)]$ or $-\infty$, or x is a fixed point itself. Therefore the restriction $\{a + b \geq ab\}$ is reasonable.

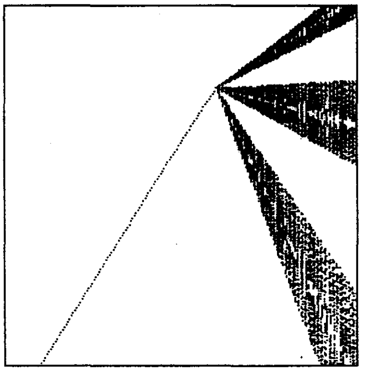

Figure 4: $a = \frac{1}{2}$, $b = 4.42$. F_μ exhibits bifurcation from an attracting fixed point to three attracting intervals. F_μ is chaotic on those intervals.

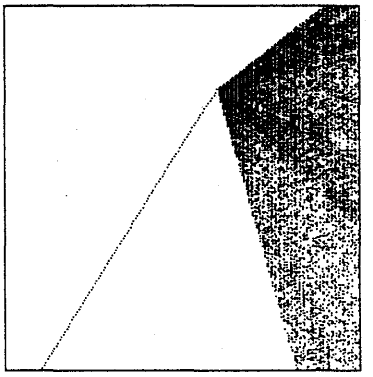

Figure 5: $a = \frac{1}{2}$, $b = 5.5$. F_μ exhibits bifurcation from an attracting fixed point to one attracting interval. F_μ is chaotic on the interval.

Hence we find that it is sufficient to examine the dynamics of $f_{a,b}$ on $I_{a,b}$, where we denote $I_{a,b} = [f_{a,b}^2(0),\ f_{a,b}(0)]$.

2 Attracting region of D

In this section, we shall characterize the regions where $f_{a,b}$ has an attracting orbit.

For $k \geq 2$, Set

$$\begin{aligned} D_k &= \left\{(a,b) \in D;\ 1 + \frac{1}{a} + \cdots + \frac{1}{a^{k-2}} < b \leq 1 + \frac{1}{a} + \cdots + \frac{1}{a^{k-1}}\right\}, \\ D_k^A &= \left\{(a,b) \in D_k;\ a^{k-1}b \leq 1\right\}. \end{aligned}$$

Proposition 1 A boundary of D_k^A is determined by algebraic curves. These curves have the following defining equations:

$$a^{k-1}b = 1, \quad b = 1 + \frac{1}{a} + \cdots + \frac{1}{a^{k-2}}.$$

These curves intersect at only one point P_k, and $\lim_{k\to\infty} P_k = (\frac{1}{2},\ \infty)$.

Proof. The equation $a^{k-1} + \cdots + a = 1$ has a unique positive root, and so P_k is unique. Obviously, it is smaller than 1. Since $1 + \dfrac{1}{2} + \dfrac{1}{2^2} + \cdots = 1$, the root converges to $a = \frac{1}{2}$ as $k \to \infty$. And the value b diverges to ∞. □

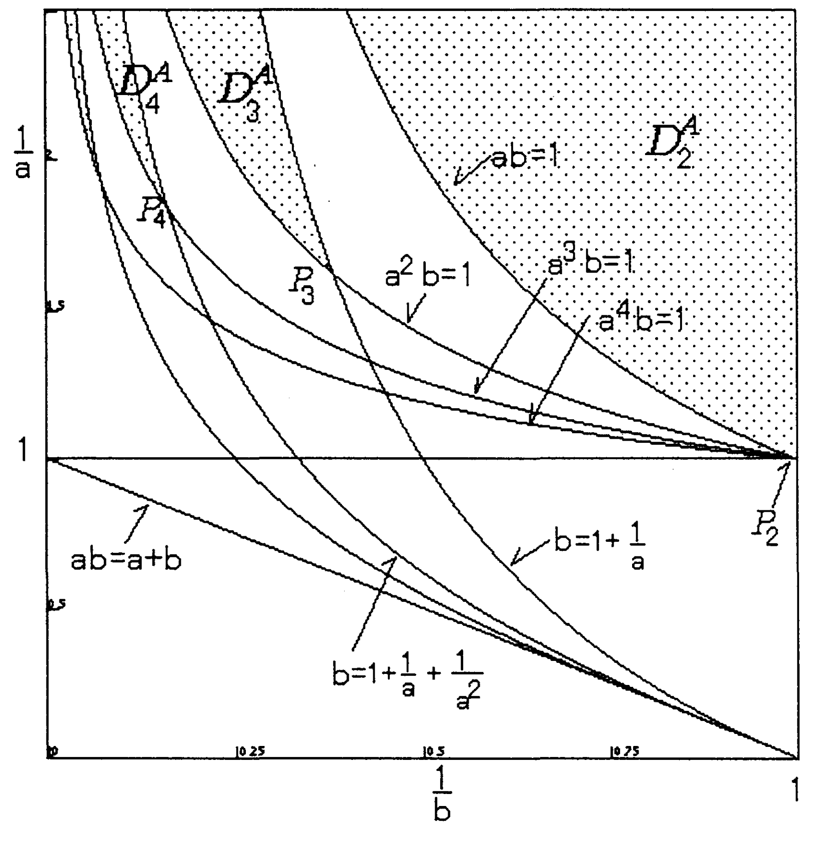

Figure 6: The domains D_k^A and boundary curves

For our purpose, we refer some results from the paper [ITN79].

First, let us prepare the following intervals and subdomains of D.

$$I_L = [f^2(0), 0],\ I_R = [0, f(0)],$$
$$D_0 = \{(a,b) \in D;\ ab > 1,\ a + b \geq ab^2\},$$
$$D_1 = \{(a,b) \in D;\ a + b < ab^2,\ b < 1 + \frac{1}{a}\}.$$

For $k \geq 3$,

$$D_k^B = \{(a,b) \in D_k;\ a^{k-1}b > 1, a+b \geq a^{k-1}b^2\},$$
$$D_k^\star = D_k \setminus (D_k^A \cup D_k^B),$$
$$D^\star = \{(a,b) \in D;\ a > 1, a+b < ab^2\}.$$

In the last two equations above, we note that $D_k^B = D_0$ and $D_k^\star = D_1$ for $k = 2$.

In the following Fact 1 and Fact 2, we suppose that (a, b) belongs in the interior of D_k for $k \geq 3$.

Fact 1 Set

$$x_\star = \frac{1-a-b+a^{k-1}b}{(1-a)(1+a^{k-1}b)}, \quad x^\star = \frac{-a^{k-2}b^2+b^2-b+ab+1-a}{(1-a)(1-a^{k-2}b^2)},$$
$$x_0 = \frac{-1+a^{k-2}}{a^{k-2}(1-a)}, \quad \text{and} \quad I_0 = [f_{a,b}^2(0),\ x^\star].$$

Then the point x_0 is mapped to $x = 0$ under $f_{a,b}^{k-2}$, and both $x_\star$ and $x^\star$ are k-periodic points of $f_{a,b}$ with $f_{a,b}^2(0) < x_\star < x_0 < x^\star$. Moreover, if (a, b) is in $D \setminus D_k^\star$, then almost all points in $I_{a,b}$ tend to the interval I_0 under iteration of $f_{a,b}$.

Fact 2 In order to analyze the dynamics of $f_{a,b}^k|_{I_0}$, a function $g_{\alpha,\beta}$ is introduced as follows:

$$g_{\alpha,\beta}(x) = \begin{cases} \alpha x, & \text{for } 0 \leq x \leq \frac{1}{\alpha} \\ -\beta x + \frac{\alpha+\beta}{\alpha}, & \text{for } \frac{1}{\alpha} \leq x \leq 1 \end{cases}$$

where $(\alpha,\ \beta)$ such that $\alpha > 1$, $\beta > 0$ and $\dfrac{1}{\alpha} + \dfrac{1}{\beta} \geq 1$. As I_0 is invariant under $f_{a,b}^k$, $f_{a,b}^k|_{I_0}$ is topologically conjugate to $g_{\alpha,\beta}$ with some $\beta < 1$. If $\beta < 1$, then the orbits of almost all points in $[0, 1]$ tend to the fixed point $x = \frac{\alpha+\beta}{\alpha(\beta+1)}$ under the iteration of $g_{\alpha,\beta}$. The attracting fixed point for $g_{\alpha,\beta}$ corresponds to $x_\star$ for $f_{a,b}^k|_{I_0}$. Therefore almost all orbits in I_0 under $f_{a,b}^k$ tend to $x_\star$.

According to J. Milnor and W. Thurston in [MT88], we will define *turning point* as follows.

Let f be a continuous map on the interval $I = [c_0, c_l]$. We suppose that I is decomposed into finite maximal intervals, $J_1 = [c_0, c_1], \cdots,\ J_l = [c_{l-1}, c_l]$

with $c_0 < c_1 < \cdots < c_l$, and each restriction $f|_{J_j}$ is monotone map. Then these endpoints $c_0, c_1, \cdots, c_l$ will be called *turning points* of f.

We obtain the following result for the turning point $x = 0$ of the skew tent maps. This result is quite different from the case of differentiable map.

Proposition 2 The periodic turning point of $f_{a,b}$ is not attractive.

Proof. Put $f_{a,b} = f$. If the turning point of f^j for some $j \geq 1$ is a fixed point, then the orbit of $x = 0$ and $f^2(0)$ also have the period j. For each region in D, we will prove the claim.

- $(a, b) \in D_0$

  There exist subintervals of $I_{a,b}$ $L_0 = [f^2(0), f^4(0)]$, $L_1 = [f^3(0), f(0)]$ such that $L_0 \cap L_1 = \emptyset$, $fL_0 = L_1$ and $fL_1 = L_0$. There is no attracting periodic point in the interval $(f^4(0), f^3(0))$ since each orbit in $I_{a,b} \setminus \{1 - \frac{1}{b}\}$ tends to L_0 or L_1. If an attracting periodic point exists in $L_0 \cup L_1$, it must have even period as L_0 and L_1 are mapped to each other under f. Let us suppose the orbit of $x = 0$ has $2j$–period. Then on any differentiable point x in $L_0 \cup L_1$, $|(f^{2j})'(x)| \geq (ab)^j > 1$ since $(a, b) \in D_0$ and $fL_0 \subset I_R$. Hence even if $x = 0$ is a periodic point, it is not attractive. The graph of f^{2j} on the neighborhood of $x = 0$ is like as Fig.7 or Fig. 8.

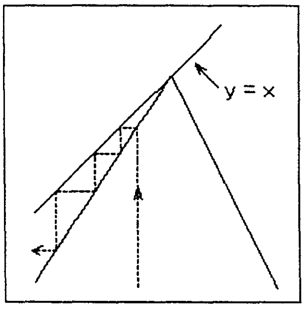

Figure 7:

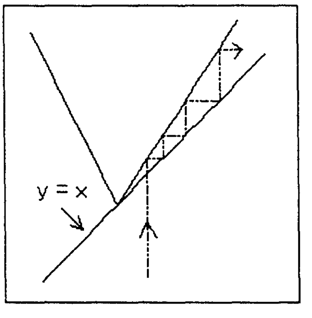

Figure 8:

- $(a, b) \in D_1$

  If f has j–periodic orbits, then we have $j \geq 2$ and $|(f^j)'(x)| > 1$ for the point x on which f is differentiable, since $fI_L \subset I_R$ and $b > 1$. Hence even if $x = 0$ is a periodic point, it is not attractive. The graph of f^j on the neighborhood of $x = 0$ is like as Fig. 7 or Fig. 8.

- $(a,b) \in D_k^B$ for $k \geq 3$

  Recall that there exists k–periodic point of f $x^\star$ and the interval $C_0 = [f^2(0), x^\star]$ is f^k–invariant. Therefore the orbit of $f^2(0)$ has a period with the form $j \times k$ for some $j \geq 0$, so does the turning point 0. Because, for any orbit, at most $k-1$ succesive images can stay on the interval I_L, $|(f^{kj})'(x)| \geq (a^{k-1}b)^j > 1$ for the point x on which f is differentiable. Hence even if $x = 0$ is a periodic point, it is not attractive. The graph of f^{jk} is like as Fig. 7 or Fig. 8.

- $(a,b) \in D_k^\star$ for $k \geq 3$

  It is easy to see that $[f^2(0), x^\star]$ is not invariant. Hence $f^2(0)$ is j–periodic point with $j \geq k$. Because, for any orbit, at most $k-1$ succesive images can stay on the interval I_L, $|(f^{kj})'(x)| \geq (a^{k-1}b)^j > 1$ for the point x on which f is differentiable. Hence even if $x = 0$ is a periodic point, it is not attractive. The graph of f^{jk} is like as Fig. 7 or Fig.8.

- $(a,b) \in D^\star$

  Since $a > 1$ and $b > 1$, the graph on the neighborhood of *turning point* $x = 0$ is like as Fig. 7 or Fig. 8. Therefore x is not attractive.

- $(a,b) \in D_k^A$ and $a^{k-1}b \neq 1$

  For $k \geq 3$, if $a^{k-1}b \neq 1$, the orbit of $f^2(0)$ under f^k is attracted to $x_\star$ by Fact 1. This contradicts that turning point is periodic.

  For $k = 2$, it is easy to check by setting $x_\star = \dfrac{1-b}{1+ab}$, that $x_\star$ is 2–periodic point and that $f^2(0)$ tend to $x_\star$ under $f^2_{a,b}$. Therefore turning point cannnot be attractive and periodic.

- $(a,b) \in D_k^A$ and $a^{k-1}b = 1$

  For both case of $k = 2$ and $k \geq 3$, the orbit of $x = 0$ has period $2k$. But the graph of f^{2k} is like as Fig. 9 or Fig. 10, it is not attractive in the sense of [Mil85].

□

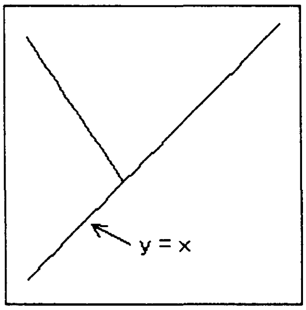

Figure 9:

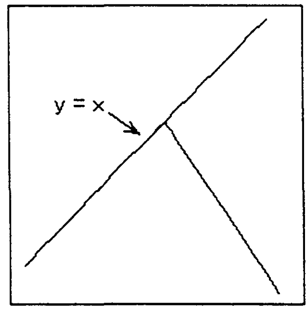

Figure 10:

Definition The periodic orbit will be called *of stair type* when it satisfies the following conditions:

$$\begin{aligned} O(x) &= \left\{x, f(x), \cdots, f^{k-1}(x)\right\}, \\ x\ (= f^k(x)) &< f(x) < \cdots < f^{k-2}(x) \leq 0 < f^{k-1}(x). \end{aligned}$$

We remark that the following result is implicitly stated in [ITN79].

Proposition 3 Skew tent map $f_{a,b}$ has attractive periodic orbit if and only if (a,b) is in the interior of D_k^A.

Proof. If (a, b) is in the interior of D_k^A for some $k \geq 3$, then $f_{a,b}$ has an attractive k–periodic point $x_\star$ by Fact 2. If (a, b) is in D_2^A, by setting $x_\star = \dfrac{1-b}{1+ab}$, it is proved that $x_\star$ is 2–periodic point and that almost all points in $I_{a,b}$ tend to $x_\star$ under $f_{a,b}^2$. And the orbit is of stair type.

Now we prove that only if (a, b) is in the interior of D_k^A, $f_{a,b}$ has an attracting k–periodic point. By Proposition 2, the orbit of turning point is not attractive. Therefore in neighborhood of the periodic orbit, there exists an open interval where the itinerary is same. This cannot happen if $(a, b) \notin \cup_{k=2}^{\infty} D_k^A$ by virtue of Lemma 2.1 in [ITN79]. Hence if $f_{a,b}$ has an attracting periodic orbit, there exists unique k such that $(a, b) \in D_k^A$. On the other hand, for (a, b) on the boundary of D_k^A, there is no attractor in the sense of [Mil85] (see Fig. 9 and Fig. 10). □

For the case of differentiable map with one critical point, the following result is well known by using the method of Schwarzian derivative (see for example, [Dev89], p.74). But despite of the indefferentiablity, it is worthy of notice that we have the same result.

Lemma 1 $f_{a,b}$ has at most one attracting periodic orbit.

Proof. Suppose that $f_{a,b}$ has attracting k–periodic orbit. By Proposition 3, it is the orbit of $x_\star$. By considering this and Fact 2, $f_{a,b}$ has at most one attractive periodic orbit. □

By the argument above, the attracting periodic orbit of skew tent map is characterized as follows.

Theorem The attracting periodic orbit of $f_{a,b}$ is of stair type.

Proof. The proof will immediately follow by Lemma 3 and Lemma 1. □

3 Monotonicity of kneading sequences

We first introduce the concept of symbolic dynamics.

For $x \in I_{a,b}$ and $j \geq 0$, we associate $f^j_{a,b}(x)$ with a symbol $s^j(x)$ as follows:

$$s^j(x) = \begin{cases} L, & \text{if } f^j_{a,b}(x) < 0 \\ C, & \text{if } f^j_{a,b}(x) = 0 \\ R, & \text{if } f^j_{a,b}(x) > 0 \end{cases}$$

For $x \in I_{a,b}$, the sequence $\{s^j(x)\}_{i\geq 0}$ is called *itinerary* and denoted as $S_{a,b}(x)$. *Kneading sequence* of $f_{a,b}$ is defined as $S_{a,b}(1)$ and will be denoted by $K(a,b)$. If there exists $i \geq 0$ such that $f^i_{a,b}(1) = 0$, then we regard that $K(a,b)$ is a finite sequence, which is ended with C.

In this section we will mention the monotonicity property of kneading sequence in the domain

$$\tilde{D} = \{(a,b) \in D;\ a \geq 1\}.$$

Let us define the order for parameter pairs as follows, according to M. Misiurewicz and E. Visinescu [MV91]:

$(a,b) \succ (a',b') \Leftrightarrow a' \geq a,\ b' \geq b$, and at least one of these inequalities is strict.

Kneading sequences are monotone increasing with respect to this order.

Monotonicity Theorem (Theorem A in [MV91]) For $(a',b'), (a,b)$ in $\tilde{D}$ with $(a',b') \succ (a,b)$, it holds that $K(a',b') \succ K(a,b)$.

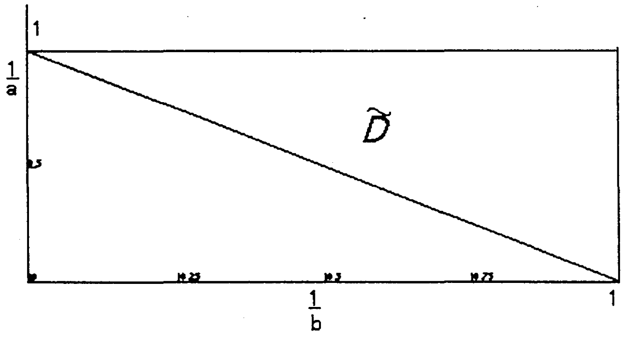

Figure 11: The domain $\tilde{D}$

This theorem is already proved in [MV91]. M. Misiurewicz and E. Visinescu showed the claim by using the estimation of topological entropy. But we shall reprove it by using only thier results for $D^\star$ in [MV91], and renormalization method, not via the topological entropy. For that purpose, we show Proposition 5, Proposition 6 and Proposition 7.

3.1 Preliminaries

In order to prove our Monotonicity Theorem, we summarize some definitions and facts for symbolic dynamical systems.

- Finite symbolic sequence $\underline{M}$ is said to be *even* (resp. *odd*) if $\underline{M}$ has an even (resp. odd) number of symbol R.
- Order between two itineraries is defined as follows:

  Set $\underline{I} = I_0 I_1 \cdots I_n I_{n+1} \cdots$, and $\underline{J} = J_0 J_1 \cdots J_n J_{n+1} \cdots$.
  Assume that $I_k = J_k$ for $0 \le k \le n$, $I_{n+1} \ne J_{n+1}$.

  $$\underline{I} \prec \underline{J} \Leftrightarrow \begin{cases} I_{n+1} < J_{n+1}, & \text{if } I_0 I_1 \cdots I_n \text{ is even} \\ I_{n+1} > J_{n+1}, & \text{if } I_0 I_1 \cdots I_n \text{ is odd} \end{cases}$$

  where $L < C < R$.

- Symbolic sequence $\underline{M}$ is said to be *admissible* if $\underline{M}$ is an infinite sequence of L and R, or if it is a finite sequence of L and R, ended with C.

- Symbolic sequence $\underline{M}$ is said to be *maximal* if it is *admissible* and $\sigma^k(\underline{M}) \preceq \underline{M}$ for all $k \geq 0$, where σ is *shift map i.e.*

  $\sigma(\underline{M}) = (M_1 M_2 \cdots)$ for $\underline{M} = (M_0 M_1 \cdots)$.

3.2 $\star$–product

According to P. Collet and J. P. Eckmann[CE85], we define $\star$–*product.*

For a finite sequence $\underline{A}$ of R, L and an admissible sequence $\underline{B}$, $\underline{A} \star \underline{B}$ is defined as follows:

(1) If $\underline{\text{A}}$ is *even* and $\underline{B}$ is infinite $\underline{B} = B_0 B_1 \cdots$,

$\underline{A} \star \underline{B} = \underline{A} B_0 \underline{A} B_1 \underline{A} B_2 \cdots$.

(2) If $\underline{\text{A}}$ is *odd* and $\underline{\text{B}}$ is infinite $\underline{B} = B_0 B_1 \cdots$,

$\underline{A} \star \underline{B} = \underline{A} \check{B}_0 \underline{A} \check{B}_1 \underline{A} \check{B}_2 \cdots$.

(3) If $\underline{\text{A}}$ is *even* and $\underline{\text{B}}$ is finite $\underline{B} = B_0 B_1 \cdots B_{n-1} C$,

$\underline{A} \star \underline{B} = \underline{A} B_0 \underline{A} B_1 \cdots \underline{A} B_{n-1} \underline{A} C$.

(4) If $\underline{\text{A}}$ is *odd* and is $\underline{\text{B}}$ is finite $\underline{B} = B_0 B_1 \cdots B_{n-1} C$,

$\underline{A} \star \underline{B} = \underline{A} \check{B}_0 \underline{A} \check{B}_1 \cdots \underline{A} \check{B}_{n-1} \underline{A} C$

Here we define $\check{B}_i$ as follows

$$\check{B}_i = \begin{cases} L, & \text{if } B_i = R \\ R, & \text{if } B_i = L \end{cases}$$

We remark that $\star$–product holds associative law:

If $\underline{A_1} C$, $\underline{A_2} C$ and $\underline{B}$ is admissible and $\underline{A} C = \underline{A_1} \star (\underline{A_2} C)$, then it is proved that $\underline{A_1} \star (\underline{A_2} \star \underline{B}) = \underline{A} \star \underline{B}$. Hence the sequence $R^{\star n}$ is well defined such that

$$R^{\star n} \star \underline{B} = \overbrace{R \star (R \star (\cdots R \star (R \star \underline{B})) \cdots)}^{n \text{ times}}.$$

The sequence $R^{\star\infty}$ is defined to be $\lim_{n \to \infty} R^{\star n}$.

Example.

$R^{\star 1} = R,$
$R^{\star 2} = RLR,$
$R^{\star 3} = RLRR\,RLR,$
$R^{\star 4} = RLRR\,RLRL\,RLRR\,RLR.$

Remark. Related to $R^{\star n}$, similar sequences are introduced in [Dev89].

$\tau_0 = R,$
$\tau_1 = RL,$
$\tau_2 = RLRR,$
$\vdots$
$\tau_{n+1} = \tau_n \hat{\tau}_n$, where $\hat{\tau}_n = t_0 \cdots \check{t_k}$ when $\tau_n = t_0 \cdots t_k$.

Note that $R^{\star n}$ is equal to the sequence whose last entry is removed from τ_{n+1}.

Proposition 4 The length of $R^{\star n}$ is odd for all n. Moreover $R^{\star n}$ is an odd (*resp.* even) sequence if n is odd(*resp.* even).

Proof. For the first part of the proposition, we can show that the length of $R^{\star n}$ is odd for n by induction of n. For the second part, we shall show the claim by induction of n.

(i) When $n = 1$, $R^{\star n} = R$ and it is odd.

(ii) When $n = 2$, $R^{\star 2} = RLR$ and it is even.

(iii) For $n \le 2k$, we assume the claim holds.

Recall that $R^{\star 2k+1}C = R \star (R^{\star 2k}C)$.

Let us denote $R^{\star 2k}C = S_0 \cdots S_{2j}C$ for some $j \ge 0$ since the length of $R^{\star 2k}$ is odd. By assumption, $R^{\star 2k}$ is even and has an odd number of L. It follows that $\check{S_0} \cdots \check{S_{2j}}$ is odd. Recall that

$$R \star (R^{\star 2k}C) = R\check{S_0} \cdots R\check{S_{2j}}RC.$$

And this sequence is odd. Hence $R^{\star 2k+1}$ is odd.

(iv) Let us check that $R^{\star 2k+2}$ is even. Recall that $R^{\star 2k+2}C = R\star(R^{\star 2k+1}C)$. We can denote $R^{\star 2k+1}C = T_0 \cdots T_{2i}C$ for some $i \ge 0$ since the length of $R^{\star 2k+1}$ is odd. By (iii), $R^{\star 2k+1}$ is odd and has an even numer of L. It follows that $\check{T_0} \cdots \check{T_{2i}}$ is odd. Recall that

$$R \star (R^{\star 2k+1}C) = R\check{T_0} \cdots R\check{T_{2i}}RC.$$

And this sequence is even. Hence $R^{\star 2k+2}$ is even.

By (i)$\sim$(iv), the proof is done. $\square$

Proposition 5 Let $\underline{A}$ and $\underline{B}$ be symbolic sequences with $\underline{A} \succ \underline{B}$. Then for all $n \geq 1$, it holds that $R^{\star n} \star \underline{A} \succ R^{\star n} \star \underline{B}$.

Proof. Set $\underline{A} = A_0 A_1 \cdots A_{k-1} A_k \cdots$, $\underline{B} = B_0 B_1 \cdots B_{k-1} B_k \cdots$.
Let $A_j = B_j$, for $0 \leq j \leq k-1$ and $A_k \neq B_k$.

(I) If n is an even number, then $R^{\star n}$ is an even sequence.

If $A_0 \cdots A_{k-1} = B_0 \cdots B_{k-1}$ is even, then $A_k > B_k$ since $\underline{A} \succ \underline{B}$. Recall that

$$R^{\star n} \star \underline{A} = R^{\star n} A_0 \cdots R^{\star n} A_{k-1} R^{\star n} A_k \cdots.$$

Then it is easy to see that $R^{\star n} A_0 \cdots R^{\star n} A_{k-1} R^{\star n} A_k$ is even. Hence we have $R^{\star n} \star \underline{A} \succ R^{\star n} \star \underline{B}$.

If $A_0 \cdots A_{k-1} = B_0 \cdots B_{k-1}$ is odd, then $A_k < B_k$ since $\underline{A} \succ \underline{B}$. Recall that

$$R^{\star n} \star \underline{A} = R^{\star n} A_0 \cdots R^{\star n} A_{k-1} R^{\star n} A_k \cdots.$$

Therefore $R^{\star n} A_0 \cdots R^{\star n} A_{k-1} R^{\star n} A_k$ is odd. And we have that

$R^{\star n} \star \underline{A} \succ R^{\star n} \star \underline{B}$.

(II) If n is an odd number, then $R^{\star n}$ is an odd sequence.

(a) Assume that k is an even number.

If $A_0 \cdots A_{k-1} = B_0 \cdots B_{k-1}$ is even, then $A_k > B_k$ since $\underline{A} \succ \underline{B}$. Moreover we have that $\breve{A}_0 \cdots \breve{A}_{k-1} = \breve{B}_0 \cdots \breve{B}_{k-1}$ is even and that $\breve{A}_k < \breve{B}_k$. By recalling that

$$R^{\star n} \star \underline{A} = R^{\star n} \breve{A}_0 \cdots R^{\star n} \breve{A}_{k-1} R^{\star n} \breve{A}_k \cdots,$$

it is easy to see that $R^{\star n} \breve{A}_0 \cdots R^{\star n} \breve{A}_{k-1} R^{\star n}$ is odd. Hence we have that $R^{\star n} \star \underline{A} \succ R^{\star n} \star \underline{B}$.

If $A_0 \cdots A_{k-1} = B_0 \cdots B_{k-1}$ is odd, then $A_k < B_k$ since $\underline{A} \succ \underline{B}$. Moreover we have that $\breve{A}_0 \cdots \breve{A}_{k-1} = \breve{B}_0 \cdots \breve{B}_{k-1}$ is odd and that $\breve{A}_k > \breve{B}_k$. By recalling that

$$R^{\star n} \star \underline{A} = R^{\star n} \breve{A}_0 \cdots R^{\star n} \breve{A}_{k-1} R^{\star n} \breve{A}_k \cdots,$$

it is easy to see that

$R^{\star n}\check{A}_0 \cdots R^{\star n}\check{A}_{k-1}R^{\star n}$ is even. Hence we have that

$R^{\star n} \star \underline{A} \succ R^{\star n} \star \underline{B}$.

(b) Assume that k is an odd number.

If $A_0 \cdots A_{k-1} = B_0 \cdots B_{k-1}$ is even, then $A_k > B_k$ since $\underline{A} \succ \underline{B}$. Moreover we have $\check{A}_0 \cdots \check{A}_{k-1} = \check{B}_0 \cdots \check{B}_{k-1}$ is odd and that $\check{A}_k < \check{B}_k$. By recalling that

$$R^{\star n} \star \underline{A} = R^{\star n}\check{A}_0 \cdots R^{\star n}\check{A}_{k-1}R^{\star n}\check{A}_k \cdots,$$

it is easy to see that $R^{\star n}\check{A}_0 \cdots R^{\star n}\check{A}_{k-1}R^{\star n}$ is odd. Hence we have that $R^{\star n} \star \underline{A} \succ R^{\star n} \star \underline{B}$.

If $A_0 \cdots A_{k-1} = B_0 \cdots B_{k-1}$ is odd, then $A_k < B_k$ since $\underline{A} \succ \underline{B}$. Moreover we have that$\check{A}_0 \cdots \check{A}_{k-1} = \check{B}_0 \cdots \check{B}_{k-1}$ is even and that $\check{A}_k > \check{B}_k$. By recalling that

$$R^{\star n} \star \underline{A} = R^{\star n}\check{A}_0 \cdots R^{\star n}\check{A}_{k-1}R^{\star n}\check{A}_k \cdots,$$

it follows that

$R^{\star n}\check{A}_0 \cdots R^{\star n}\check{A}_{k-1}R^{\star n}$ is odd. Hence we conclude that

$R^{\star n} \star \underline{A} \succ R^{\star n} \star \underline{B}$. □

3.3 Monotonicity in $D^\star$

M. Misiurewicz and E. Visinescu proved in [MV91] that $K(a', b') > K(a, b)$ for $(a', b'), (a, b) \in D^\star$ such that $(a', b') \succ (a, b)$, where $D^\star$ is defined by

$$D^\star = \{(a, b) \in D\,;\, a + b < ab^2, a > 1\}.$$

This domain $D^\star$ is characterlized by the following lemma.

Fact 3 (Lemma 2.1 in [MV91]) $(a, b) \in D^\star \Leftrightarrow K(a, b) \succ RLR^\infty$.

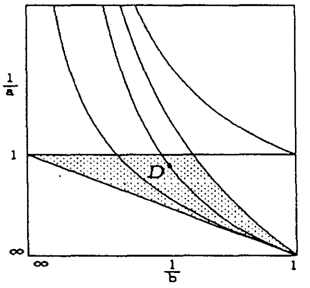

Figure 12: The shadowed part is domain of $D^\star$.

Using a parameter $t \in [0,1]$, set

$$f_t = f_{a't+a(1-t),\ b't+b(1-t)}.$$

First, the monotonicity of kneading sequence is shown as to parameter t.

Fact 4 (Lemma 4.1 in [MV91]) If $0 \leq v < \omega \leq 1$ then $K(f_v) < K(f_\omega)$, where $K(f_t)$ denotes kneading sequence of f_v.

Fact 5 (Lemma 4.2 in [MV91]) $K(f_t)$ is not constant on $[0,1]$.

From the two lemmas above, monotone increasing property of kneading sequence is proved in $D^\star$.

Fact 6 (Proposition 4.3 in [MV91]) If (a,b) and (a',b') are in $D^\star$ with $(a,b) \prec (a',b')$, then it holds that $K(a,b) \prec K(a',b')$.

3.4 Renormalization and $\star$–product

The aim of this section is to prove Monotonicity Theorem by using only renormalization method. For the details of *renormalization* and *prime sequence*, see the sequel paper [ITO] in this volume.

Proposition 6 Let (a,b) be in D. The following three conditions are equivalent mutually.

(i) $(a,b) \in D_0$.

(ii) There exists a unique number $m \geq 1$ and a prime sequence $\underline{B}$ whose length is longer than 2 such that $K(a,b) = R^{\star m} \star \underline{B}$.

(iii) There exists some number $m \geq 1$ such that $\varphi^m(a,b) \in D^\star$, where $\varphi(a,b) = (b^2, ab)$.

Furthermore, there exist closed subintervals of $I_{a,b}$ $\{I_i\}_{i=0,\ \cdots,\ 2^m-1}$ such that their interiors are disjoint mutually, $f_{a,b}I_i = I_{i+1}$ for $0 \leq i \leq 2^m - 2$ and $f_{a,b}I_{2^m-1} = I_0$, $I_{2^m-1} \ni 0$, and $f_{a,b}^{2^m}|_{I_i} \sim f_{\varphi^m(a,b)}$.

Proof. Recall that $D_0 = \{(a,b) \in D;\ ab > 1,\ a+b \geq ab^2\}$.

First, we will show that (i)$\Rightarrow$ (iii).

Let $(a,b) \in D_0$. Set $\varphi^n(a,b) = (\tilde{a}_n, \tilde{b}_n)$ and assume that $\tilde{a}_n \leq \dfrac{\tilde{b}_n}{\tilde{b}_n^2 - 1}$ for all $n \geq 1$. Fix any $k \geq 1$. By the definition of φ, we have that $\tilde{b}_{k+p} > (\tilde{a}_k\tilde{b}_k)(\tilde{b}_k)^{p-1}$ for all $p \geq 1$. If we take $p \to \infty$, then we have that $\lim_{p\to\infty} \tilde{b}_{k+p} = \infty$ and $\lim_{n\to\infty} \tilde{b}_n = \infty$. If we take $n \to \infty$ on the assumption $\tilde{a}_n \leq \frac{\tilde{b}_n}{\tilde{b}_n^2-1}$, then we have that

$$\frac{\tilde{b}_n}{\tilde{b}_n^2 - 1} = \frac{\frac{1}{\tilde{b}_n}}{1 - \frac{1}{\tilde{b}_n^2}} \to 0\,.$$

This contradicts that $\tilde{a}_n > 1$ for $n \geq 2$. Hence, for each $(a,b) \in D_0$, there exists $m \geq 1$ such that $\varphi^m(a,b) \in D^\star$.

We remark that once we have $\varphi^m(a,b) \in D^\star$, $\varphi^{m+1}(a,b)$ does not belong to our domain D any longer. For the second part, it is clear by [ITN79] and [IN97b].

Second, we will show that (iii)$\Rightarrow$(i).

If $(a,b) \in D_k^A \cup D_k^\star$, then it is proved that there is no subinterval where $f_{a,b}^{2^m}$ is surjective, except for $I_{a,b}$.

If $(a,b) \in D_k^B$ for some $k \geq 3$, then we have that $a+b < ab^2$ and that $\varphi(a,b)$ is in $\{(a,b);\ a+b < ab\}$. Because the map φ is proved to be surjective on this domain, it follows that $\varphi^m(a,b) \notin D^\star$ for any $m \geq 1$.

Hence (a,b) is in $D_2^B = D_0$.

To see that (i)$\Rightarrow$ (ii), it is clear by virtue of Lemma 5.1, Lemma 5.2 in [MV91]. Moreover, $\underline{B}$ is equal to $K(\varphi^m(a,b))$ with $\varphi^m(a,b) \in D^\star$.

If we suppose that $\underline{B}$ is not prime, then we can show a contradiction because of Theorem 3 in [ITO] and of that $\varphi^m(a,b) \in D^\star$.

Last, we will show that (ii)$\Rightarrow$ (i).

If $(a,b) \in D_2^A$, then we have either $K(a,b) = (RL)^\infty$ or $K(a,b) = R\star(RC)$.

If $(a,b) \in D_1$, then we have $K(a,b) = RLRRL\cdots$, or $K(a,b) = RLRRC$.

If $(a,b) \in D_k$ for some $k \geq 3$, then we have $K(a,b) = RLL\cdots$.

Hence the kneading sequence cannot be written in the form $R^{\star m} \star \underline{B}$ for any $m \geq 1$. Therefore we have $(a,b) \in D_0$.

We remark that the function φ is defined only on D_0. □

Proposition 7 Let $(a,b),(a',b') \in \tilde{D} \setminus D^\star$ such that $(a,b) \prec (a',b')$. If $\varphi^m(a,b) \in D^\star$ and $\varphi^n(a',b') \in D^\star$, then $m \geq n$.

Proof. Assume that $n > m$. Then $\varphi^m(a,b)$ is in D. But it cannot be in $D^\star$. Because of that $(a,b) \prec (a',b')$ and of the definition of φ, we have that $\varphi^m(a',b') \succ \varphi^m(a,b)$. Setting

$$\varphi^m(a',b') = (\tilde{a'},\tilde{b'}) \quad \text{and} \quad \varphi^m(a,b) = (\tilde{a},\tilde{b}),$$

we have the inequalities $\tilde{a'} \geq \tilde{a}$, $\tilde{b'} \geq \tilde{b}$ and that $\tilde{a} > \dfrac{\tilde{b}}{\tilde{b}^2 - 1}$ since $\varphi^m(a,b)$ is in $D^\star$.

On the other hand, since $s \mapsto \frac{s}{s^2-1}$ is strictly decreasing, we have

$$\frac{\tilde{b}}{\tilde{b}^2 - 1} > \frac{\tilde{b'}}{\tilde{b'}^2 - 1} > \tilde{a'} \geq \tilde{a}.$$

Hence this is a contradiction. Therefore we conclude that $n \leq m$. □

3.5 Proof of Monotonicity Theorem

Assume that $(a,b) \prec (a',b')$.

(i) If both (a,b) and (a',b') belong to $D^\star$, then the proof is already given by Fact 6.

(ii) Assume that either (a,b) or (a',b') belongs to $D^\star$. Then (a',b') is in $D^\star$ because $(a,b) \prec (a',b')$. By virtue of Fact 3, it follows that

$$K(a,b) \preceq RLR^\infty \prec K(a',b').$$

We have that $K(a,b) \prec K(a',b')$ since an order relation "$\prec$" is total.

(iii) Assume that (a,b) and (a',b') both belong to $\tilde{D} \setminus D^\star$. Then by Proposition 7, their kneading sequences are written as, for some $n \leq m$,

$$K(a,b) = R^{\star m} \star K(\varphi^m(a,b)) \text{ and } K(a',b') = R^{\star n} \star K(\varphi^n(a,b)).$$

If $m = n$, then we have that $\varphi^n(a,b) \prec \varphi^n(a',b')$ since φ is an increasing function. Because $K(\varphi^n(a,b)) \prec K(\varphi^n(a',b'))$ and from Proposition 5, we have that $K(a,b) \prec K(a',b')$.

If $n < m$, then we have that $\varphi^n(a,b) \neq D^\star$ and $\varphi^n(a',b') \in D^\star$. By virtue of Fact 3, it follows that

$$K(\varphi^n(a,b)) \preceq RLR^\infty \prec K(\varphi^n(a',b')).$$

By Proposition 5, we have that $K(a,b) \prec K(a',b')$. □

4 Topological Entropy of $f_{a,b}$

J. Milnor and W. Thurston studied in the paper [MT88] that for piecewise monotone mapping, the kneading sequence determines its topological entropy. In this section, we will summarize some, well–known but important results and give some correct for a statement in [MV92].

4.1 Topological entropy of $f_{a,b}$ for $(a,b) \in \tilde{D}$

In the paper [MV91], the following theorem is also proved as Theorem B, and as a result, strict monotoicity property for the topological entropy is obtained as well.

Fact 7 (Theorem B in [MV91]) If $(a,b) \in \tilde{D}$, then $K(a,b) \in \mathcal{M}$,

where $\mathcal{M}$ denotes the set of kneading sequence of *tent maps*, that is,

$$f_{a,a} = \begin{cases} ax, & \text{for } x < 0 \\ a(1-x), & \text{for } x > 0 \end{cases} \text{ with } 1 < a \leq 2.$$

Fact 8 (see for example [MW80] and [M89]) The entropy of $f_{a,a}$ is proved to equal to $\log a$, namely it is monotone increasing for a.

By taking the aboves and our Monotonicity Theorem into consideration, it is found that in $\tilde{D}$, the topological entropy is also strictly monotone increasing with respect to the order "$\succ$" (See Corollary in [MV91]).

We remark that the topological entropy of $f_{a,b}$ is equal to $\log \beta_{K(a,b)}(\mu)$ with μ such that $\beta_{K(a,b)}(\mu) = \mu$. This fact is implicitly shown by Fact 7 and Theorem C in [MV91]. Theorem C is as follows.

Fact 9 (Theorem C in [MV91]) For each $\underline{M} \in \mathcal{M}$, there exists a number $\gamma(\underline{M})$ and a continuous decreasing function $\beta_{\underline{M}} : (1, \gamma(\underline{M})] \to [1, \infty)$ (with one exeption $\underline{M} = RL^\infty$ when $\gamma(\underline{M}) = \infty$) such that for $(a,b) \in \tilde{D}$, $K(a,b) = \underline{M}$ if and only if $a = \beta_{\underline{M}}(b)$. The function γ is increasing. The graphs of the functions $\beta_{\underline{M}}$ fill up the whole set $\tilde{D}$. Moreover the following are shown:

$$\lim_{\underline{M}\to R^{\star\infty}} \gamma(\underline{M}) = 1,$$

$$\lim_{\underline{M}\to RL^{\infty}} \gamma(\underline{M}) = \infty,$$

$$\lim_{b\searrow 1} \beta_{\underline{M}}(b) = \infty \quad \text{if } \underline{M} \succeq RLR^{\infty},$$

$$\lim_{b\searrow 1} \beta_{\underline{M}}(b) = \gamma(\underline{J}) \quad \text{if } \underline{M} \prec RLR^{\infty},$$

$\underline{J}$ is given by

$$\underline{M} = R \star \underline{J},$$

$$\beta_{\underline{M}}(\gamma(\underline{M})) = 1 \quad \text{if } \underline{M} \neq RL^{\infty},$$

$$\lim_{b\searrow 1} \beta_{\underline{M}}(b) = +\infty \quad \text{if } \underline{M} = RL^{\infty}.$$

4.2 Topological Entropy of $f_{a,b}$ for $(a, b) \in D \setminus \tilde{D}$

Let us denote the topological entropy of $f_{a,b}$ by $h(a, b)$.

According to J. C. Marcuard and E. Visinescu, topological entropy of $f_{a,b}$ for $(a, b) \in D \setminus \tilde{D}$ is studied in [MV92]. But we claim that the following result, Corollary in [MV92] is not true:

The function $h(a, b)$ is constant in $(D_k^B \cup D_k^A) \cap (D \setminus \tilde{D})$, for $k \geq 2$.

Proposition 8 The function $h(a, b)$ is not constant in $(D_k^B \cup D_k^A) \cap (D \setminus \tilde{D})$, for $k \geq 2$.

Proof. We can give the following counter example.
Consider the case of $k = 2$. Recall that $D_2^B \cap (D \setminus \tilde{D}) = \{(a, b) \in D_0 \text{ and } a < 1\}$ and that, for $(a, b) \in D_0$, $f_{a,b}$ is renormalizable of level, at least, two. By renormalization, namely by map φ, the parameter domain $\{(a, b) \in D_0 \text{ and } a < 1\}$ is mapped into a subdomain of $\tilde{D}$,

$$\{(a, b) \in \tilde{D};\ a = 1,\ a = b^2,\ a + b = ab\}.$$

See also Fig. 13 and Fig. 14.

If (a, b) is in D_0, then it is easily proved that $2h(a, b) = h(\varphi(a, b))$.

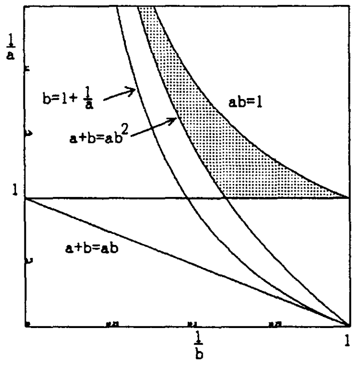

Figure 13: The shadowed part is the domain $D_2^B \cap (D \setminus \tilde{D})$ $(0.4 \leq a \leq 1)$.

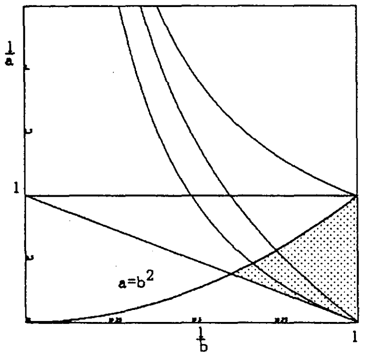

Figure 14: In this figure, the shadowed part corresponds the domain where $(D_2^B \cap D \setminus \tilde{D})$ is mapped by map φ is depicted .

Consequently, the entropy is not constant in the domain

$$\{(a, b) \in D_0;\ a < 1\}.$$

□

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