Renormalization and topological entropy of skew tent maps

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Abstract

We study the dynamics and monotonicity of entropy of skew tent maps $f_{a,b}$ having two slopes a, b as parameters. By the behavior of $f_{a,b}$, the domain D is divided into subdomains D_k defined by some algebraic curves : $D = \sum_{k=2}^{\infty} D_k$. In each D_k there are subdomains D_k^A where $f_{a,b}$ has unique attracting periodic orbit of period k ([IN97a][ITN79b]) and D_k^B where $f_{a,b}$ has $2k$ or k chaotic intervals. We analyze D_k^B by the method of renormalization, that sheds light to the structure of having $2k$ and k chaotic intervals. For this family Misiurewicz and Visinescu in [MV91] and Marcuard and Visinescu in [MV92] get some results of monotonicity of topological entropy. We correct their statements of Theorem 1 and Corollary of Theorem 2 in [MV92].

1 Introduction

For skew tent maps

$$f_{a,b}(x) = \begin{cases} ax+1, & (x \le 0) \\ -bx+1, & (x \ge 0) \end{cases}$$

depending on parameter-pair (a, b) in the domain of definition $D := \{(a,b) : a > 0, b > 1, a+b \ge ab\}$, we divide D into subdomains according to the dynamical behaviors : $D = \cup_{k=2}^{\infty} D_k$ such that

$$D_k := \{(a,b) \in D\,;\, 1 + \frac{1}{a} + \cdots + \frac{1}{a^{k-2}} < b \le 1 + \frac{1}{a} + \cdots + \frac{1}{a^{k-1}}\}.$$

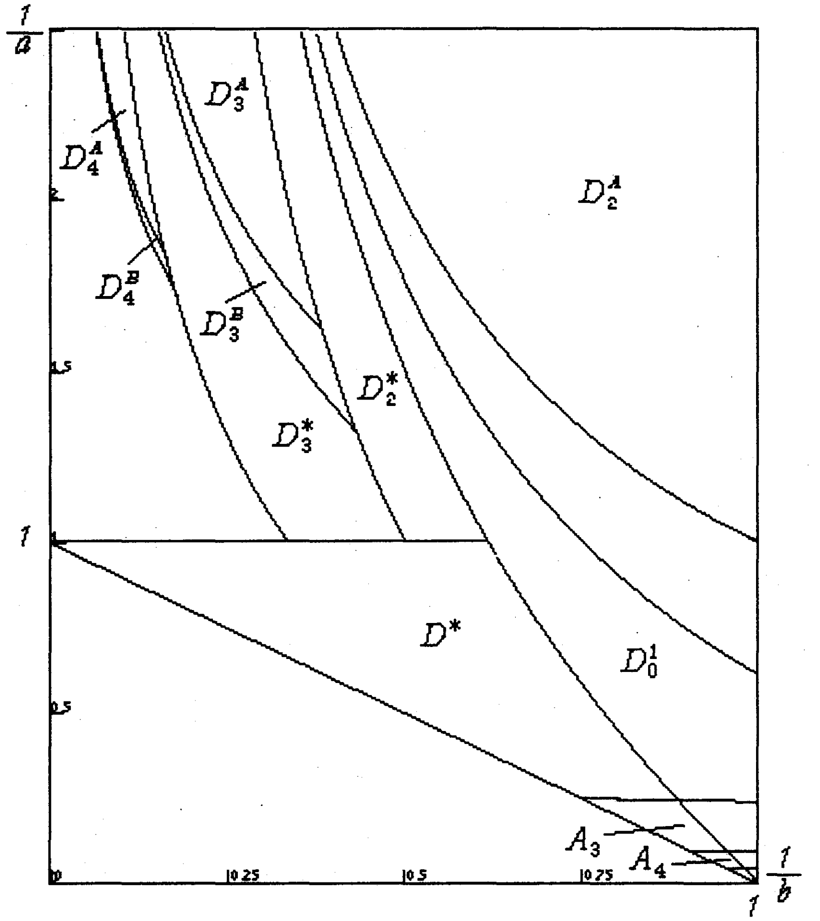


Figure 1: The parameter space of skew tent maps.
D_k^A : attractor domain (k-periodic attractor),
D_k^B : renormalizable domain (k or $2k$ chaotic bands),
$D_k^\star$: chaotic domain (1 chaotic band),
D_0^1 : renormalizable domain (2 chaotic bands),
A_k : renormalization domain (1 or 2 chaotic bands).

In each D_k there are subdomains D_k^A, D_k^B and $D_k^\star$:

$$D_k^A := \left\{(a,b) \in D_k\,;\; a^{k-1}b \le 1\right\},$$

$$D_k^B := \left\{(a,b) \in D_k\,;\; a^{k-1}b > 1, a+b \ge a^{k-1}b^2\right\},$$

$$D_k^\star := \left\{(a,b) \in D_k\,;\; a+b < a^{k-1}b^2\right\}.$$

For $D_k^\star$ it is known that $f_{a,b}$ has one chaotic interval ([ITN79b]). For D_k^A the following results is obtained ([ITN79b][IN97a]) :

A map $f_{a,b}$ of D_k^A has unique attracting periodic orbit with period k.

In this paper we analyze D_k^B by the method of renormalization. In [NY95] they have bifurcation diagrams by computer experiment related to our skew tent maps. We can see in them that one attractor bifurcates k-periodic attractor or $2k$, k, 1 chaotic intervals (see [Ich]). The first case is D_k^A and the last case is $D_k^\star$. The second and third case is $D_k^B\,(k \geq 3)$: by the method of renormalization, the structure of having $2k$ and k chaotic intervals will be made clear. Let

$$A_k := \left\{(a,b) \in D\,;\, a > b^2\,,\, (a+b)b^{\frac{k-2}{k}} > (b+1)a^{\frac{k-1}{k}}\right\} \quad \text{for } k \geq 3$$

where a skew tent map has 2 or 1 chaotic intervals.

We obtain a result as Theorem 2 that any skew tent map of D_k^B is renormalized to one of A_k : namely, for $f_{a,b}$ of D_k^B there exist some $f_{a',b'}$ of A_k and some homeomorphic function h satisfying $f_{a,b}^k|_U \circ h = h \circ f_{a',b'}$. We rename D_2^B D_0 and have to define $A_2 = \{(a,b) \in D\,;\, a > 1\}$ where a skew tent map has 2^m chaotic intervals for $m \geq 1$. There also exists some $f_{a',b'}$ of A_2 for any $f_{a,b}$ of D_0 such that $f_{a,b}^2|_U \sim f_{a',b'}$.

For monotonicity of kneading sequence and topological entropy for this family, some results are obtained in [MV91] and in [MV92]. We also give the relation between kneading sequence and $\star$-product as Theorem 3 and correct statements of [MV92] at the end of this paper.

2 Dynamics of $f_{a,b}$

First we prepare some definitions and notations.

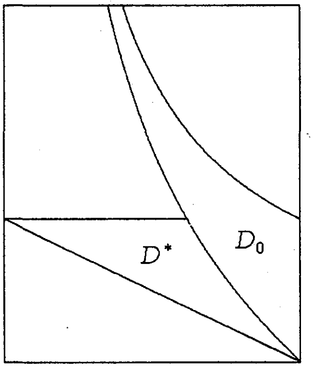
Let $I_{a,b}$ be $[f^2(0), f(0)]$. We shall analyze dynamics of $f_{a,b}$ restricted only on $I_{a,b}$ because one is obvious on $\mathbf{R}\backslash I_{a,b}$ (see [Ich]). Let X be an interval of $\mathbf{R}$. A map $f : X \to X$ is said to be *chaotic* on X if f is *sensitively dependent on initial conditions* and *topologically transitive*, and its periodic points are dense in X ([Dev89]).

Definition ([IN97a]). A closed subinterval J of $I_{a,b}$ will be called a *chaotic interval* if $f_{a,b}$ is chaotic on J.

Definition. We will say that $f_{a,b}$ is n - *renormalizable* or *renormalizable of level* n if there exist some closed interval U and positive integer n such that $f_{a,b}^n : U \longrightarrow U$ and $f_{a,b}^n|_U$ is also skew tent map. If $f_{a,b}$ is n - renormalizable, there exists topological conjugacy h satisfying $f_{a,b}^n|_U \circ h = h \circ f_{a',b'}$ for some (a', b') in D. Then we denote it by $f_{a,b}^n|_U \overset{h}{\sim} f_{a',b'}$. We shall call these maps $f_{a,b}^n|_U$ or $f_{a',b'}$ n - *renormalization* of $f_{a,b}$.

2.1 The case of $D_0 \,(= D_2^B)$

We introduce some facts from [ITN79b] for our purpose : we define



$$D^\star := \left\{(a, b) \in D\,;\; a + b < ab^2, a > 1\right\}$$

where $f_{a,b}$ has unique chaotic interval (i.e., $I_{a,b}$ is a chaotic interval). For $(a, b) \in D_0$, set

$L_0 = [f_{a,b}^2(0),\ f_{a,b}^4(0)]$ and $L_1 = [f_{a,b}^3(0),\ f_{a,b}(0)]$.

We have $int(L_0) \cap int(L_1) = \emptyset$, $f_{a,b}L_0 = L_1$ and $f_{a,b}L_1 = L_0$. $int(L_i)$ is the interior of L_i.

Moreover there exist topological conjugacy

$$h_0(x) = \frac{f_{b^2,ab}(0) - f_{b^2,ab}^2(0)}{f_{a,b}^4(0) - f_{a,b}^2(0)} \left(f_{a,b}^4(0) - x\right) + f_{b^2,ab}^2(0) \quad \text{for } f_{a,b}^2 \text{ on } L_0$$

and

$$h_1(x) = \frac{f_{b^2,ab}(0) - f_{b^2,ab}^2(0)}{f_{a,b}(0) - f_{a,b}^3(0)} \left(x - f_{a,b}^3(0)\right) + f_{b^2,ab}^2(0) \quad \text{for } f_{a,b}^2 \text{ on } L_1$$

such that

$$f_{a,b}^2|_{L_i} \overset{h_i}{\sim} f_{b^2,ab} \qquad (i = 0, 1). \tag{1}$$

First we remark that $f_{a,b}^2|_{L_0} \sim f_{a,b}^2|_{L_1}$.

Orbit of any point in $I_{a,b} \backslash (L_0 \cup L_1)$ except of a fixed point for $f_{a,b}$ is attracted to $L_0 \cup L_1$. Therefore, we shall study dynamics of $f_{a,b}$ restricted on $L_i\,(i = 0, 1)$.

The relation (1) motivates us to define the domain

$$\begin{aligned} A_2 : &= \left\{(b^2, ab)\,;\, (a, b) \in D_0\right\} \\ &= \{(a, b) \in D\,;\, a > 1\}\,. \end{aligned}$$

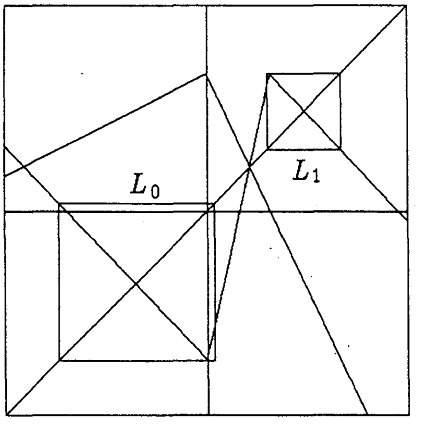


Figure 2: Invariant intervals L_0 and L_1 under the map f^2.

A_2 is contained in the union of D^* and D_0.

We know that D_0 can be divided into subdomains $\{D_0^m\}$ by some function $p(m)$ ([ITN79b]). Hence we have the following lemma.

key-lemma (*renormalization*). $p(m)$ is defined as follows

$$p(m) = \begin{cases} 1, & \text{if } m = 1 \\ 2p(m-1), & \text{if } m \text{ is even} \\ 2p(m-1)-1, & \text{if } m \text{ is odd} \end{cases}$$

and subdomains D_0^m of D_0 is

$$D_0^m := \left\{(a,b) \in D_0; a^{p(m)}b^{p(m+1)} \leq a+b < a^{p(m+1)}b^{p(m+2)}\right\}.$$

Then $f_{a,b}$ of D_0^m is renormalizable of level 2^m.

Proof. The division of D_0 by a function $p(m)$, which means repeat of renormalization, is showed in theorem 1.2 in [ITN79b]. We remark that each D_0^m means renormalizable of level 2^m.

2.2 The case of D_k^B $(k \geq 3)$

For D_k^B $(k \geq 3)$, we can also use the renormalization method.

We define other subdomains in D for a family of skew tent map having chaotic intervals.

Definition.

$$A_2 := \{(a,b) \in D; a > 1\},$$
$$A_k := \left\{(a,b) \in D; a > b^2,\ (a+b)b^{\frac{k-2}{k}} > (b+1)a^{\frac{k-1}{k}}\right\}\ (k \geq 3).$$

We have the following relation of inclusion for $\{A_k\}$.

Theorem 1. $A_2 \subset D^\star \cup \{\cup_{m=1}^{\infty} D_0^m\}$ and $A_k \subset D^\star \cup D_0^1\,(k \geq 3)$. Moreover $A_i \supset A_{i+1}\,(i \geq 2)$.

Proof. We have the following algebraic curves from definition of A_k:

$$a - b^2 = 0 \quad (2)$$
$$(a+b)b^{\frac{k-2}{k}} - (b+1)a^{\frac{k-1}{k}} = 0 \quad (3)$$
$$\Leftrightarrow (a+b)^k b^{k-2} - (b+1)^k a^{k-1} = 0 \quad (4)$$

It is sufficient to observe these curves where $1 < b < \frac{1+\sqrt{5}}{2}$ and $a > b^2$. Then they do not intersect each other (see Figure 3).

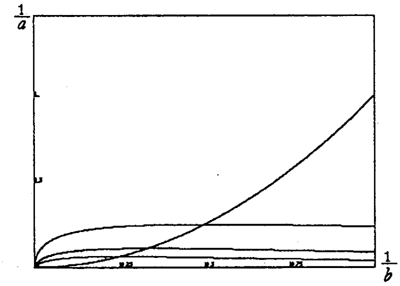


Figure 3 : Boundary curves of the renormalization domains A_3, A_4 and A_5.

For fixed $b\,(1 < b < \frac{1+\sqrt{5}}{2})$, the equation (4) as function of variable a has unique root α_k. We have that α_k tends monotonely to infinite when k varies to infinite. Hence $A_i \supset A_{i+1}$ $(i \geq 2)$. As it is easy to show that $A_3 \subset D^\star \cup D_0^1$, we have $A_k \subset D^\star \cup D_0^1$ for $k \geq 3$.

Each boundary curve of D_0^m and $b = 1$ cross at the point β_m which is maximal root of $a^{p(m)} - a - 1 = 0$ and β_m tends decreasingly to 1 when m varies to infinity. Hence we have $A_2 \subset D^\star \cup \{\cup_{m=1}^{\infty} D_0^m\}$. □

The equation (4) has the equation (2) as a factor for each $k \geq 3$, so we call the cofactor of (4) a curve Γ_k. A_k and Γ_k for $k = 2, \cdots, 5$ are pictured in Figure 4.

$\Gamma_k(a,b)$ $(k = 3,4,5)$ are

$$\Gamma_3(a,b) : -b^2 + (a^2 - 3a)b - a = 0, \quad (5)$$
$$\Gamma_4(a,b) : -b^4 - 4ab^3 + (a^3 - 6a^2 - a)b^2 - 4a^2b - a^2 = 0, \quad (6)$$
$$\Gamma_5(a,b) : -b^6 - 5ab^5 + (-10a^2 - a)b^4 + (a^4 - 10a^3 - 5a^2)b^3 + (-10a^3 - a^2)b^2 - 5a^3b - a^3 = 0. \quad (7)$$

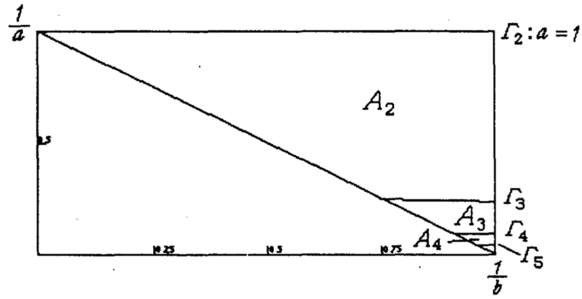


Figure 4: A_i and Γ_i for $i = 2, 3, 4$.

These curves are absolutely irreducible. It is proved by algorithms based in [YNT90], [L85], and implemented in symbolic algebraic computing system Risa/Asir. Detailed procedure is given in Table 1 of Appendix.

Fact (Theorem 2.3 in [ITN79b]). Assume that $(a, b) \in D_k^B$ $(k \geq 3)$. Set $J_j = [f_{a,b}^{j+2}(0),\ f_{a,b}^{k+j+2}(0)]$ $(0 \leq j \leq k-2)$ and $J_{k-1} = [f_{a,b}^{k+1}(0), 1]$. Then we have

(1) J_j's are disjoint and $f_{a,b}J_j = J_{j+1}$ $(0 \leq j \leq k-2)$, $f_{a,b}J_{k-1} = J_0$.

(2) $f_{a,b}^k|_{J_i} \sim f_{a^{k-2}b^2, a^{k-1}b}$.

(3) For almost all $x \in I_{a,b} - \cup_{j=0}^{k-1} J_j$ there exists integer n such that $f_{a,b}^n(x) \in \cup_{j=0}^{k-1} J_j$.

From the above fact we know that f^k on each J_i is topologically conjugate to $f_{a^{k-2}b^2, a^{k-1}b}$. Therefore we obtain the following theorem, which indicate the renormalization between D_k^B and A_k for $k \geq 3$.

Lemma. Each $D_k^B (k \geq 3)$ is divided into the following two subdomains :

$$B_k^2 = \left\{(a, b) \in D_k^B\ ;\ a + b \geq a^{2k-2}b^3\right\},$$

$$B_k^1 = \left\{(a, b) \in D_k^B\ ;\ a + b < a^{2k-2}b^3\right\}.$$

Proof. We consider an algebraic curve $a+b = a^{2k-2}b^3$ pulled back of the boundary curve $a + b = ab^2$ of D_0^1 and $D^\star$ by renormalization. It divides D_k^B into two subdomains and does not intersect two boundary curves of D_k^B : $a^{k-1}b = 1$ and $a + b = a^{k-1}b^2$. □

Theorem 2. If $(a, b) \in B_k^2 (k \geq 3)$, then there exists some $(a', b') \in A_k \cap D_0^1$ such that $f_{a,b}^k$ restricted on each invariant interval is topological

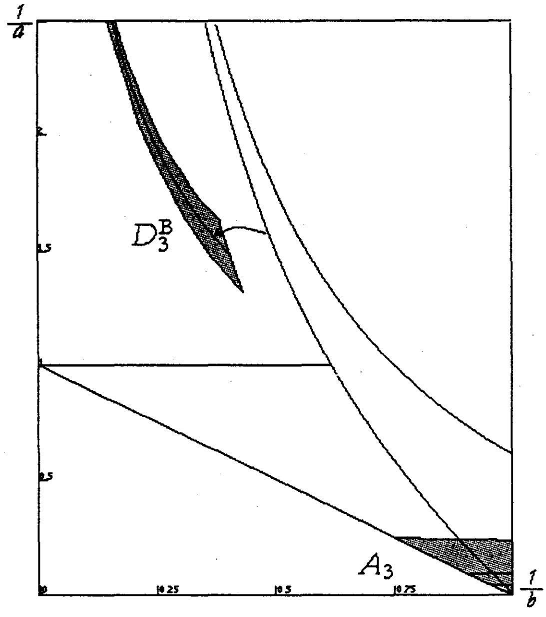


Figure 5: D_3^B and A_3.

conjugate to $f_{a',b'}$. If $(a,b) \in B_k^1\,(k \geq 3)$, then there exists some $(a',b') \in A_k \cap D^\star$ such that $f_{a,b}^k$ restricted on each invariant interval is topological conjugate to $f_{a',b'}$.

Proof. We have $A_k \subset D^\star \cup D_0^1\,(k \geq 3)$ from Theorem 1. The renormalization map is continuous and bijection from D_k^B to A_k. The three boundary curves of D_k^B correspond to ones of A_k as follows

$$a^{k-1}b = 1 \text{ to } b = 1,\ a + b = a^{k-1}b^2 \text{ to } a + b = ab$$

$$\text{and } 1 + \frac{1}{a} + \cdots + \frac{1}{a^{k-2}} = b \text{ to } \Gamma_k.$$

Hence we have the results. □

We also have next corollary, which is stated implicitly in Corollary 3.2. in [ITN79b].

Corollary. If $(a,b) \in B_k^2$, then $f_{a,b}$ has $2k$ chaotic intervals. If $(a,b) \in B_k^1$, then $f_{a,b}$ has k chaotic intervals.

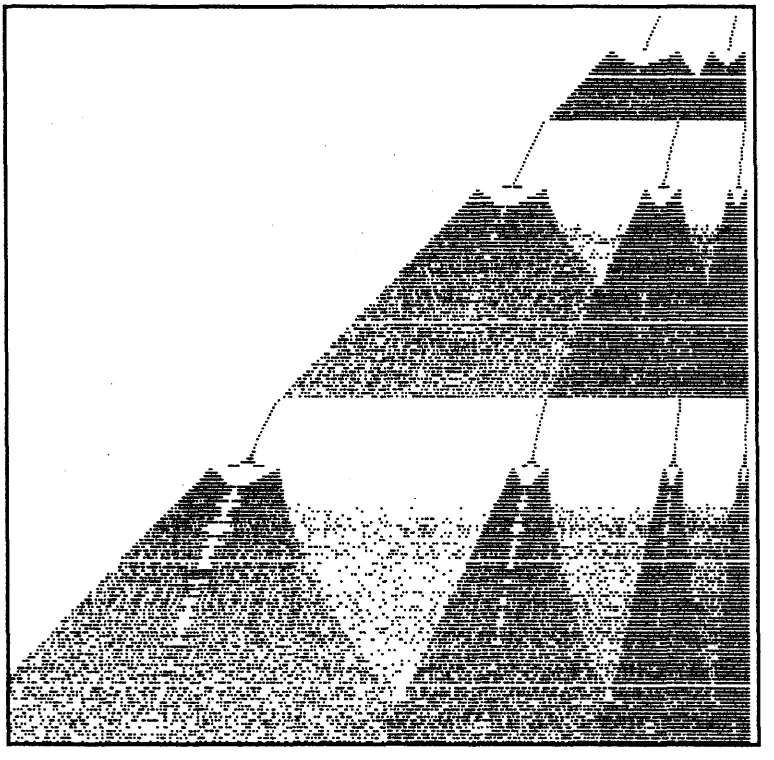


Figure 6: Bifurcation diagram of the skew tent maps in the case that a is fixed at 0.5 and b varies from 1.5 to 12.

3 Kneading sequence and topological entropy of skew tent maps

We classified the parameter domain of definition D in the previous sections. Bifurcation diagram is pictured in Figure 6 where $a = 0.5$ and $b = 1.5 \sim 14$ (i.e., b varies in D_2, D_3 and D_4). In this section we analyze kneading sequence $K(a, b)$ and topological entropy $h(a, b)$ of skew tent map $f_{a,b}$ of D. Some results for monotonicity of them are given today in [MV91] and [MV92]. For the subdomain, $a \geq 1$, of D, the monotonicity holds, as Theorem A,B in [MV91]:

$$(a, b) < (a', b') \Leftrightarrow K(a, b) < K(a', b') \text{ and } h(a, b) < h(a', b')$$

under the following order :

$$(a,b) < (a',b') \quad \text{if}$$

$$a \le a',\ b \le b' \text{ and at least one of these inequalities is sharp.}$$

They also give the relation between kneading sequences of skew tent maps and ones of tent maps (Theorem B). For the rest subdomain, $a \le 1$, of D, [MV92] stated properties of kneading sequence and topological entropy in their Theorem 1, Theorem 2 and Corollary of Theorem 2. Now we add here the following Theorem 3 from a view point of the relation between renormalization and $\star$-product (see p.72 in [CE80]) and correct some of their results at the end of this section. We refer basic definitions and notations of symbolic dynamics from [Ich][CE80].

3.1 Renormalization and $\star$-product

We denote $f_{a,b}$ by f in the rest of this section. For getting maximal level of renormalization, we assume sequence $\underline{B}$ is prime. Let $|\underline{A}|$ be the length of sequence $\underline{A}$ and $int(J)$ interior of an interval J.

Definition. A sequence $\underline{S}$ is called *prime* if $\underline{S}$ does not have any finite sequence $\underline{A}\,(\neq \emptyset)$ of L's and R's and any finite or infinite sequence $\underline{B}\,(\neq C)$ such that $\underline{S} = \underline{A} \star \underline{B}$.

Theorem 3. $K(a,b) = \underline{A} \star \underline{B}$ where $\underline{A}\,(\neq \emptyset)$ is finite sequence of L's and R's and $\underline{B}\,(\neq C)$ is prime if and only if there exist invariant closed intervals $\{J_i\}_{i=0,\cdots,|\underline{A}|}$ such that $J_{|\underline{A}|} \ni 0$, $fJ_i = J_{i+1}\,(i = 0,\cdots,|\underline{A}|-1)$, $fJ_{|\underline{A}|} = J_0$ and $int(J_i) \cap int(J_{i'}) = \emptyset\,(i \neq i')$. f can not have any refinement of $\{J_i\}$.

Proof. Assume $K(a,b) = \underline{A} \star \underline{B}$ where $\underline{A}\ (\neq \emptyset)$ is finite sequence of L's or R's and $\underline{B}\,(\neq C)$ is prime. Set $x_n = f^n(1)\,(n \ge 0)$, $p = |\underline{A}|$ and $\underline{A} = A_0 A_1 \cdots A_{p-1}$. Let J_i be convex hull of $\{x_{i+k(p+1)} : k = 0, 1, \cdots\}$ for $i = 0, \cdots, p$. Then, we have $fJ_i = J_{i+1}\,(i = 0, \cdots, p-1)$ and f is monotone on each J_i except of $i = p$ because symbol of $x_{i+k(p+1)}$ for all $k\,(\ge 0)$ is A_k. Remark that f^{p+1} on each J_i has same slopes. It follows that $f^{p+1}|_{J_i} \sim f^{p+1}|_{J_{i'}}\,(i \neq i')$. We consider the following two cases.

The first case : $\underline{B}$ does not contain both L and R.

$\underline{B}$ is finite in this case. It follows that J_p contains a turning point 0 as an end point of it. Hence, $fJ_p = J_0$. As f is monotone on J_i for all $i\,(0 \le i \le p)$, f^{p+1} restricted on J_i is monotone and surjective on J_i. Hence, its slope is -1. Then $\{J_i\}$'s are disjoint (see Figure 7) or there would exist some i, i' such that $J_i = J_{i'}$ from continuity of f. The latter can not occur

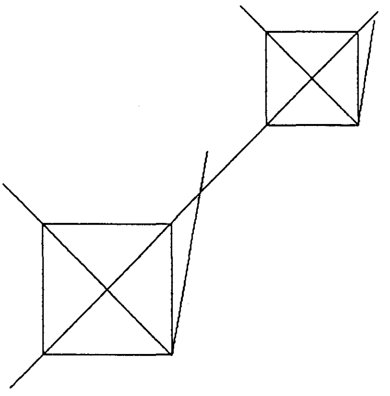


Figure 7: The graph of f^{p+1} on J_i of the first case.

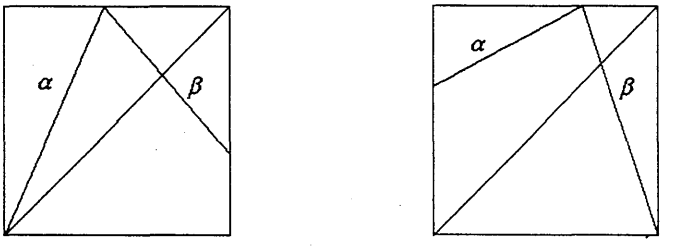


Figure 8: The graph of f^{p+1} on J_i of the second case.

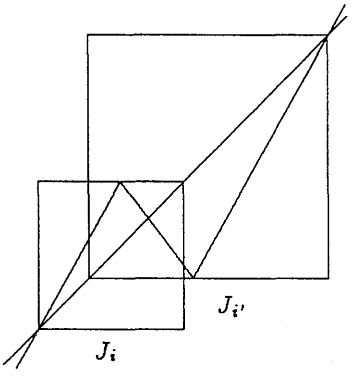


Figure 9: The graph of f^{p+1} on $J_{i'}$ having two turning points.

because of the assumption of $\underline{A}$. Therefore $\{J_i\}$ are disjoint. Notice that the first case corresponds to boundary curve of D_k^A and D_k^B.

The second case : $\underline{B}$ contains both L and R.

In this case $fJ_p = J_0$ and $f^{p+1}|_{J_i}$ has unique turning point c_i inside J_i. We set two slopes of $f^{p+1}|_{J_i}$ $\alpha\,(>0)$, $\beta\,(<0)$. We divide J_i into two subintervals I_{α_i} and I_{β_i} corresponding to slope α and β. As $f^{p+1}|_{J_i}$ is surjective on J_i, we have that $\sup\{|\alpha|, |\beta|\} > 1$.

If $|\beta| < 1$, then turning point is attracted to a fixed point on I_β. It follows that $\underline{B} = L^\infty$ or R^∞. This contradicts assumption of this case.

If $|\beta| = 1$, we reduce to the first case.

If $|\beta| > 1$ and $int(J_i) \cap int(J_{i'}) \neq \emptyset$, there exists $J_{i'}$ such that $f^{p+1}|_{J_{i'}}$ has two turning points (see Figure 9) or there exist i, i' such that $J_i = J_{i'}$. In the latter case we have J_m equals J_p for some $m\,(m \neq p)$. This contradicts that f is monotone on J_m because J_p includes turning point in it. Hence we obtain $int(J_i) \cap int(J_{i'}) = \emptyset$. Notice that the second case corresponds to D_k^B.

Conversely, if there exist disjoint invariant closed intervals $\{J_i\}_{i=0,\cdots,|\underline{A}|}$ in theorem, we have $K(a,b) = \underline{A} \star \underline{B}$ with $\underline{A} = A_{J_0}A_{J_1}\cdots A_{J_{p-1}}$. If $\underline{B}$ is not prime, f has refinement of $\{J_i\}$. Hence, $\underline{B}$ is prime. □

Now we have the relation of our renormalization (i.e., $(|\underline{A}|+1)$ - renormalization is a skew tent map of D) and $\star$ - product.

Corollary 1. If $|\underline{B}| \neq 2$ in above theorem, then f is renormalizable of level $|\underline{A}| + 1$.

Proof. Let p be $|\underline{A}|$. In the first case, we have $|\underline{B}| = 2$ because a turning point of f on J_p is 2-periodic point of f^{p+1}. In the second case, we have $|\beta| > 1$ and $f^{p+1}J_i = J_i$. It follows $(\alpha, \beta) \in D$. Therefore f is $(p+1)$-renormalizable on $[c_i, f^{p+1}(c_i)]$ (resp. $[f^{p+1}(c_i), c_i]$) if $c_i < f^{p+1}(c_i)$ (resp. $f^{p+1}(c_i) < c_i$). □

It is well known that for a smooth unimodal map g, n-periodic g-admissible sequence implies the existence of n or $2n$ - periodic point ([Dev89]). This fact is proved by Schwarzian derivative. But we have the following analogous fact for skew tent maps.

Corollary 2. If $K(a,b) = \underline{A} \star \underline{B}$ where $\underline{A}\,(\neq \emptyset)$ is finite sequence of L's and R's and $\underline{B}\,(\neq C)$ is prime, then f has periodic points of period $|A| + 1$. Moreover if $|\underline{B}| = 2$, then f also has periodic points of period $2(|A| + 1)$.

Remark. For showing Corollary 1 and 2, we need only the assumption $\underline{B} \neq C, L^\infty, R^\infty$ instead of primarity of $\underline{B}$.

3.2 Renormalization and topological entropy

Now we correct two statements of [MV92].

First : **kneading sequence for boundary curve of $A_m(= D^A_{m+1})$ and $B_m(= D^B_{m+1})$.**

In Theorem 1 of the paper [MV92], they say ;

$$(\lambda, \beta)(= (a, b)) \in A_m \Leftrightarrow K(\lambda, \beta) = (RL^m)^\infty,$$

$$(\lambda, \beta) \in B_m \Leftrightarrow K(\lambda, \beta) = RL^{m-1} \star \underline{B} \text{ with } \underline{B} \in M$$

where M is set of kneading sequence for tent map $f_{\lambda,\lambda}$ $(1 < \lambda \leq 2)$.

A_m and B_m have common boundary curve : $\lambda^m \mu = 1$. In our opinion this curve should be discussed separately from A_m and from B_m. We find our reason in the fact that the kneading sequence on this curve is $RL^m RL^{m-1}C$, not admitted by one on A_m and on B_m.

Second : **topological entropy of $B_1(= D_0)$ is not constant.**

In Corollary in [MV92], they say ;

$$\text{let } (\lambda, \beta), (\lambda', \beta') \in \{(\lambda, \beta) \in D\, ;\, \lambda \leq 1\} \text{ such that } (\lambda, \beta) < (\lambda', \beta'),$$

$$(\lambda, \beta), (\lambda', \beta') \in A_m \cup B_m \;\Rightarrow\; h(\lambda, \beta) = h(\lambda', \beta').$$

Namely, topological entropy on B_m is constant for all $m\,(\geq 1)$. But we can show the followings :

Proposition. Let (λ, β), $(\lambda', \beta') \in \{(\lambda, \beta) \in D\, ;\, \lambda \leq 1\}$. If $(\lambda, \beta) < (\lambda', \beta')$,

$$h(\lambda, \beta) < h(\lambda', \beta').$$

Proof. From [MT88] we obtain that topological entropy of $f_{a,b}$ for B_1 naturally follows from one of its renormalized map of subdomain $a \geq 1$ where the strictly monotonicity holds. □

A counter example to this statement is given in [Ich] in this volume.

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References

[ITN79a] S. Ito, S. Tanaka, and H. Nakada. On unimodal linear transformations and chaos I. *Tokyo J. Math.*, 2(2):221–239, 1979.

[ITN79b] S. Ito, S. Tanaka, and H. Nakada. On unimodal linear transformations and chaos II. *Tokyo J. Math.*, 2(2):241–259, 1979.

[NY95] H. E. Nusse and J. A. Yorke. Border-collision bifurcations for piecewise smooth one-dimensional maps. *International Journal of Bifurcation and Chaos*, 5(1):189–207, 1995.

[CE80] P. Collet and J. P. Eckmann. *Iterated maps on the interval as dynamical systems. Progress in Physics.* Birkhauser, Boston, 1980.

[Dev89] R. L. Devaney. *An introduction to chaotic dynamical systems.* Addison-Wesley, 1989.

[MV91] J. C. Misiurewicz and E. Visinescu. Kneading sequences of skew tent maps. *Ann. Inst. Henri Poincare*, 27:125–140, 1991.

[MV92] J. C. Marcuard and E. Visinescu. Monotonicity properties of some skew tent maps. *Ann. Inst. Henri Poincare*, 28:1–29, 1992.

[MT88] J. Milnor and W. Thurston. *On iterated maps on the interval.* Springer LNM 1342 (1988),465-563.

[IN97a] K. Ichimura and K. Nishizawa. Bifurcations for skew tent maps I (stair type). *RIMS Kokyuroku*, 986:41–48. Kyoto Univ., 1997.

[IN97b] M. Ito and K. Nishizawa. Bifurcations for skew tent maps II (renormalization). *RIMS Kokyuroku*, 986:49–56. Kyoto Univ., 1997.

[Ich] K. Ichimura. Dynamics and kneading sequences of skew tent maps. *In this paper.*

[L85] Landau. Factoring polynomials over algebraic number fields. *SIAM J. Comput.*, 14:184–195, 1985.

[YNT90] Yokoyama, Noro and Takeshima. On factoring multi-variate polynomials over algebraically closed fields. *ISSAC'90* (extended abstract), 1990.

Appendix

Table 1: Calculation by Risa/Asir

```
This is Asir, Version 940420.
Copyright (C) FUJITSU LABORATORIES LIMITED.
3 March 1994. All rights reserved.
[0] F3=-b^2+(a^2-3*a)*b-a;
b*a^2+(-3*b-1)*a-b^2
[1] F4=-b^4-4*a*b^3+(a^3-6*a^2-a)*b^2-4*a^2*b-a^2;
b^2*a^3+(-6*b^2-4*b-1)*a^2+(-4*b^3-b^2)*a-b^4
[2] F5=-b^6-5*a*b^5+(-10*a^2-a)*b^4+(a^4-10*a^3-5*a^2)*b^3+(-10*a^3-a^2
)*b^2-5*a^3*b-a^3;
b^3*a^4+(-10*b^3-10*b^2-5*b-1)*a^3+(-10*b^4-5*b^3-b^2)*a^2+(-5*b^5-b^4)
*a-b^6
[3] fctr(F3);
[[1,1],[b*a^2+(-3*b-1)*a-b^2,1]]
[4] fctr(F4);
[[1,1],[b^2*a^3+(-6*b^2-4*b-1)*a^2+(-4*b^3-b^2)*a-b^4,1]]
[5] fctr(F5);
[[1,1],[b^3*a^4+(-10*b^3-10*b^2-5*b-1)*a^3+(-10*b^4-5*b^3-b^2)*a^2+(-5*
b^5-b^4)*a-b^6,1]]
[6] FF3=subst(F3,a,1);
-b^2-2*b-1
[7] fctr(FF3);
[[-1,1],[b+1,2]]
[8] FF3=subst(F3,a,-1);
-b^2+4*b+1
[10] fctr(res(c,subst(F3,b,b+c),subst(FF3,b,c)));
[[1,1],[(b^2+4*b-1)*a^4+(-6*b^2-26*b+2)*a^3+(-2*b^3-3*b^2+28*b+8)*a^2+(
6*b^3+38*b^2+50*b+6)*a+b^4+8*b^3+14*b^2-8*b+1,1]]
[11] FF4=subst(F4,a,-1);
-b^4+4*b^3-6*b^2-4*b-1
[12] fctr(res(c,subst(F4,b,b+c),subst(FF4,b,c)));
[[1,1],[(b^8+8*b^7+28*b^6+40*b^5+6*b^4-40*b^3+28*b^2-8*b+1)*a^12+(-24*b
^8-208*b^7-788*b^6-1320*b^5-604*b^4+800*b^3-492*b^2+152*b-12)*a^11+(-16
*b^9+68*b^8+1408*b^7+7032*b^6+14848*b^5+11782*b^4-3752*b^3+2988*b^2-744
*b+2)*a^10+(-4*b^10+248*b^9+1812*b^8+3520*b^7-10668*b^6-53800*b^5-75228
*b^4-9664*b^3-10516*b^2-600*b+164)*a^9+(168*b^10+48*b^9-8550*b^8-51120*
b^7-116904*b^6-89216*b^5+68840*b^4+40928*b^3+19376*b^2+5472*b+495)*a^8+
(48*b^11-1044*b^10-11544*b^9-42876*b^8-34800*b^7+169984*b^6+495088*b^5+
423656*b^4+208384*b^3+75784*b^2+13040*b+872)*a^7+(6*b^12-760*b^11-5156*
b^10-664*b^9+117222*b^8+524688*b^7+1063752*b^6+1069952*b^5+658828*b^4+3
13840*b^3+87384*b^2+15792*b+1052)*a^6+(-264*b^12-48*b^11+21784*b^10+169
280*b^9+612248*b^8+1249648*b^7+1491472*b^6+1170672*b^5+691144*b^4+27065
6*b^3+80792*b^2+12112*b+872)*a^5+(-48*b^13+988*b^12+17488*b^11+114920*b
^10+412656*b^9+897335*b^8+1213176*b^7+1071556*b^6+652520*b^5+278934*b^4
```

Table 1: Calculation by Risa/Asir

```
+106968*b^3+22396*b^2+4152*b+495)*a^4+(-4*b^14+488*b^13+7164*b^12+46624
*b^11+174060*b^10+403304*b^9+582252*b^8+504320*b^7+220020*b^6+12472*b^5
-26892*b^4-22816*b^3-9564*b^2-264*b+164)*a^3+(120*b^14+1744*b^13+11762*
b^12+46296*b^11+113612*b^10+170040*b^9+135038*b^8+18544*b^7-42304*b^6-1
4656*b^5+5198*b^4-6856*b^3+2876*b^2-72*b+2)*a^2+(16*b^15+244*b^14+1736*
b^13+7228*b^12+18656*b^11+28436*b^10+19448*b^9-6852*b^8-14448*b^7+3900*
b^6+8120*b^5-7532*b^4+3200*b^3-836*b^2+136*b-12)*a+b^16+16*b^15+120*b^1
4+528*b^13+1436*b^12+2256*b^11+1352*b^10-1328*b^9-1722*b^8+1328*b^7+135
2*b^6-2256*b^5+1436*b^4-528*b^3+120*b^2-16*b+1,1]]
```