On the boundaries of Baker domains

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1 Introduction

Let f be a transcendental entire function, $F_f \subset \mathbb{C}$ the Fatou set of f. We call a connected component U of F_f a Fatou component. Then U is either a wandering domain (that is, $f^m(U) \cap f^n(U) = \emptyset$ for all $n, m \in \mathbb{N}$) or eventually periodic (that is, $f^m(U)$ is periodic for an $m \in \mathbb{N}$). If it is periodic, it is well known that there are four possibilities; U is either an attractive basin, a parabolic basin, a Siegel disk, or a Baker domain.

Now in what follows let U be an unbounded invariant (that is, $f(U) \subseteq U$) Fatou component. Then it is known that U is simply connected ([**B**], [**EL**]) and so let $\varphi : \mathbb{D} \to U$ be a Riemann map of U. The boundary ∂U of U can be very complicated. For example, consider the exponential family $E_\lambda(z) := \lambda e^z$. If the parameter λ satisfies $\lambda = te^{-t}$ $|t| < 1$, then there exists a unique unbounded completely invariant attractive basin U which is equal to the Fatou set F_{E_λ} and ∂U is equal to the Julia set J_{E_λ} which is so called a Cantor bouquet. Moreover,

$$\Theta_\infty := \left\{ e^{i\theta} \;\middle|\; \varphi(e^{i\theta}) := \lim_{r \nearrow 1} \varphi(re^{i\theta}) = \infty \right\} \subset \partial\mathbb{D}$$

is dense in $\partial\mathbb{D}$ ([**DG**]). This implies that φ is highly discontinuous on $\partial\mathbb{D}$ and hence ∂U has a very complicated structure.

Baker and Weinreich investigated the boundary behavior of φ generally in the case of attractive basins, parabolic basins and Siegel disks and showed the following:

Theorem (Baker-Weinrech, [BW]) The point ∞ belongs to the impression of every prime end of U. $\square$

From the classical theory of prime end by Carathéodory it is well known that there is a 1 to 1 correspondence between $\partial\mathbb{D}$ and the set of all the prime ends of U. Let us denote $P(e^{i\theta})$ the prime end corresponding to the point $e^{i\theta} \in \partial\mathbb{D}$. The impression $\mathrm{Im}(P(e^{i\theta}))$ of a prime end $P(e^{i\theta})$ is a subset of ∂U which is known to be written as follows:

$$\mathrm{Im}(P(e^{i\theta})) = \{p \in \partial U \mid \text{for } {}^{\exists}z_n \in \mathbb{D} \ \text{ s.t. } z_n \to e^{i\theta},\ \varphi(z_n) \to p\}.$$

For the details of the theory of prime end, see for example, [**CL**]. Define the set $\mathrm{I}_\infty \subset \partial\mathbb{D}$ by

$$\mathrm{I}_\infty := \{e^{i\theta} \in \partial\mathbb{D} \mid \infty \in \mathrm{Im}(P(e^{i\theta}))\},$$

then the above result asserts that $I_\infty = \partial\mathbb{D}$ in the case of unbounded attractive basins, parabolic basins and Siegel disks. This shows that ∂U is extremely complicated.

On the other hand, ∂U can be very "simple" in the case when U is a Baker domain. For example,

$$f(z) := 2 - \log 2 + 2z - e^z$$

has a Baker domain U on which f is univalent and whose boundary ∂U is a Jordan curve (i.e. $\partial U \cup \{\infty\} \subset \widehat{\mathbb{C}}$ is a Jordan curve and $\partial U \subset \mathbb{C}$ is a Jordan arc, [**Ber**, Theorem 2]). In this case I_∞ consists of only a single point.

Then what can we say about the set I_∞ in general when U is a Baker domain? In this paper we give an answer to this problem.

2 Classification of Baker domains

In this section we classify Baker domains from the dynamical point of view. Now let U be an invariant Baker domain. By definition $f^n|U \to \infty$ $(n \to \infty)$ locally uniformly, so put

$$g := \varphi^{-1} \circ f \circ \varphi : \mathbb{D} \to \mathbb{D},$$

then g is conjugate to $f|U : U \to U$ and from the dynamics of $f|U$, g has no fixed point in $\mathbb{D}$. By the theorem of Denjoy and Wolff, there exists a unique point $p \in \partial\mathbb{D}$ (which is called Denjoy-Wolff point) and $g^n \to p$ locally uniformly. It is known that there exists a radial limit $c := \lim_{r \nearrow 1} g'(rp)$ with $0 < c \leq 1$, which means that p is either an attracting or a parabolic fixed point of the boundary map of g. Next let

$$z_n := g^n(0) \quad \text{and} \quad q_n := \frac{z_{n+1} - z_n}{1 - \overline{z_n} z_{n+1}},$$

then by the Schwarz-Pick's lemma $\{|q_n|\}_{n=1}^{\infty}$ turned out to be a decreasing sequence and hence there exists a limit $\lim_{n\to\infty} |q_n|$ ([P]). By using this limit and the value c, the dynamics of g on $\mathbb{D}$ can be classified for three different classes as follows. This result is essentially due to Baker and Pommerenke ([**BP**], [P]).

Theorem (1) If $c < 1$, then g is semi-conjugate to a hyperbolic Möbius transformation $\psi : \mathbb{D} \to \mathbb{D}$ with $\psi(z) = \dfrac{(1+c)z + 1 - c}{(1-c)z + 1 + c}$.
(2) If $c = 1$ and $\lim_{n\to\infty} |q_n| > 0$, then g is semi-conjugate to a parabolic Möbius transformation $\psi : \mathbb{D} \to \mathbb{D}$ with $\psi(z) = \dfrac{(1 \pm 2i)z - 1}{z - 1 \pm 2i}$.
(3) If $c = 1$ and $\lim_{n\to\infty} |q_n| = 0$, then g is semi-conjugate to a parabolic Möbius transformation $\psi : \mathbb{C} \to \mathbb{C}$ with $\psi(z) = z + 1$. □

König investigated the relation between the above classification and the dynamics of $f|U : U \to U$ and obtained the following result:

Theorem (König, [K]) Let $w_0 \in U$ and define

$$w_n := f^n(w_0) \quad \text{and} \quad d_n := \text{dist}\ (w_n, \partial U),$$

where "dist" is a Euclidean distance. Then

(1) $f|U$ is semi-conjugate to a hyperbolic Möbius transformation $\psi : \mathbb{D} \to \mathbb{D}$ if and only if there exists a constant $\beta = \beta(f) > 0$ such that

$$\frac{|w_{n+1} - w_n|}{d_n} \geq \beta \quad (n \in \mathbb{N})$$

holds for any $w_0 \in U$.

(2) $f|U$ is semi-conjugate to a parabolic Möbius transformation $\psi : \mathbb{D} \to \mathbb{D}$ if and only if

$$\liminf_{n\to\infty} \frac{|w_{n+1} - w_n|}{d_n} > 0$$

holds for any $w_0 \in U$ but

$$\inf_{w_0 \in U} \limsup_{n\to\infty} \frac{|w_{n+1} - w_n|}{d_n} = 0.$$

(3) $f|U$ is semi-conjugate to a parabolic Möbius transformation $\psi : \mathbb{C} \to \mathbb{C}$ with $\psi(z) = z + 1$ if and only if

$$\lim_{n\to\infty} \frac{w_{n+1} - w_n}{d_n} = 0$$

holds for any $w_0 \in U$. □

For each cases König also gave concrete examples satisfying the above conditions:

(1) $f(z) = 3z + e^{-z}$,

(2) $f(z) = z + 2\pi i\alpha + e^z$, where $\alpha \in (0, 1)$ satisfies the Diophantine condition,

(3) $f(z) = e^{\frac{2\pi i}{p}} \left(z + \int_0^z e^{-\zeta^p} d\zeta \right)$, where $p \in \mathbb{N},\ p \geq 2$.

Note that in the case (3), the function f above has a Baker domain of period $p \geq 2$, not an invariant one. Of course, if we consider f^p instead of f, f^p has an invariant Baker domain.

3 Result and the outline of the proof

With the above classification, we can state our main theorem as follows:

Main Theorem Let f be a transcendental entire function and suppose that f has an invariant Baker domain U. Let $\varphi : \mathbb{D} \to U$ be a Riemann map of U and the set I_∞ as above. Assume that $f|U : U \to U$ is not univalent.

(1) If $f|U$ is semi-conjugate to a hyperbolic Möbius transformation $\psi : \mathbb{D} \to \mathbb{D}$, then I_∞ contains a perfect set $K \subset \partial\mathbb{D}$.

(2) If $f|U$ is semi-conjugate to a parabolic Möbius transformation $\psi : \mathbb{D} \to \mathbb{D}$, then I_∞ contains a perfect set $K \subset \partial\mathbb{D}$.

(3) If $f|U$ is semi-conjugate to a parabolic Möbius transformation $\psi : \mathbb{C} \to \mathbb{C} \quad z \mapsto z+1$, then $\mathrm{I}_\infty = \partial\mathbb{D}$.

If $f|U$ is univalent, then $^{\#}\mathrm{I}_\infty = 1, 2$ or ∞.

(Outline of the Proof) : Since $U \subset \mathbb{C}$ is unbounded, we have $I_\infty \neq \emptyset$ and it is easy to see that I_∞ is a closed subset of $\partial\mathbb{D}$. Then $\partial\mathbb{D} \setminus I_\infty$ is open and it can be shown that g can be analytically continued over $\partial\mathbb{D} \setminus I_\infty$. So in particular g is analytic on $\partial\mathbb{D} \setminus I_\infty$ and we have

$$g(\partial\mathbb{D} \setminus I_\infty) \subseteq \partial\mathbb{D} \setminus I_\infty.$$

If g is a d to 1 map ($2 \leq d < \infty$), then g is a finite Blaschke product of degree d and its Julia set J_g is either $\partial\mathbb{D}$ or a Cantor set (in particular, it is a perfect set) in $\partial\mathbb{D}$. Assume that $J_g \cap (\partial\mathbb{D} \setminus I_\infty) \neq \emptyset$, then from the general property of the dynamics of rational maps and the g-invariance of $\partial\mathbb{D} \setminus I_\infty$ we have

$$\partial\mathbb{D} \subset \partial\mathbb{D} \setminus I_\infty,$$

that is, $I_\infty = \emptyset$, which is a contradiction. Therefore we have $J_g \subset I_\infty$. This proves the case (1) and (2) with a further assumption that g is a finite to one map.

If g is an ∞ to 1 map, we can show that

$$\overline{\bigcup_{n=1}^{\infty} g^{-n}(z_0)} \cap \partial\mathbb{D} \subset I_\infty$$

holds for every $z_0 \in \mathbb{D}$ (there may be some exception) and the set $\overline{\bigcup_{n=1}^{\infty} g^{-n}(z_0)} \cap \partial\mathbb{D}$ is either equal to $\partial\mathbb{D}$ or at least contains a certain perfect set $K \subset \partial\mathbb{D}$. This result comes from a property of g as a boundary map $g : \partial\mathbb{D} \to \partial\mathbb{D}$. This completes the proof for the case (1) and (2).

For the case (3), since we have $\lim_{n\to\infty} |q_n| = 0$, we can obtain that

$$\overline{\bigcup_{n=1}^{\infty} g^{-n}(z_0)} \cap \partial\mathbb{D} = \partial\mathbb{D} \subset I_\infty,$$

and hence $I_\infty = \partial\mathbb{D}$. This fact comes from the ergodic property of g as an inner function. This completes the proof for the case (3).

If g is univalent, then g is either hyperbolic or parabolic Möbius transformation. g has either one or two fixed point and the every orbit of a point other than the fixed points has infinitely many points. On the other hand, we have

$$g(\partial\mathbb{D} \setminus I_\infty) \subseteq \partial\mathbb{D} \setminus I_\infty,$$

so we can conclude that $^{\#}\mathrm{I}_\infty = 1, 2$ or ∞. $\square$

Of course, we can obtain the same result when U is a periodic Baker domain.

References

[B] I. N. Baker, Wandering domains in the iteration of entire functions, *Proc. London Math. Soc. (3),* **49** (1984), 563–576.

[BP] I. N. Baker and CH. Pommerenke, On the iteration of analytic functions in a halfplane II, *J. London Math. Soc. (2),* **20** (1979), 255–258.

[BW] I. N. Baker and J. Weinreich, Boundaries which arise in the dynamics of entire functions, *Revue Roumaine de Math. Pures et Appliquées,* **36** (1991), 413–420.

[Ber] W. Bergweiler, Invariant domains and singularities, *Math. Proc. Camb. Phil. Soc.* **117** (1995), 525–532.

[CL] E. F. Collingwood and A. J. Lohwater, *The theorey of cluster sets*, Cambridge University Press, 1966.

[DG] R. L. Devaney and L. R. Goldberg, Uniformization of attracting basins for exponential maps, *Duke. Math. J.* **55** No.2 (1987), 253–266.

[EL] A. E. Eremenko and M. Yu. Lyubich, The dynamics of analytic transformations, *Leningrad Math. J.* 1 No.3 (1990), 563–634.

[K] H. König, Konforme Konjugation in Baker-Gebieten, PhD Thesis, Universität Hannover (1996), 1–79.

[P] CH. Pommerenke, On the iteration of analytic functions in a halfplane, I, *J. London Math. Soc. (2)*, **19** (1979), 439–447.