Topological Pressure and Conformal Measures in Semigroup Dynamics

Hiroki Sumi *
Graduate School of Human and Environmental Studies,
Kyoto University, kyoto 606, Japan
e-mail; sumi@math.h.kyoto-u.ac.jp

June 1997

Abstract

We consider dynamics of semigroups of rational functions on Riemannn sphere. First, we will define hyperbolic rational semigroups and show the metrical property. We will also define subhyperbolic rational semigroups and show that if G is finitely generated subhyperbolic rational semigroup containing an element with the degree at least two and each Möbius transformation in G is hyperbolic or loxodromic, then there is no wandering domain. Also we can show the continuity of the Julia set with respect to the pertubation of the generators.

Next, we will consider constructing pseudo δ-conformal measures on the Julia sets. If a finitely generated semigroup satisfies the strong open set condition, then we can construct δ-conformal measures on the Julia set. Using this measures, we get an upper estimate of the Hausdorff dimension of the Julia sets of finitely generated expanding semigroups.

Considering conformal measures in a skew product, with a method of the thermodynamical formalism, we can get another upper estimate of the Hausdorff dimension of the Julia sets of finitely generated expanding semigroups.

In more general cases than the cases in which semigroups are hyperbolic or satisfy the strong open set condition, we can construct generalized Brolin-Lyubich's invariant measures or self-similar measures in the Julia sets and can show the uniqueness. We will get a lower estimate of the metric entropy of the invariant measures. With these facts and a generalization of Mañé's result, we get a lower estimate of the Hausdorff dimension of any finitely generated rational semigroups such that the backward images of the Julia sets by the generators are mutually disjoint.

1 Introduction

For a Riemann surface S, let $\text{End}(S)$ denote the set of all holomorphic endomorphisms of S. It is a semigroup with the semigroup operation being composition of functions. A *rational semigroup* is a subsemigroup of $\text{End}(\overline{\mathbb{C}})$ without any constant elements.Similarly, an *entire semigroup* is a subsemigroup of $\text{End}(\mathbb{C})$ without any constant elements. A rational semigroup G is called a *polynomial semigroup* if each $g \in G$ is a polynomial. When a rational or entire semigroup G is generated by $\{f_1, f_2, \dots f_n, \dots\}$, we denote this situation by

$$G = \langle f_1, f_2, \dots f_n, \dots \rangle.$$

A rational or entire semigroup generated by a single function g is denoted by $\langle g \rangle$. We denote the n th iterate of f by f^n.

The studies of dynamics of rational semigroups were introduced by W.Zhou and F.Ren[ZR], Z.Gong and F.Ren[GR] and Hinkkanen and Martin[HM1]. Some properties of dynamics of rational semigroups were studied in [HM1], [HM2], [S1], [S2].

In [S3], dynamics of hyperbolic rational semigroups are investigated and it is shown that all limit functions of finitely generated rational semigroups on the Fatou sets are constant functions that take their values in the post critical sets. Also with respect to pertubations of generators of any finitely generated hyperbolic rational semigroup, the hyperbolicity is kept and the Julia set moves continuously.

In this paper, we will define *subhyperbolic* rational semigroups and show that if G is finitely generated subhyperbolic rational semigroup containing an element with the degree at least two and each Möbius transformation in G is hyperbolic or loxodromic, then there is no wandering domain. Also we will discuss about the continuity of the Julia set.

In [S4], we will show that if a finitely generated rational semigroup contains an element of degree at least two and each Möbius transformation in it is neither the identity nor an elliptic element, then the hyperbolicity and expandingness are equivalent. If the sets of backward images of the Julia set by generators are almost disjoint, then the Julia set has no interior points. We construct a generalized δ-conformal measure on the Julia set of any rational semigroup which satisfies the *strong open set condition.* We show that if the semigroup is hyperbolic, then the Hausdorff dimension of the Julia set coincides with the unique value δ that allows us to construct a δ-conformal measure and it is strictly less than 2. Also the δ- Hausdorff measure of the Julia set is a finite value strictly bigger than zero. Considering the convergent series of the norm of the derivative at the backward images, With the similar method to the! ! construction of the Patterson-Sullivan measures on the limit sets of Kleinian groups we get a pseudo δ-conformal measure in more general case and we will show that if a finitely generated rational semigroup is expanding, then the Hausdorff dimension of the Julia set is less than the exponent δ.

Generalized Brolin-Lyubich's invariant measures on the Julia set of any rational semigroup which is hyperbolic or satisfying the strong open set condition are constructed in [S5] and a lower estimate of the Hausdorff dimension of the rational semigroups is given.

In this paper and [S6], the author will discuss about the existance and uniquness of the conformal measures and self-similar measures of rational semigroups in more general cases. We use the thermodynamic formalism and give an upper bound of the Hausdorff dimension of the Julia sets of finitely generated hyperbolic rational semigroups. Also we construct invariant measures or self-similar measures on Julia sets of any finitely generated rational semigroups and will estimate the metric entropy of the invariant measures of the skew product maps. If $G = \langle f_1, f_2, \dots f_m \rangle$ is finitely generated rational semigroup and the sets $\{f_i^{-1}(J(G))\}_i$ are mutually disjoint, then by a generalization of Mañé's relult and the estimate of the metric entropy, we will get

$$\dim_H(J(G)) \geq \frac{\log(\sum_{j=1}^{m} \deg(f_j))}{\int_{J(G)} \log \|f'(z)\| d\mu(z)},$$

where the map $f : J(G) \to J(G)$ is defined by $f(z) = f_i(z)$ if $z \in f_i^{-1}(J(G))$.

Definition 1.1 Let G be a rational semigroup.

$$F(G) \stackrel{\text{def}}{=} \{z \in \overline{\mathbb{C}} \mid G \text{ is normal in a neighborhood of } z\}$$

$$J(G) \stackrel{\text{def}}{=} \overline{\mathbb{C}} \setminus F(G)$$

$F(G)$ is called the *Fatou set* for G and $J(G)$ is called the *Julia set* for G.

$J(G)$ is backward invariant under G but not forward invariant in general. If $G = \langle f_1, f_2, \dots f_n \rangle$ is a finitely generated rational semigroup , then $J(G)$ has the *backward self-similarity*. That is, we have

$$J(G) = \cup_{i=1}^{n} f_i^{-1}(J(G)).$$

The Julia set of any rational semigroup is a perfect set, backward orbit of any point of the Julia set is dense in the Julia set and the set of repelling fixed points of the semigroup is dense in the Julia set. For more detail about these properties, see [ZR], [GR], [HM1], [HM2], [S1] and [S2]. In general Julia sets may have non-empty interior points and be not the Riemann sphere. For example, $J(\langle z^2, 2z \rangle)$ is the closure of the unit disc. In [HM2], it was shown that if G is a finitely generated rational semigroup, then each super attracting fixed point of any element of $g \in G$ does not belong to the boundary of the Julia set. So we can construct many examples such that the Julia sets have non-empty interior points.

In [S4], it was shown that if $G = \langle f_1, f_2, \dots f_n \rangle$ is a finitely generated rational semigroup and the set $\cup_{(i,j):i \neq j} f_i^{-1}(J(G)) \cap f_j^{-1}(J(G))$ does not contain any continuum, then the Julia set $J(G)$ has no interior points. Note that this result solves the Problem 3,4 in [Re].

2 (Sub)hyperbolicity and Strong Open Set Condition

Definition 2.1 Let G be a rational semigroup. We set

$$P(G) = \overline{\bigcup_{g \in G} \{ \text{ critical values of } g\}}$$

and we say that G is hyperbolic if $P(G) \subset F(G)$. We call $P(G)$ the post critical set of G.

Definition 2.2 Let $G = \langle f_1, f_2, \dots f_n \rangle$ be a finitely generated rational semigroup. We say that G satisfies strong open set condition if there is an open neighborhood O of $J(G)$ such that each set $f_j^{-1}(O)$ is included in O and is mutually disjoint.

In [S3], dynamics of hyperbolic rational semigroups were investigated and it was shown that if a finitely generated rational semigroup is hyperbolic and each Möbius transformation in the semigroup is neither identity nor elliptic, then all limit functions of the semigroup on any component of the Fatou set are constant functions that take their values in the post critical sets. Also with respect to pertubations of generators of any finitely generated hyperbolic rational semigroup, the hyperbolicity is kept and the Julia set moves continuously.

If a finitely generated rational semigroup satisfies the strong open set condition, then the Julia set has no interior points.

Now we consider the expandingness of hyperbolic rational semigroups, which gives us an information about the analytic property of them.

Theorem 2.3 ([S4]) *Let $G = \langle f_1, f_2, \dots f_n \rangle$ be a finitely generated hyperbolic rational semigroup . Assume that G contains an element with the degree at least two and each Möbius transformation in G is neither the identity nor an elliptic element. Let K be a compact subset of $\overline{\mathbb{C}} \setminus P(G)$. Then there are a positive number c, a number $\lambda > 1$ and a conformal metric ρ on an open subset V of $\overline{\mathbb{C}} \setminus P(G)$ which contains $K \cup J(G)$ and is backward invariant under G such that for each k*

$$\inf\{\|(f_{i_k} \circ \cdots \circ f_{i_1})'(z)\|_\rho \mid z \in (f_{i_k} \circ \cdots \circ f_{i_1})^{-1}(K), (i_k, \dots, i_1) \in \{1, \dots, n\}^k\}$$

$\geq c\lambda^k$, *here we denote by $\|\cdot\|_\rho$ the norm of the derivative measured from the metric ρ to it.*

Now we will show the converse of Theorem 2.3.

Theorem 2.4 ([S4]) *Let $G = \langle f_1, f_2, \dots f_n \rangle$ be a finitely generated rational semigroup. If there are a positive number c, a number $\lambda > 1$ and a conformal metric ρ on an open subset U containing $J(G)$ such that for each k*

$$\inf\{\|(f_{i_k} \circ \cdots \circ f_{i_1})'(z)\|_\rho \mid z \in (f_{i_k} \circ \cdots \circ f_{i_1})^{-1}(J(G)), (i_k, \dots, i_1) \in \{1, \dots, n\}^k\}$$

$\geq c\lambda^k$, *where we denote by* $\|\cdot\|_\rho$ *the norm of the derivative measured from the metric* ρ *on* V *to it, then* G *is hyperbolic and for each* $h \in G$ *such that* $\deg(h)$ *is one the map* h *is not elliptic.*

Remark. Because of the compactness of $J(G)$, we can show, with an easy argument, which is familiar to us in the iteration theory of rational functions, that even if we exchange the metric ρ to another conformal metric ρ_1, the enequality of the assumption holds with the same number λ and a different constant c_1.

Definition 2.5 Let $G = \langle f_1, f_2, \dots f_n \rangle$ be a finitely generated rational semigroup. We say that G is expanding if the assumption in Theorem 2.4 holds.

Theorem 2.6 *Let* $G = \langle f_1, f_2, \dots f_m \rangle$ *be a finitely generated expanding rational semigroup. Assume that*

- G *is expanding, and*
- *there is an open set* O *such that* $\sharp(\partial O \cap J(G)) < \infty$, *for each* j, $f_j^{-1}(O) \subset O$ *and* $\{f_j^{-1}(O)\}_{j=1,\dots,m}$ *are mutually disjoint.*

Then 2-dimensional Lebesgue measure of $J(G)$ *is equal to* 0.

Proof. With the assumption of our theorem, we can show that for each $x \in J(G) \setminus (G^{-1}G(\partial O))$, the orbit $G(x) \cap J(G)$ has an accumulation point in $J(G) \setminus (G(\partial O) \cup \partial O)$. By Koebe theorem, the statement holds. □

Definition 2.7 Let G be a rational semigroup and U be a component of $F(G)$. For every element g of G, we denote by U_g the connected component of $F(G)$ containing $g(U)$. We say that U is a wandering domain if $\{U_g\}$ is infinite.

Next theorem follows from the argument in Theorem 2.2.4 in [S3].

Theorem 2.8 *Let* G *be a rational semigroup with* $F(G) \neq \emptyset$. *Assume that* G *contains an element with the degree at least two and* $P(G) \cap \partial J(G) = \emptyset$. *Then there is no wandering domain. Moreover, if* G *is finitely generated, each Möbius transformation in* G *is hyperbolic or loxodromic and there exists no element* $g \in G$ *such that* g *has Siegel disks or Herman rings, then there exists a non-empty compact set* K *in* $P(G) \cap F(G)$ *such that for each* $z \in F(G)$, *the orbit* $G(z)$ *can accumulate only in* K.

Propotition 2.9 *Let G be a rational semigroup. Assume that G contains an element g with the degree at least two such that g has no Siegel disks or Herman rings. If $P(\langle g\rangle)$ is included in the interior of $J(G)$, then $J(G)$ is equal to $\overline{\mathbb{C}}$.*

Proof. Assume $F(G) \neq \emptyset$. Let U be a connected component of $F(G)$. Considering $\{g^n(U)\}$, it is a contradiction. □

Definition 2.10 Let G be a rational semigroup. We say that G is subhyperbolic if $\sharp P(G) \cap J(G) < \infty$ and $P(G) \cap F(G)$ is compact.

Theorem 2.11 *Let $G = \langle f_1, f_2, \dots f_m\rangle$ be a finitely generated rational semigroup which is subhyperbolic. Assume that there is an element of G with the degree at least two and each Möbius transformation in G is hyperbolic or loxodromic. Then if $F(G) \neq \emptyset$, there exists a non-empty compact set K in $P(G) \cap F(G)$ such that for each $z \in F(G)$, the orbit $G(z)$ can accumulate only in K. In particular, there is no wandering domain.*

Proof. Using a similar argument in Theorem 2.2.8 in [S3], we have only to show that for each connected component U of $F(G)$, $\sharp\{U_g \mid g \in G\} < \infty$. Now assume $\sharp\{U_g \mid g \in G\} = \infty$. By Theorem 2.2.4 in [S3], there exists a sequence (g_n) of mutually distinct elements in G and a point $\zeta \in P(G) \cap \partial J(G)$ such that (g_n) converges to ζ locally uniformly in U. Since G is finitely generated, we can assume that for each n, there exists an element $h_n \in G$ such that $g_{n+1} = h_n g_n$. Then for each sufficiently large n, $h_n(\zeta) = \zeta$. Now we consider $|h_n'(\zeta)|$. By [HM2], there is no super attracting fixed point of any element of G in $\partial J(G)$. With the fact, since G is subhyperbolic, it follows that ζ is a repelling fixed point of h_n. But this is a contradiction by Koebe theorem. □

Theorem 2.12 *Let G be a finitely generated rational semigroup which contains an element with the degree at least two. Assume that $\sharp P(G) < \infty$ and $P(G) \subset J(G)$. Then $J(G) = \overline{\mathbb{C}}$.*

Proof. By [HM2], there is no super attracting fixed point of any element of G in $\partial J(G)$. Now we can show the statement in the same way as Propotition 2.9. □

By Theorem 2.11 and Theorem 2.3.4 in [S3], we get the following result.

Theorem 2.13 *Let M be a complex manifold. Let $\{G_a\}_{a\in M}$ be a holomorphic family of rational semigroups (See the definition in [S3]) where $G_a = \langle f_{1,a}, \cdots, f_{n,a}\rangle$. We assume that for a point $b \in M$, G_b is subhyperbolic, contains an element of the degree at least two and each Möbius transformation in G_b is hyperbolic or loxodromic. Then the map*

$$a \mapsto J(G_a)$$

is continuous at the point $a = b$ with respect to the Hausdorff metric.

3 δ-Conformal Measure

We construct δ-conformal measures on Julia sets of rational semigroups. δ-conformal measures on Julia sets of rational functions were introduced in [Sul].

Definition 3.1 Let $G = \langle f_1, f_2, \dots f_n\rangle$ be a finitely generated rational semigroup satisfying the strong open set condition and let δ be a non-negative number. We say that a probability measure μ on $J(G)$ is δ-conformal if for each $j = 1, \dots, n$ and for each measurable set A included in $f_j^{-1}(J(G))$ where f_j is injective on A,

$$\mu(f_j(A)) = \int_A \|f_j'(z)\|^\delta d\mu,$$

where $\|\cdot\|$ denotes the norm of the derivative with respect to the spherical metric. And we set

$$\delta(G) = \inf\{\delta \mid \text{there is a } \delta\text{-conformal measure on } J(G)\}.$$

Theorem 3.2 ([S4]) *Let $G = \langle f_1, f_2, \dots f_n\rangle$ be a finitely generated rational semigroup satisfying the strong open set condition. We assume that when n is equal to one the degree of f_1 is at least two. Then there are a number $0 < \delta \leq 2$ and a probability measure μ whose support is equal to $J(G)$ such that μ is δ-conformal. Also $\delta(G) > 0$.*

If G is finitely generated hyperbolic rational semigroup and satisfies the strong open set condition, then $\dim_H(J(G)) = \delta(G)$ ([S4]). In [DU], M.Denker and M.Urbański gave a conjecture which states that for any rational map f with $\deg(f) \geq 2$,

$$\dim_H(J(\langle f\rangle)) = \delta(\langle f\rangle).$$

Similary we give the following conjecture.

Conjecture 3.3 ([S4]) *Let $G = \langle f_1, f_2, \ldots f_n\rangle$ be a finitely generated rational semigroup satisfying the strong open set condition. We assume that when n is equal to one the degree of f_1 is at least two. Then*

$$\dim_H(J(G)) = \delta(G).$$

4 Pseudo δ-Conformal Measure

Definition 4.1 Let G be a rational semigroup and δ be a non-negative number. We say that a probability measure μ on $J(G)$ is pseudo δ-conformal if for each $g \in G$ and for each measurable set A included in $g^{-1}(J(G))$ where g is injective on A,

$$\mu(g(A)) \leq \int_A \|g'(z)\|^\delta d\mu.$$

For each $x \in \overline{\mathbb{C}}$ we set

$$S(\delta,\ x) = \sum_{g\in G}\sum_{g(y)=x} \|g'(y)\|^{-\delta}$$

counting multiplicities and

$$S(x) = \inf\{s \mid S(s,\ x) < \infty\}.$$

If there is not s such that $S(s,\ x) < \infty$, then we set $S(x) = \infty$.Also we set

$$s_0(G) = \inf\{S(x)\},\ s(G) = \inf\{\delta \mid \exists\mu : \text{pseudo } \delta\text{-conformal measure}\}$$

where the former infimum is taken over all points x such that $O^-(x)$ does not accumulate at any point of $F(G)$.

Using the same method of the proof of Theorem 3.2, we can show the following result.

Theorem 4.2 ([S4]) *Let G be a rational semigroup which has at most countably many elements. If there exists a point $x \in \overline{\mathbb{C}}$ such that $S(x) < \infty$ and $O^-(x)$ does not accumulate at any point of $F(G)$, then there is a pseudo $S(x)$-conformal measure whose support is equal to $J(G)$. In particular, there is a pseudo $s_0(G)$-conformal measure. Also we have $s(G) > 0$.*

Theorem 4.3 ([S4]) *Let $G = \langle f_1, f_2, \dots f_n \rangle$ be a finitely generated rational semigroup which is expanding. Then $s_0(G) < \infty$ and*

$$\dim_H(J(G)) \leq s(G) \leq s_0(G).$$

5 Conformal Measures in a Skew Product

Let m be a positive integer. We denote by Σ_m the one-sided word space, that is

$$\Sigma_m = \{1, \dots, m\}^{\mathbb{N}}$$

and denote by $\sigma : \Sigma_m \to \Sigma_m$ the shift map, that is

$$(w_1, \dots) \mapsto (w_2, \dots).$$

Let $G = \langle f_1, f_2, \dots f_m \rangle$ be a finitely generated rational semigroup. We define a map $\tilde{f} : \Sigma_m \times \overline{\mathbb{C}} \to \Sigma_m \times \overline{\mathbb{C}}$ by

$$\tilde{f}((w, x)) = (\sigma w, f_{w_1} x).$$

$\tilde{f}$ is a finite-to-one and open map. We have that a point $(w, x) \in \Sigma_m \times \overline{\mathbb{C}}$ satisfies $f'_{w_1}(x) \neq 0$ if and only if $\tilde{f}$ is a homeomorphism in a small neighborhood of (w, x). Hence the map $\tilde{f}$ has infinitely many critical points. We set $\tilde{J} = \cap_{n=0}^{\infty}(\Sigma_m \times J(G))$. Then by definition, $\tilde{f}^{-1}(\tilde{J}) = \tilde{J}$. Also from the backward self-similarity of $J(G)$, we can show that $\pi(\tilde{J}) = J(G)$ where $\pi : \Sigma_m \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is the second projection.

For each $j = 1, \dots, m$, let φ_j be a Hölder continuous function on $f_j^{-1}(J(G))$. We set for each $(w, x) \in \tilde{J}$, $\varphi((w, x)) = \varphi_{w_1}(x)$. Then φ is a Hölder continuous function on $\tilde{J}$. We define an operater L on $C(\tilde{J}) = \{\psi : \tilde{J} \to \mathbb{C} \mid \text{continuous}\}$ by

$$L\psi((w, x)) = \sum_{\tilde{f}((w', y)) = (w, x)} \frac{\exp(\varphi((w', y)))}{\exp(P)} \psi((w', y)),$$

counting multiplicities, where we denote by $P = P(\tilde{f}|_{\tilde{J}}, \varphi)$ the pressure of $(\tilde{f}|_{\tilde{J}}, \varphi)$.

Lemma 5.1 *With the same notations as the above, let $G = \langle f_1, f_2, \dots f_m \rangle$ be a finitely generated expanding rational semigroup. Then for each set of Hölder continuous functions $\{\varphi_j\}_{j=1,\dots,m}$, there exists a unique probability measure τ on $\tilde{J}$ such that*

- $L^* \tau = \tau$,
- *for each $\psi \in C(\tilde{J})$, $\|L^n \psi - \tau(\psi)\alpha\|_{\tilde{J}} \to 0, n \to \infty$, where we set $\alpha = \lim_{l \to \infty} L^l(1) \in C(\tilde{J})$ and we denote by $\|\cdot\|_{\tilde{J}}$ the supremum norm on $\tilde{J}$,*
- *$\alpha\tau$ is an equilibrium state for $(\tilde{f}|_{\tilde{J}}, \varphi)$.*

Lemma 5.2 *Let $G = \langle f_1, f_2, \dots f_m \rangle$ be a finitely generated expanding rational semigroup. Then there exists a unique number $\delta > 0$ such that if we set $\varphi_j(x) = -\delta \log(\|f_j'(x)\|), j = 1, \dots, m$, then $P = 0$.*

From Lemma 5.1, for this δ there exists a unique probability measure τ on $\tilde{J}$ such that $L_\delta^* \tau = \tau$ where L_δ is an operator on $C(\tilde{J})$ defined by

$$L_\delta \psi((w, x)) = \sum_{\tilde{f}((w', y)) = (w, x)} \frac{\psi((w', y))}{\|(f_{w_1'})'(y)\|^\delta}.$$

Also δ satisfies that

$$\delta = \frac{h_{\alpha\tau}(\tilde{f})}{\int_{\tilde{J}} \tilde{\varphi} \alpha d\tau} \leq \frac{\log(\sum_{j=1}^m \deg(f_j))}{\int_{\tilde{J}} \tilde{\varphi} \alpha d\tau},$$

where $\alpha = \lim_{l \to \infty} L_\delta^l(1)$, we denote by $h_{\alpha\tau}(\tilde{f})$ the metric entropy of $(\tilde{f}, \alpha\tau)$ and $\tilde{\varphi}$ is a function on $\tilde{J}$ defined by $\tilde{\varphi}((w, x)) = \log(\|f_{w_1}'(x)\|)$.

By these argument, we get the following result.

Theorem 5.3 *Let $G = \langle f_1, f_2, \dots f_m \rangle$ be a finitely generated expanding rational semigroup and δ the number in the above argument. Then*

$$\dim_H(J(G)) \leq s(G) \leq \delta.$$

Moreover, if the sets $\{f_j^{-1}(J(G))\}$ are mutually disjoint, then $\dim_H(J(G)) = \delta < 2$ and $0 < H_\delta(J(G)) < \infty$, where we denote by H_δ the δ-Hausdorff measure.

Corollary 5.4 *Let $G = \langle f_1, f_2, \dots f_m \rangle$ be a finitely generated expanding rational semigroup. Then*

$$\dim_H(J(G)) \leq \frac{\log(\sum_{j=1}^m \deg(f_j))}{\log \lambda},$$

where λ denotes the number in Definition 2.5.

6 Generalized Brolin-Lyubich's Invariant Measure, Self-Similar Measure

With the same notation as the previous section, we define an operator $\tilde{A}$ on $C(\tilde{J})$ by

$$\tilde{A}\tilde{\psi}((w,x)) = \frac{1}{\sum_{j=1}^m \deg(f_j)} \sum_{\tilde{f}((w',y))=(w,x)} \tilde{\psi}((w',y)), \text{ for each } \tilde{\psi} \in C(\tilde{J}),$$

and an operator A on $C(J(G)) = \{\psi : J(G) \to \mathbb{C} \mid \text{ continuous}\}$ by

$$A\psi(x) = \frac{1}{\sum_{j=1}^m \deg(f_j)} \sum_{j=1}^m \sum_{f_j(y)=x} \psi(y), \text{ for each } \psi \in C(J(G)).$$

Then $\tilde{A} \circ \pi^* = \pi^* \circ A$, where π^* is the map from $C(J(G))$ to $C(\tilde{J})$ defined by $(\pi^*\psi)((w,x)) = \psi(x)$. Note that since $\pi(\tilde{J}) = J(G)$, we have that for each $\psi \in C(J(G))$,

$$||\pi^*\psi||_{\tilde{J}} = ||\psi||_{J(G)}. \tag{1}$$

Now we consider a condition such that the invariant measures are unique.

Definition 6.1 Let $G = \langle f_1, f_2, \dots f_m \rangle$ be a finitely generated rational semigroup. With the same notation as the previous section, we say that G satisfies condition $*$ if for any $z \in \tilde{J} \setminus \text{per}(\tilde{f})$, for any $\epsilon > 0$, there exists a positive integer $n_0 = n_0(z, \epsilon)$ such that

$$\frac{\sharp\{\tilde{f}^{-n_0}(z) \cap Z_\infty\}}{(\sum_{j=1}^m \deg(f_j))^{n_0}} < \epsilon, \tag{2}$$

counting multiplicities, where we set

$$Z_\infty = \cup_{n=1}^\infty \tilde{f}^n(\{\text{ critical points of } \tilde{f}\} \cap \tilde{J}). \tag{3}$$

Remark. Let $G = \langle f_1, f_2, \dots f_m \rangle$ be a finitely generated rational semigroup. In each case of the following, the condition $*$ holds.

- There exists an element f such that for each $j = 1, \dots, m$, $f_j = f$.
- The sets $\{f_i^{-1}(J(G))\}_{i=1,\dots,m}$ are mutually disjoint.
- $J(G) \setminus \overline{\cup_{g \in G}\{\text{critical values of } g\} \cap J(G)} \neq \emptyset$.

Therefore we have many finitely generated rational semigroups satisfying condition $*$. It seems to be true that the condition $*$ holds if a finitely generated rational semigroup G satisfies that $J(G) \cap E(G) = \emptyset$, where $E(G)$ denotes the exceptional set of G, that is $E(G) = \{z \in \overline{\mathbb{C}} \mid \sharp\{\cup_{g \in G} g^{-1}(z)\} < \infty\}$.

Theorem 6.2 *Let $G = \langle f_1, f_2, \dots f_m \rangle$ be a finitely generated rational semigroup. Assume that $F(H) \supset J(G)$, where we set $H = \{g^{-1} \in Aut\ (\overline{\mathbb{C}}) \mid g \in Aut\ (\overline{\mathbb{C}}) \cap G\}$, and condition $*$ holds. Then we have the following:*

1. *There exists a unique probability measure $\tilde{\mu}$ on $\tilde{J}$ such that*

$$||\tilde{A}^n \tilde{\varphi} - \tilde{\mu}(\tilde{\varphi})\, \mathbf{1}_{\tilde{J}}||_{\tilde{J}} \to 0,\ n \to \infty,\ \text{for any } \tilde{\varphi} \in C(\tilde{J}),$$

   where we denote by $\mathbf{1}_{\tilde{J}}$ the constant function on $\tilde{J}$ taking its value 1, *and exists a unique probability measure μ on $J(G)$ such that*

$$||A^n \varphi - \mu(\varphi)\, \mathbf{1}_{J(G)}||_{J(G)} \to 0,\ n \to \infty,\ \text{for any } \varphi \in C(J(G)),$$

   where we denote by $\mathbf{1}_{J(G)}$ the constant function on $J(G)$ taking its value 1.

2. *$\pi_* \tilde{\mu} = \mu$ and $\tilde{\mu}$ is $\tilde{f}$-invariant.*
3. *$(\tilde{f}, \tilde{\mu})$ is exact. In particular, $\tilde{\mu}$ is ergodic.*
4. *μ is non-atomic. supp (μ) is equal to $J(G)$.*
5. *$h(\tilde{f}|_{\tilde{J}}) \geq h_{\tilde{\mu}}(\tilde{f}) \geq \log(\sum_{j=1}^{m} \deg(f_j))$, where $h(\tilde{f}|_{\tilde{J}})$ denotes the topological entropy of $\tilde{f}$ on $\tilde{J}$.*

Proof. We will show the statement in the similar way to [L]. By [HM3], the family of all holomorphic inverse branches of any elements of G in any open set U which has non-empty intersection with $J(G)$ is normal in U. With this fact, we can show that the operator $\tilde{A}$ is almost periodic, i.e. for each $\tilde{\psi} \in C(\tilde{J})$, $\{\tilde{A}^n\tilde{\psi}\}_n$ is relative compact in $C(\tilde{J})$. Hence, by [L], $C(\tilde{J})$ is the direct sum of the attractive basin of 0 for $\tilde{A}$ and the closure of the space generated by unit eigenvectors. It is easy to see that 1 is the unique eigenvalue and the eigenvectors are constant. Therefore 1. holds.

Because of the condition $*$, $E(G)$ is included in $F(G)$. With the fact, we can show that μ is non-atomic, which implies 5. □

Remark1. If $\tilde{f}|_{\tilde{J}}$ is expansive,(in particular, if G is expanding,) then

$$h(\tilde{f}|_{\tilde{J}}) = h_{\tilde{\mu}}(\tilde{f}) = \log(\sum_{j=1}^{m} \deg(f_j)).$$

Remark2. We can also construct *self-similar measures* on $J(G)$ and show the uniqueness under a similar assumption to condition $*$. For example, in each case of the Remark after Definition 6.1, we can show that.

Now we consider a generalization of Mañé's result([Ma]).

Theorem 6.3 *Let $G = \langle f_1, f_2, \dots f_m\rangle$ be a finitely generated rational semigroup. Assume that the sets $\{f_i^{-1}(J(G))\}_{j=1,\dots,m}$ are mutually disjoint. We define a map $f : J(G) \to J(G)$ by $f(x) = f_i(x)$ if $x \in f_i^{-1}(J(G))$. If μ is an ergodic invariant probability measure for $f : J(G) \to J(G)$ with $h_\mu(f) > 0$, then*

$$\int_{J(G)} \log(\|f'\|)\ d\mu > 0$$

and

$$HD(\mu) = \frac{h_\mu(f)}{\int_{J(G)} \log(\|f'\|)\ d\mu},$$

where we set

$$HD(\mu) = \inf\{\dim_H(Y) \mid Y \subset J(G),\ \mu(Y) = 1\}.$$

Proof. We can show the statement in the same way as [Ma]. Note that the Ruelle's inequality([Ru]) also holds for the map $f : J(G) \to J(G)$. □

¿From the remark after Definition 6.1, Theorem 6.2 and Theorem 6.3, we get the following result. This solves the Problem 12 in [Re] of F.Ren's.

Theorem 6.4 *Let $G = \langle f_1, f_2, \dots f_m \rangle$ be a finitely generated rational semigroup. Assume that the sets $\{f_i^{-1}(J(G))\}_{j=1,\dots,m}$ are mutually disjoint. Then*

$$\dim_H(J(G)) \geq \frac{\log(\sum_{j=1}^{m} \deg(f_j))}{\int_{J(G)} \log(\|f'\|)\, d\mu},$$

where μ denotes the probability measure in Theorem 6.2 and $f(x) = f_i(x)$ if $x \in f_i^{-1}(J(G))$.

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