The Bryuno condition and the Yoccoz theorem

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Abstract

In this paper, we want to survay the proof of the Bryuno theorem, due to Yoccoz, and the Yoccoz theorem.

1 Preparation

We consider the modified continuous fraction expantion of $\alpha \in \mathbb{R} - \mathbb{Q}$. Namely for $\alpha \in \mathbb{R} - \mathbb{Q}$, we can define $\{a_n\}$, $\{\varepsilon_n\}$, and $\{\alpha_n\}$ such that a_0 is an integer which is the closest to α and $\alpha_0 = |\alpha - a_0|$ and $\varepsilon_0 = \pm 1$ which satisfies $\alpha = a_0 + \varepsilon_0 \alpha_0$ and for $n \geq 1$, a_n is an integer which is the closest to $1/\alpha_{n-1}$, $\alpha_n = |1/\alpha_{n-1} - a_n|$, $\varepsilon_n = \pm 1$ which satisfies $1/\alpha_{n-1} = a_n + \varepsilon_n \alpha_n$. Then we have

$$\alpha = a_0 + \varepsilon_0 \cfrac{1}{a_1 + \varepsilon_1 \cfrac{1}{a_2 + \varepsilon_2 \cfrac{1}{a_3 + \cdots}}}.$$

Definition 1.1. *We define a function* $\Psi : \mathbb{R} - \mathbb{Q} \to \mathbb{R}_+ \cup \{+\infty\}$ *such that*

$$\Phi(\alpha) = \sum_{i \geq 0} \beta_{i-1} \log \alpha_i^{-1}.$$

We call the number α *a Bryuno number if* $\Phi(\alpha)$ *is finite.*

2 Known results

Consider a germ of holomorphic map $(\mathbb{C}, 0) \to (\mathbb{C}, 0)$

$$f(z) = \lambda z + a_2 z^2 + a_3 z^3 + \cdots$$

with multiplier λ at $z = 0$. And we consider the case that $|\lambda| = 1$ but λ is not a root of unity. Thus the multiplier λ can be written as

$$\lambda = e^{2\pi i \alpha} \text{ for an } \alpha \in \mathbb{R} - \mathbb{Q}.$$

The origin is said to be an irrationally indifferent fixed point.

The linearization problem for f is whether or not there exists a local change of coordinate $z = h(w)$ with $h(0) = 0$ which conjugates f to the irrational rotation $w \mapsto \lambda w$, so that

$$h(\lambda w) = f(h(w))$$

near the origin. We say that an irrationally indifferent fixed point is a Siegel point or a Cremer point according as the local linearization is possible or not (cf. [1]).

2.1 Known results for univalent functions

We define

$$S := \{f; \text{univalent map on } \mathbb{D}\ ,\ f(0) = 0\ ,\ \text{and } |f'(0)| = 1\},$$
$$S_\lambda := \{f \in S; \quad f'(0) = \lambda\}.$$

For $f \in S_\lambda$, we can define the formal linearizing map H_f which is a formal power series satisfying the following

$$H_f(0) = 0, \quad DH_f(0) = 1, \quad f(H_f(z)) = H_f(\lambda z).$$

Let $R(f)$ be the radius of convergence of H_f. If f is linearizable, we have $R(f) > 0$. If not, we define $R(f) = 0$. We define

$$R(\alpha) := \inf_{f \in S_\lambda} R(f).$$

We state the known results.

Theorem 2.1 (Bryuno). *If $\Phi(\alpha) < +\infty$, then $R(\alpha) > 0$. Therefore all $f \in S_\lambda$ are linearizable at the origin.*

And Yoccoz proved that this result is optimal.

Theorem 2.2 (Yoccoz[2]). *If $\Phi(\alpha) = +\infty$, then there exists a map $f \in S_\lambda$ which is nonlinearizable at the origin.*

Remark 2.1. *In the case that f is a germ of holomorphic map of $(\mathbb{C}, 0) \to (\mathbb{C}, 0)$ with a multiplier λ at $z = 0$, a map $z \mapsto \frac{1}{t} f(tz)$ belongs to S_λ for a suitable $t > 0$. So the same results for f as the above theorems hold.*

2.2 Reduction

We define

$$\begin{aligned} E(z) :=& \exp(2\pi i z), \\ T(z) :=& z + 1. \end{aligned}$$

For $\alpha \in \mathbb{R} - \mathbb{Q}$, we define

$$\begin{aligned} \hat{S}_\alpha :=& \{F; \text{ holomorphic and univalent on } \mathbb{H},\ F \circ T = T \circ F \\ & \text{and } \lim_{\Im z \to +\infty} (F(z) - z) = \alpha\}. \end{aligned}$$

For $f \in S_{E(\alpha)}$, there exists the unique lifting of f which belongs to $\hat{S}_\alpha$. We define the following

$$\begin{aligned} K_F :=& \{z \in \mathbb{H};\ F^n(z) \in \mathbb{H} \text{ (for all } n > 0)\} \text{ and} \\ d_F :=& \sup_{z \in \mathbb{C} - K_F} \Im z \ \in \mathbb{R}_+ \cup \{+\infty\}. \end{aligned}$$

For proving the Yoccoz Theorem, it is sufficient to prove the following.

Proposition 2.1. *If $\Phi(\alpha) = +\infty$, there exists a map $F \in \hat{S}_\alpha$ such that $d_F = +\infty$.*

We would like to survey the proof of this proposition.

3 The renormalization construction

Before surveying the Proposition 2.1, we would like to state about the proof of the Bryuno Theorem, due to Yoccoz. It is based on a *renormalization construction*, due to Douady and Ghys (See also [1] p8-9). Here is an outline of it. $\beta \in (0, 1/2) \cap \mathbb{R} - \mathbb{Q}$ is given. There exists a sufficiently large $t_0 > 0$ such that for all $F \in \hat{S}_\beta$, we can take a connected open set $\mathcal{U}_F$ which is bounded on the left by the vertical line $\hat{l} := \{it; t > t_0\}$, on the right by its image $F(\hat{l})$, and from below by the straight line $\hat{l}'$ from it_0 and $iF(t_0)$. For $z \in \mathcal{U}_F$, there exists a unique number $n(z) \in \mathbb{N}$ such that $F^{n(z)}(z) \in T(\mathcal{U}_F)$. Then we can define *the first return map* $\tilde{G} : \mathcal{U}_F \to \mathcal{U}_F$ (Figure 1) which satisfies that

$$\tilde{G}(z) = T^{-1} \circ F^{n(z)}(z),$$

and *the uniformizing map* K which is holomorphic and univalent on $\mathcal{U}_F$ which satisfies that

$$\Im K(z) = 0 \text{ on } \hat{l}' \text{ and } K(F(z)) = K(z) + 1 \text{ on } \hat{l}.$$

The last formula permits to prolong K on some domain. (Very roughly speaking, if $\Im z$ is large, K is $1/\beta$ times expantion in the direction of the real axis (Figure 1).) Finally we define *the renormalized map* G of F such that $G := K \circ \tilde{G} \circ K^{-1} \in \hat{S}_{-1/\beta}$.

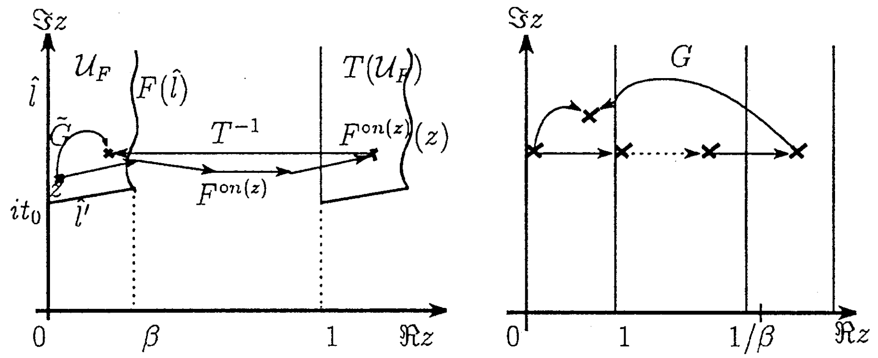

Figure 1: the first return map $\hat{G}$ and the uniformizing map K

4 The unrenomalization machine

The proof of the Yoccoz Theorem is based on the *unrenomalization machine.* Suppose that $\alpha > A$ (The number A is a sufficiently large number determined by precise estimate). We can take a smooth function $\eta : \mathbb{R} \to [0,1]$ which is analytic without $\pm 1/2$ and identically 0 on $(-\infty, -1/2]$ and identically 1 on $[1/2, +\infty)$. Let $J : \mathbb{H} \to \mathbb{C}$ be the map such that $J - \Delta\alpha \in \hat{S}_\alpha$. ($\Delta\alpha \in \mathbb{C}$, $|\triangle \alpha| < c_1$, $c_1 > 0$ is small enough.) We define

$$F_0(z) := F(z+i) - i, \quad J_0(z) := J(z+i) - i \text{ and } J_1(z) := \overline{J_0(\bar{z})}.$$

And we define $\chi : i\mathbb{R} \to \mathbb{C}$ such that

$$\chi(is) := \eta(s)F_0(is) + (1 - \eta(s))J_1(is) \quad (s \in \mathbb{R}).$$

Let $\boldsymbol{\chi}$ be a closed domain which is bounded on the left by the vertical line $i\mathbb{R}$, on the right by its image $\chi(i\mathbb{R})$. Grueing the bords of $\boldsymbol{\chi}$ by χ, we obtain the Riemann surface which is isomorphic to $\mathbb{C}^*$ (Figure 2). So there exists the grueing map K_0 which is continuous on $\boldsymbol{\chi}$, holomorphic and univalent on the interior of $\boldsymbol{\chi}$ and satisfies the following

$$K_0(\chi(is)) = K_0(is) \quad (s \in \mathbb{R}) \ , \ \lim_{s \to +\infty} K_0(is) = 0 \text{ and } K_0(0) = 1$$

(Figure 2). Let R be a square of which the vertices are $\pm i/2$, $1 \pm i/2$. We define $G_0 : \boldsymbol{\chi} - \mathbb{R} \to \boldsymbol{\chi}$ such that

$$G_0(z) := \begin{cases} z - 1, & \text{if } \Re z \geq 1: \\ F_0(z) - 1, & 0 \leq \Re z \leq 1 \text{ and } \Im z \geq 1/2: \\ J_1(z) - 1 & 0 \leq \Re z \leq 1 \text{ and } \Im z \leq -1/2 \end{cases}$$

and $G_1 : \mathbb{C}^* - K_0(R) \to \mathbb{C}^*$ which satisfies

$$G_1(K_0(z)) = K_0(G_0(z)) \quad (z \in \boldsymbol{\chi} - R)$$

(Figure 2). It is easy to see that we can extend G_1 to $G_1 : \hat{\mathbb{C}} - K_0(R) \to \hat{\mathbb{C}}$, and 0 and ∞ are fixed points of G_1 and $G_1'(0) = E(-1/\alpha)$, $G_1'(\infty) = E(1/\alpha')$. At last, we consider a Mobius transformation $k(z) = t\frac{z}{z-1}$ $(t > 0)$. There exists $c > 0$ such that $\mathbb{D} \cap k \circ K_0(R) = \emptyset$ if $t = c/\alpha$. We define $g := k \circ G_1 \circ k^{-1}$. We can see that g is holomorphic and univalent map on $\hat{\mathbb{C}} - k \circ K_0(R)$, and 0 and t are fixed points of g such that $g'(0) = E(-1/\alpha)$, $g'(t) = E(1/\alpha')$. Therefore the map g belongs to $S_{-1/\alpha}$. We note that g has another fixed point t near the origin. Let $G \in \hat{S}_{-1/\alpha}$ be the lift of g to $\mathbb{H}$. We call $G \in \hat{S}_{-1/\alpha}$ be *the unrenormalized map* of $F \in \hat{S}_\alpha$.

5 Construction of nonlinearizable map

From a map $F \in \hat{S}_\alpha$, we can obtain a map $G \in \hat{S}_{-1/\alpha}$ by *renormalization construction* or *unrenormalization machine.*

Proposition 5.1. *Suppose $\alpha < 1/2$. By renormalization construction, we obtain the renormalized map $G \in \hat{S}_{-1/\alpha}$ of F. If there exists $n \in \mathbb{N}$ such that $F^n(z) \notin \mathbb{H}$, there exists $m \in \mathbb{N}$ such that $0 \leq m < n$ and $G^m(K(z)) \notin \mathbb{H}$.*

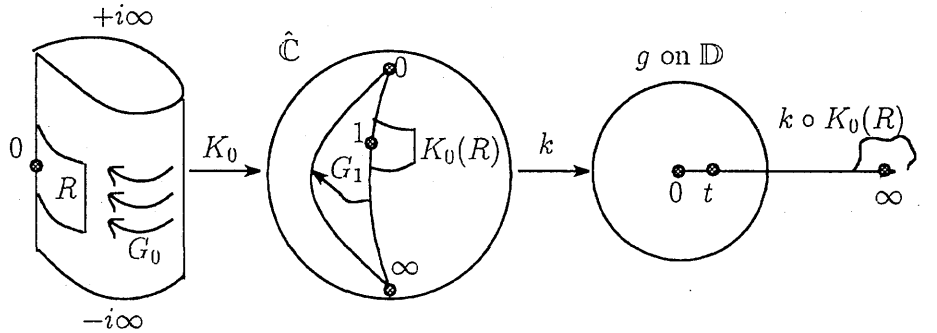

Figure 2: the unrenormalized machine

Proposition 5.2. *Suppose $\alpha > A$. By unrenormalization machine, we obtain the unrenormalized map $G \in \hat{S}_{-1/\alpha}$ of F. The following holds.*

(1) The result similar to that in proposition 5.1 holds.

(2) There exists $z_1 \in \mathbb{H}$ such that

$$\Im z_1 \geq \frac{1}{2\pi} \log t^{-1} = \frac{1}{2\pi} \log \alpha - \frac{1}{2\pi} \log c$$

and $0 \leq m < 2\alpha$ such that $G^m(K(z)) \notin \mathbb{H}$.

Proposition 5.3. *In other cases, By renormalization construction, we obtain the renormalized map $G \in \hat{S}_{-1/\alpha}$ of F. Then the result similar to that in proposition 5.1 holds.*

5.1 Construction of $\{F_n\}_{n \in \mathbb{N}}$

By the modified continuous fraction expantion of $\alpha \in \mathbb{R} - \mathbb{Q}$, we obtain

$$\{a_n\}_{n \geq 0},\ \{\alpha_n\}_{n \geq 0} \text{ and } \{\varepsilon_n\}_{n \geq 0} \quad (1/\alpha_n = a_{n+1} + \varepsilon_{n+1}\alpha_{n+1}).$$

For $F \in \hat{S}_\alpha$, $n \in \mathbb{N}$ and $\varepsilon \in \{+1, -1\}$, we define the operator $T_{n,\varepsilon} : \hat{S}_\alpha \to \hat{S}_{\alpha\varepsilon + n}$ such that

$$T_{n,\varepsilon}(F)(z) = \begin{cases} F(z) + n, & \text{if } \varepsilon = +1: \\ -\overline{F(-\bar{z})} + n & \text{if } \varepsilon = -1. \end{cases}$$

For all $n \in \mathbb{N}$, we construct a map F_n belonging to $\hat{S}_\alpha$ by the following procedure.

(1) $F_{n,n+1}(z) := z + \alpha_{n+1} \in \hat{S}_{\alpha_{n+1}}$.

(2) Suppose we have constructed $F_{n,l+1} \in \hat{S}_{\alpha_{l+1}}$ $(0 \leq l \leq n)$. Let $\tilde{F}_{n,l+1} := T_{a_{l+1},\varepsilon_{l+1}}(F_{n,l+1}) \in \hat{S}_{1/\alpha_l}$. Then We are able to obtain a map $G_{n,l} \in \hat{S}_{-\alpha_l}$ by *renormalization* when $1/\alpha_l \leq A$, or by *unrenormalization* when $1/\alpha_l > A$. And we define a map $F_{n,l} := T_{0,1}(G_{n,l}) \in \hat{S}_{\alpha_l}$.

(3) Since $\alpha = a_0 + \varepsilon_0\alpha_0$, we can define $F_n := T_{a_0,\varepsilon_0}(F_{n,0}) \in \hat{S}_\alpha$.

5.2 The end of construction

We extract the subsequence from $\{F_n\}_{n\in\mathbb{N}}$ which converges locally uniformly to a limit F on $\mathbb{H}$, and we underestimate d_F.

Lemma 5.1. *If α is not a Bryuno number, there exists $\mathcal{I}(\alpha) \subset \mathbb{N}$ which is an infinite set and satisfies the following: For all $0 \leq n' \leq n$ with $n', n \in \mathcal{I}(\alpha)$, there exist*

$$m(n', n) \in \mathbb{N} \text{ and } z(n', n) \in \mathbb{H}$$

such that

$$\begin{aligned} m(n', n) &< C(n') \quad (C(n') \text{ depends on only } n'), \\ \Im z(n', n) &> \frac{1}{2\pi}\sum_{i=0}^{n'} \beta_{i-1}\log\alpha_i^{-1} + Const. \text{ and} \\ F_n^{m(n',n)}(z(n', n)) &\notin \mathbb{H}. \end{aligned}$$

Fix $n' \in \mathcal{I}(\alpha)$. It follows that

- Since $\{m(n', n)\}$ is bounded, there exists a number $m(n')$ appearing in it in infinite times.
- We can consider that $\{z(n', n)\}$ is also bounded. So there exists the accumulation point $z(n')$.

We can extract the subsequence from $\{F_n\}_{n\in\mathbb{N}}$ converging locally uniformly on $\mathbb{H}$ to the limit F which possesses the following property : for all $n' \in \mathcal{I}(\alpha)$, it follows that $F^{m(n')}(z(n')) \notin \mathbb{H}$. On the other hand, by the lemma 5.1, we have

$$\Im z(n') \geq \frac{1}{2\pi}\sum_{i=0}^{n'} \beta_{i-1}\log\alpha_i^{-1} + \text{Const.}$$

Considering n' tend to $+\infty$, we have $d_F = +\infty$. Therefore the proposition 2.1 holds.

References

[1] MILNOR, J. *Dynamics in one complex variable*, Stony Brook IMS Preprint (1990).

[2] YOCCOZ, J. C. Thèorém de Siegel, nombres de Bruno et polynômes quadratiques, *Astérisque*, **231** (1996), 3–88.