On Quasionformal Equivalence on the Boundary of the Tricorn

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Abstract

In this note, we shall show quasiconformal equivalence on the boundaries of hyperbolic components of the tricorn. This fact is quite different from that on the boundary of the Mandelbrot set.

1 Introduction

In this paper, we shall consider the dynamics of the family of antiquadratic polynomials of the form :

$$f_c(z) = \bar{z}^2 + c, \quad c \in \mathbf{C}.$$

The tricorn or the Mandelbar set is defined as the connectedness locus of this family. It was founded independently by Milnor [Mil1] and Rippon et al. [Rip]. Rippon et al. considered it as an analogy with the Mandelbrot set, the connectedness locus of the family of quadratic polynomials :

$$P_c(z) = z^2 + c, \quad c \in \mathbf{C}.$$

On the other hand, Milnor founded it in the real slice of the connectedness locus of the family of cubic polynomials :

$$P_{a,b}(z) = z^3 - 3a^2 z + b, \quad a, b \in \mathbf{C},$$

and through the study of their critical orbits, arrived at the family $\{f_c\}$. In the sense that it appears not only in the antiquadratic family but also in the families of cubics or any more, it is said to be a universal object.

The study of the dynamics of a family of antiholomorphic mappings goes, to some extent, analogously as in case of the family of holomorphic mappings. The difficulty or the difference lies in the lack of analyticity with respect to the parameter. In fact, as is easily seen,

$$Q_c(z) \equiv f_c^2(z) = (z^2 + \bar{c})^2 + c = P_c \circ P_{\bar{c}}(z)$$

no longer depends analytically on the parameter c. Here we also remark that f_c^k is holomorphic for even k and is antiholomorphic for odd k.

For antiholomorphic or antipolynomial-like mappings, we can also show analogous results as in Douady-Hubbard [DH2]. The first difference appears in Lemma 7 in [DH2], which says that, on the boundary of the Mandelbrot set, two different quadratics are never quasiconformally equivalent to each other. We remark that this lemma plays an essential role in the proof of the self-similarity of the Mandelbrot set. That is, it assures the continuity of the mapping from the baby Mandelbrot set to the whole Mandelbrot set on its boundary.

In case of the tricorn, we have a quite different result. That is, on a real analytic arc contained in the boundaries of hyperbolic components of odd periods of the tricorn, antiquadratics are quasiconformally equivalent to each other. On the contrary, if an antiquadratic is quasiconformally equivalent to another lying on a real analytic arc as above, it is contained in that arc up to trivial affine equivalence. This is our main result. See Theorem 2.15. The proof relies on the characterization of quasiconformal equivalence in Mañé-Sad-Sullivan [MSS] and the method of quasiconformal deformation in [DH2]. On the boundaries of hyperbolic components of even periods of the tricorn, we conjecture that the same results as in the Mandelbrot set holds.

An abbreviated version of this paper is announced in [Nak2].

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2 Statement of Results

In this section, we give some definitions and state our main results.

Definition 2.1 *z_0 is said to be periodic if there exists a k such that $f_c^k(z_0) = z_0$. Minimum of such k is called its period and z_0 is called a k-periodic point* *of f_c. In this case, we define its eigenvalue $\rho = \rho(z_0)$ by*

$$\rho = \begin{cases} \frac{\partial f_c^k}{\partial \bar{z}}|_{z=z_0}, & \text{if } k \text{ is odd,} \\ \frac{\partial f_c^k}{\partial z}|_{z=z_0}, & \text{if } k \text{ is even.} \end{cases}$$

We say z_0 is attracting (resp. superattracting, repelling, indifferent) if $|\rho| < 1$ (resp. $= 0$, > 1, $= 1$). It is called hyperbolic if it is not indifferent.

REMARK. If k is odd and z_0 is a k-periodic point of f_c, z_0 is also a k-periodic point of Q_c. Its eigenvalue Λ as a periodic point of Q_c is always non-negative:

$$\Lambda = (Q_c^k)'(z_0) = \prod_{j=0}^{\frac{k-1}{2}} 4\overline{f_c^{2j+1}(z_0)} f_c^{2j}(z_0) = |\rho|^2 \geq 0.$$

Definition 2.2 *$K(f_c) = \{z \in \mathrm{C}; \lim_{n\to\infty} f_c^n(z) \neq \infty\}$ is said to be the filled-in Julia set of f_c and its boundary $J(f_c)$ is called the Julia set of f_c. For general polynomials, we define them in the same way. $M = \{c \in \mathrm{C}; \lim_{n\to\infty} P_c^n(0) \neq \infty\}$ and $T = \{c \in \mathrm{C}; \lim_{n\to\infty} f_c^n(0) \neq \infty\}$ are called respectively the Mandelbrot set and the tricorn. They are the connectedness loci of the corresponding families. That is,*

$$\begin{aligned} M &= \{c \in \mathrm{C}; K(P_c) \text{ and } J(P_c) \text{ are connected}\}, \\ T &= \{c \in \mathrm{C}; K(f_c) \text{ and } J(f_c) \text{ are connected}\}. \end{aligned}$$

Definition 2.3 *We define a hyperbolic component of period k of the tricorn by a connected component of the set*

$$H_k = \{c \in T; f_c \text{ has an attracting } k\text{-periodic point}\}.$$

For example, $H_1 = \{c = z - \bar{z}^2; |z| < 1/2\}$ is the unique hyperbolic component of period one. The tricorn is compact ([Rip]) and connected (Nakane [Nak1]) just as in the Mandelbrot set. But there are differences between them.

Theorem 2.4 (Winters [Win]) *We have $\{c; |c - 1/4| < \epsilon\} \cap \partial T \subset \partial H_1$ for sufficiently small $\epsilon > 0$.*

REMARK. Recently Shishikura [Shi] showed that the Hausdorff dimension of the boundary of the Mandelbrot set is equal to two. Above theorem implies that the boundary of the tricorn contains a smooth arc, hence a part of Hausdorff dimension one.

Definition 2.5 *Let U and U' be open sets in* $\mathbf{C}$. *An orientation preserving homeomorphism $\varphi : U \to U'$ is called quasiconformal if its first derivatives in distribution sense are locally integrable and satisfies $|\frac{\partial}{\partial \bar{z}}\varphi| \leq k|\frac{\partial}{\partial z}\varphi|$ for some $k < 1$.*

Definition 2.6 *Let f and g be polynomials or antipolynomials. Suppose there exist open neighborhoods U and V of $K(f)$ and $K(g)$ respectively and a homeomorphism $\varphi : U \to V$ satisfying $\varphi \circ f = g \circ \varphi$. Then we say f is topologically equivalent to g and denote $f \sim_{top} g$. Furthermore, if φ is quasiconformal (resp. holomorphic, affine) in U, we say f is quasiconformally (resp. holomorphically, affinely) equivalent to g and denote $f \sim_{qc} g$(resp. $f \sim_{hol} g, f \sim_{affine} g$). If φ is quasiconformal in U and satisfies $\frac{\partial}{\partial \bar{z}}\varphi = 0$ on $K(f)$, we say f is hybrid equivalent to g and denote $f \sim_{hb} g$. If φ is only a homeomorphism $\varphi : J(f) \to J(g)$, we say f is J-equivalent to g and denote $f \sim_J g$.*

Though it easily follows from the definitions:

$$f \sim_{affine} g \Rightarrow f \sim_{hol} g \Rightarrow f \sim_{hb} g \Rightarrow f \sim_{qc} g \Rightarrow f \sim_{top} g \Rightarrow f \sim_J g,$$

an inverse follows in some cases.

Theorem 2.7 ([DH2]) *Suppose $K(f)$ and $K(g)$ are connected. Then*

$$f \sim_{hb} g \quad \textit{implies} \quad f \sim_{affine} g.$$

REMARK. Though Theorem 2.7 is shown only for polynomials in [DH2], its antipolynomial version can be proved similarly.

For the family $\{f_c\}$, we have a trivial affine equivalence.

Lemma 2.8 ([Rip],[Win]) *$f_{c'} \sim_{affine} f_c$ if and only if $c' = \omega^j c$, for some $j = 0, 1$ or 2. Here $\omega = e^{2\pi i/3}$.*

Definition 2.9 *An element g_λ of a family $\{g_\lambda\}_{\lambda\in\Lambda}$ is K-stable if and only if there exists a neighborhood U of λ in Λ such that $g_{\lambda'} \sim_K g_\lambda$ for any $\lambda' \in U$. We say "structurally stable" in stead of "topologically stable".*

For example, consider the family $P_c(z) = z^2 + c, \quad c \in \mathbf{C}$.

Theorem 2.10 *P_c is J-stable if and only if $c \in \mathbf{C} - \partial M$.*

Theorem 2.11 *Suppose $c \in W$, a hyperbolic component of M. Then P_c is quasiconformally stable if and only if c is not the center of W.*

These are obtained by applying the results in [MSS] for general analytic families of rational mappings to the above family.

Theorem 2.12 ([DH2]) *Suppose $c \in \partial M$ and $P_{c'} \sim_{qc} P_c$. Then $c' = c$.*

Since above theorems are used to prove the self-similarity of the Mandelbrot set, it is important to show similar results for the tricorn. Theorem 2.12 seems to relate to the following.

Theorem 2.13 (Naĭshul [Nai]) *The eigenvalue of an indifferent periodic point of a holomorphic mapping is a topological invariant.*

Theorems 2.11 and 2.12 imply that Theorem 2.13 does not hold for hyperbolic periodic points. The following can be proved by a similar argument as in [MSS].

Theorem 2.14 *Suppose $c \in W$, a hyperbolic component of the tricorn. Then f_c is J-stable. Moreover, f_c is quasiconformally stable if and only if c is not the center of W.*

Now, let W be a hyperbolic component of odd period k of the tricorn and $c_0 \in \partial W$. Then, there exists an indifferent k-periodic point z_0 of f_{c_0}.That is, it satisfies $f_{c_0}^k(z_0) = z_0$ and $(f_{c_0}^{2k})'(z_0) = 1$.

Theorem 2.15 *Suppose*

$$(f_{c_0}^{2k})''(z_0) \neq 0. \tag{2.1}$$

Then we have

1. $f_c \sim_{qc} f_{c_0}$· *for* $c \in \partial W$, *close to* c_0,

2. *if* $f_c \sim_{qc} f_{c_0}$, *then* $\omega^j c \in \partial W$ *for some* $j = 0, 1$ *or* 2.

REMARK.

1. Note that $(f_{c_0}^{2k})''(z_0) = 0$ implies $(f_{c_0}^{2k})'''(z_0) \neq 0$. Hence roughly speaking, assumption (2.1) implies geometrically that c_0 is not a cusp point of ∂W and dynamically that z_0 is persistently non-hyperbolic in the sense of [MSS] (see the following definition).

2. From the canonical form at rationally indifferent periodic points in Camacho [Cam], it follows that (2.1) is indispensable.

3. For $k = 1$,

$$\begin{aligned} \partial W &= \{c = z - \bar{z}^2; |z| = 1/2\} \\ &= \{c = c_t = e^{2\pi i t}/2 - e^{-4\pi i t}/4; t \in [0, 1)\} \end{aligned}$$

is a real analytic arc and (2.1) is equivalent to the fact that c_0 is not a cusp point.

Definition 2.16 *A non-hyperbolic k-periodic point* z_0 *of* f_{c_0} *is called persistent if for each neighborhood* V *of* z_0, *there is a neighborhood* W *of* c_0 *such that, for each* $c \in W$, f_c *has in* V *a non-hyperbolic k-periodic point.*

3 Proof of Theorem 2.14

The proof of Theorem 2.14 is analogous to those of Theorems 2.10 and 2.11. In order to apply the λ-lemma in [MSS], we regard $\{Q_c = f_c^2\}$ as the real part of a two-parameter analytic family $\{Q_{a,b}(z) = P_{a+bi} \circ P_{a-bi}; a, b \in \mathbf{C}\}$. Then the proofs in [MSS] work for our case if the period k of W is even. When k is odd, we use the Schröder functional equation for antiholomorphic mappings, which can be proved similarly.

Lemma 3.1 *Suppose $f(z) = \sum_{j=1}^{\infty} a_j \bar{z}^j$ is antiholomorphic in a neighborhood of the origin and $|a_1| \neq 0, 1$. Then there exists a holomorphic mapping $\varphi(z) = \sum_{j=1}^{\infty} b_j z^j$ in a neighborhood of the origin satisfying $b_1 \neq 0$ and $f \circ \varphi(z) = \varphi(a_1 \bar{z})$.*

Suppose f_c has an attracting k-periodic point z_0 with eigenvalue ρ. Then the Remark of Definition 2.1 says that z_0 is a k-periodic point of Q_c with eigenvalue $|\rho|^2$. We apply the argument in [MSS] to the family $\{Q_c\}$ and obtain that $Q_{c'} \sim_{qc} Q_c$ for c' close to c. Note that, here we use the canonical form :

$$z \mapsto |\rho|^2 z \tag{3.1}$$

of the local dynamics of Q_c near z_0, which is obtained by the usual Schröder functional equation. If we decompose this canonical form into the two fold iteration of the mapping :

$$z \mapsto \rho \bar{z}, \tag{3.2}$$

obtained in the above lemma as a canonical form of f_c near $z = z_0$, we get $f_{c'} \sim_{qc} f_c$. This completes the proof of Theorem 2.14.

4 Proof of Theorem 2.15

(1) Since ∂W consists of points c such that there exists z satisfying $f_c^k(z) = z$, $(f_c^{2k})'(z) = 1$, it is a real algebraic set and can be expressed locally by a real analytic arc $c = c_t$. Hence $F_t(z) \equiv f_{c_t}^2(z)$ is a quartic polynomial with real analytic parameter t and has an indifferent k-periodic point z_t, depending real analytically on t and satisfying $(F_t^k)'(z_t) = 1$. We can complexify t and make it a holomorphic parameter. Then it also follows that $F_t^k(z_t) = z_t$ and $(F_t^k)'(z_t) = 1$. Now (2.1) assures that z_t is persistently non-hyperbolic. In fact, if (2.1) breaks, a periodic-doubling bifurcation occures. Furthermore, the critical orbits of F_t behave continuously with respect to t. Thus Theorem D in [MSS] yields that $F_t \sim_{qc} F_0$ for sufficiently small t. Now, we have to show that this qc-equivalence is decomposed into $f_{c_t} \sim_{qc} f_{c_0}$ for real t. In this case, we use the local canonical form of F_t at rationally indifferent periodic points in [Cam]. Conjugating by the affine transformation : $z \mapsto z - z_t$, we may assume $z_t \equiv 0$. Then we have

$$\begin{aligned} f_{c_t}^k(z) &= \rho_t \bar{z} + a_t \bar{z}^2 + \dots, \quad |\rho_t| = 1, \\ F_t^k(z) &= z + b_t z^2 + \dots, \quad b_t = \rho_t \overline{a_t} + \overline{\rho_t}^2 a_t, \end{aligned}$$

Conjugating by the affine transformation $S_t : z \mapsto z/b_t$, we can take $b_t \equiv 1$. In this case $f_{c_t}^k$ is transformed into :

$$\begin{aligned} S_t^{-1} \circ f_{c_t}^k \circ S_t(z) &= \frac{\rho_t b_t \bar{z}}{\overline{b_t}} + d_t \bar{z}^2 + \ldots \\ &= \bar{z} + d_t \bar{z}^2 + \ldots, \quad d_t = \frac{a_t b_t}{\overline{b_t}^2} \end{aligned}$$

Let $\varphi(z) = -1/z$. Then

$$\varphi^{-1} \circ S_t^{-1} \circ f_{c_t}^k \circ S_t \circ \varphi(z) = \bar{z} + d_t + O(1/|z|).$$

On the other hand, since we have, from the argument in [Cam]

$$\varphi^{-1} \circ S_t^{-1} \circ F_t^k \circ S_t \circ \varphi(z) = z + 1 + O(1/|z|),$$

it follows that $d_t + \overline{d_t} = 2Re(d_t) = 1$. Conjugating by the affine transformation : $z \mapsto z + Im(d_t)i/2$, we may take $d_t \equiv 1/2$, independent of t. We carry out this procedure for real t, which becomes a real analytic parameter. Next we complexify it and make it a holomorphic parameter. Then, by the same argument as in [Cam], we get the desired qc-equivalence : $f_{c_t} \sim_{qc} f_{c_0}$ for sufficiently small t.

(2) In order to prove the latter part, we apply an antiholomorphic version of the method of quasiconformal deformation used in the proof of Theorem 2.12 in [DH2]. Let $\varphi : U \to V$ be the qc-equivalence : $\varphi \circ f_{c_0} \circ \varphi^{-1} = f_c$. Consider the Beltrami form $\mu = \frac{\bar{\partial}\varphi}{\partial\varphi} = u\frac{d\bar{z}}{dz}$. Define another Beltrami form $\mu_0 = u_0 \frac{d\bar{z}}{dz}$ by

$$u_0 = \begin{cases} u & \text{on } K(f_{c_0}), \\ 0 & \text{on } \mathbf{C} - K(f_{c_0}). \end{cases}$$

Let $m = ||\mu_0||_\infty = ||\mu||_\infty < 1$. Then, for any $t \in (-1/m, 1/m)$, there exists a unique quasiconformal mapping $\Phi_t : \mathbf{C} \to \mathbf{C}$ satisfying

$$\frac{\bar{\partial}\Phi_t}{\partial\Phi_t} = t\mu_0, \quad \Phi_t(0) = 0, \quad \lim_{z \to \infty} \Phi_t(z)/z = 1. \tag{4.1}$$

Note that Φ_t depends real analytically on t. Denote the standard conformal structure by σ_0. For general conformal structure σ, we denote by $\bar{\sigma}$, the conformal structure that gives the same Riemannian metric as σ but differs from it only in the orientation. Put $\sigma = \varphi^* \sigma_0$ and $\sigma_t = \Phi_t^* \sigma_0$.

Lemma 4.1 *We have* $(\Phi_t \circ f_{c_0} \circ \Phi_t^{-1})^* \sigma_0 = \overline{\sigma_0}$. *That is,* $\Phi_t \circ f_{c_0} \circ \Phi_t^{-1}$ *is antiholomorphic in* $\mathbf{C}$.

PROOF. Since $\sigma_t = \sigma_0$ on $\mathbf{C} - K(f_{c_0})$, Φ_t is conformal there and hence $\Phi_t \circ f_{c_0} \circ \Phi_t^{-1}$ is antiholomorphic there. On $K(f_{c_0})$, we have

$$f_{c_0}^* \sigma = f_{c_0}^* \varphi^* \sigma_0 = \varphi^* f_c^* \sigma_0 = \varphi^* \overline{\sigma_0} = \overline{\sigma},$$

which implies

$$\frac{\overline{f_{c_0, \bar{z}}}}{f_{c_0, \bar{z}}} = \frac{\bar{u}}{u \circ f_{c_0}} = \frac{\overline{tu_0}}{tu_0 \circ f_{c_0}}. \tag{4.2}$$

for real t. This means

$$f_{c_0}^* \sigma_t = \overline{\sigma_t}.$$

Then, we have

$$(\Phi_t \circ f_{c_0} \circ \Phi_t^{-1})^* \sigma_0 = \Phi_t^{*-1} f_{c_0}^* \sigma_t = \Phi_t^{*-1} \overline{\sigma_t} = \overline{\Phi_t^{*-1} \sigma_t} = \overline{\sigma_0}.$$

This completes the proof. □

Note that Lemma 4.1 does not hold for non-real t since (4.2) is no longer true for non-real t. This causes the major difference from Theorem 2.12. From the property (4.1), $\Phi_t \circ f_{c_0} \circ \Phi_t^{-1}$ turns out to be an antipolynomial of degree two, of the form $f_{c(t)}$. Now we have shown the existence of a real analytic arc : $c = c(t), t \in (-1/m, 1/m)$, in c-plane satisfying

$$c(0) = c_0, \quad \Phi_t : f_{c(t)} \sim_{qc} f_{c_0}, \quad \Phi_1 \circ \varphi^{-1} : f_{c(1)} \sim_{hb} f_c. \tag{4.3}$$

Here the last equivalence relation in (4.3) follows from the fact :

$$(\Phi_1 \circ \varphi^{-1})^* \sigma_0 = \varphi^{*-1} \Phi_1^* \sigma_0 = \varphi^{*-1} \sigma = \sigma_0,$$

on $K(f_{c_0})$. Then Theorem 2.7 implies $f_{c(1)} \sim_{affine} f_c$. From Lemma 4.1, we have $c(1) = \omega^j c, j = 0, 1$ or 2. On the other hand, from Theorem 2.13, it follows $c(t) \in \partial W, t \in (-1/m, 1/m)$, which yields the desired result. This completes the proof of Theorem 2.15. □

5 Further remarks

By using the argument of (2) in the preceding section, we get a similar result for a hyperbolic component W of even period k. Suppose $c = c_0 \in \partial W$ is not on the boundary of any hyperbolic component of odd period. Then there exists an indifferent k-periodic point $z = z_0$ of f_{c_0} with multiplier λ_0.

Theorem 5.1 *Suppose that such a point c having an indifferent k-periodic point with the same multiplier λ_0 is isolated near $c = c_0$. Then*

$$f_c \sim_{qc} f_{c_0} \text{ implies } c = \omega^j c_0 \text{ for some } j = 0, 1 \text{ or } 2.$$

REMARK. Recently, the assumption of Theorem 5.1 turns out to be satisfied. Hence Theorem 5.1 holds on the boundary of every hyperbolic component of even period off the boundary of hyperbolic component of odd period. Details will be published Nakane-Schleicher [NS].

Conjecture 5.2 *Suppose $c_0 \in \partial T$ is off the boundaries of hyperbolic components of odd periods. Then the conclusion of Theorem 5.1 holds.*

There is an another, direct proof of Theorem 2.15, by virtue of the theory of Ecalle cylinder. See [NS].

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