Monotonicity of the topological entropy on the escape locus E^-

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Abstract

We show that the level set of topological entropy is simply connected on the escape locus E^- of bimodal real cubic maps.

1 Introduction

A real cubic map f from the real line $\mathbf{R}$ to itself is called *bimodal* if it has two distinct real critical points. A bimodal real cubic map can be normalized after real affine conjugations as one of the following forms:

$$\begin{aligned} f_{a,b}(x) &:= x^3 - 3a^2x + b \ (a > 0, b \geq 0) \\ &:= -x^3 + 3a^2x + b \ (a < 0, b \leq 0) \end{aligned}$$

We remark that $\{\pm a\}$ are critical points of $f_{a,b}$. Therefore the space $P := P^+ \cup P^-$ where $P^+ := \{(a,b) \in \mathbf{R}^2 | a > 0, b \geq 0\}$ and $P^- := \{(a,b) \in \mathbf{R}^2 | a < 0, b \leq 0\}$ can be considered as the parameter space of bimodal real cubic maps. In the following we sometimes identify a bimodal real cubic map $f_{a,b}$ with the corresponding point (a,b) of P. The space P can be decomposed into two complementary subsets C and E with qualitatively different dynamical behavior: C consists of points (a,b) whose critical orbits $\{f^n_{a,b}(a)\}_{n\in\mathbf{N}}$ $\{f^n_{a,b}(-a)\}_{n\in\mathbf{N}}$ are both bounded, while E consists of points one of whose critical orbits is unbounded. We call C *the connectedness locus* and E *the escape locus* respectively. Put $C^+ := C \cap P^+$, $C^- := C \cap P^-$, $E^+ := E \cap P^+$ and $E^- := E \cap P^-$. Then C^+ and C^- are simply connected and their boundaries in P^+ and P^- are simple arcs consisting of real semialgebraic curves (see [M]). In the paper [K], we considered the topological entropy $h(f_{a,b})$ of a real bimodal cubic map $f_{a,b}$ as a function on the parameter space P and showed that in the escape loci E^+ and E^-, the level set of topological entropy is simply connected. In [K] we gave a detailed proof of this claim especially in the case of E^+. Hence in this paper we concentrate on the case of E^- and show that the level set of topological entropy is simply connectedin E^-. Before explaining the detail of our claim, first we briefly review the definition of topological entropy following [M-T]. For $f_{a,b} \in P$, *the n-th lap number* $l(f_{a,b}^n)$ is the number of the maximal subintervals of $\mathbf{R}$ on which $f_{a,b}^n$ the n-fold composite of $f_{a,b}$ is monotone. We define the topological entropy $h(f_{a,b})$ of $f_{a,b}$ by $h(f_{a,b}) := \lim_{n\to\infty} \frac{1}{n} \log l(f_{a,b}^n)$. This is well-defined and as a function on P i.e., a function h on P defined by $h(a,b) := h(f_{a,b})$ it is continuous (see [M-T] Lemma 12.3). Analogues to the monotonicity of the topological entropy for the real quadratic family $Q_c(x) = x^2 + c\,(c \in \mathbf{R})$ (see [M-T]), Milnor conjectured that on each connected components P^+ and P^- of P, the level set of the entropy function h is connected (see [M]). Dawson, Galeeva, Milnor and Tresser considered this problem on the connectedness locus C^+ in detail (see [D-G-M-T]). Our main result in this paper is

Theorem 1.1 *In the escape locus E^-, the set of bimodal real cubic maps whose topological entropy is constant is connected, in fact simply connected.*

To prove this theorem, we show the following claims.

Theorem 1.2 *Topological entropy is monotone along the boundary curve ∂C^- of C^-.*

Theorem 1.3 *There exists a homeomorphism R from $\mathbf{R}^+ \times (0,1]$ to E^- such that for $u \in (0,1]$ fixed, any bimodal real cubic maps in $R(\mathbf{R}^+, u)$ are quasi-symmetric conjugate to each other. Both ends of the ray $R(\mathbf{R}^+, u)$ have the* *following property; one goes to infinity and the other accumurates to the boundary curve ∂C^- of C^-.*

Theorem 1.2 is an analoguous statement of the monotonicity for the quadratic family. In fact on ∂C^-, one of the kneading sequences is constant. We prove theorem 1.2 by using well known properties of kneading sequences (especially "Intermediate Value Theorem 12.2" in [M-T]) and the combinatorial rigidity of post critically finite rational maps proved by Thurston (see [D-H]). To prove theorem 1.3, the work of Branner and Hubbard on the parameter space of complex cubic maps is essential. They decomposed the (complex) escape locus into stretching rays and equipotential three dimensional spheres (see [B-H]). We consider the purely imaginary locus of their decomposition. Then rays $R_-(\mathbf{R}^+, u)$ ($u \in (0,1]$) in theorem 1.3 are precisely equal to stretching rays in the sense of Branner and Hubbard. The remainder of this paper is organized as follows. In section 2 we study the behavior of topological entropy along the boundary curve ∂C^- and prove theorem 1.2. Our main tool is the kneading theory and we reduce our claim to the monotonicity of kneading sequences along ∂C^-. In section 3 after reviewing the work of Branner and Hubbard on stretching rays, we prove theorem 1.3. The point is that the stretching deformation of real bimodal cubic map commutes with the complex conjugation. In section 4 we prove theorem 1.1 by using results of the previous sections.

2 Monotonicity along the boundary curve ∂C^-

First we review the definitions and notations of the kneading theory for bimodal real cubic maps $f_{a,b}$ for $(a,b) \in E^-$. Let Σ be the set of maps from $\mathbf{N} \cup \{0\}$ to the set of symbols $\{I_1, C_L, I_2, C_R, I_3\}$ i.e., consisting of maps $A : \mathbf{N} \cup \{0\} \to \Sigma$, $A = (a_0, a_1, \cdots, a_n, \cdots)$. We define the order structure on Σ as follows; first we assume that $I_1 < C_L < I_2 < C_R < I_3$. A finite sequence of symbols $\{I_1, I_2, I_3\}$ is called *even* if it contains even number of-symbols I_1 and I_3. For $A = (a_0, a_1, \cdots)$ and $B = (b_0, b_1, \cdots)$ of Σ, we say that A is smaller than B and denote it by $A < B$ if there exists $n \in \mathbf{N}$ such that $A|_n = B|_n$ consisting of symbols $\{I_1, I_2, I_3\}$ and $a_n < b_n$ if $A|_n$ is even, or $a_n > b_n$ if $A|_n$ is not even, where $A|_n$ is the first finite subsequence of A of length n. For a bimodal map $f = f_{a,b}$, let a map $I_f : \mathbf{R} \to \Sigma$ be defined by $I_f(x) = (i_0(x), i_1(x), \cdots, i_n(x), \cdots)$ where $i_n(x) = I_1$ if $f^n(x) < a$, $i_n(x) = C_L$ if $f^n(x) = a$, $i_n(x) = I_2$ if $a < f^n(x) < -a$, $i_n(x) = C_R$ if $f^n(x) = -a$ and $i_n(x) = I_3$ if $-a < f^n(x)$. The sequence $I_f(x)$ is called *the itinerary of x for the map f*. This map I_f is order preserving; $I_f(x) < I_f(x')$ implies that $x < x'$ and $x < x'$ means that $I_f(x) \leq I_f(x')$. Especially we call the itinerary of both critical values $I_f(f(C_L))$ and $I_f(f(C_R))$ *the kneading sequences of f* and denote them by $K_L(f)$ and $K_R(f)$ respectively.

Next we consider dynamical properties of maps of the boundary ∂C^- of C^-. The boundary curve ∂C^- is a simple arc consisting of the following two semi-algebraic curves:

$$\begin{aligned} S_1 &:= \{(a,b) \in P^- \,|\, b = -2(\frac{2}{3} + a^2)^{\frac{3}{2}}\,,\ -\frac{1}{6} \leq a < 0\} \\ S_2 &:= \{(a,b) \in P^- \,|\, b = -2a^3 + a - 1\,,\ -1 \leq a \leq -\frac{1}{6}\} \end{aligned}$$

S_1 consists of maps whose critical points are both bounded and which have a 2-periodic orbit whose multiplier is equal to 1. On the other hand S_2 consists of maps whose critical points are both bounded and one of whose critical values is a 2-periodic point. Because ∂C^- is parametrized by its a-coordinate, for (a, b) of ∂C^- we denote the corresponding map $f_{a,b}$, its kneading sequences $K_L(f_{a,b})$ and $K_R(f_{a,b})$ by f_a, $K_L(a)$ and $K_R(a)$ respectively.

Lemma 2.1 *For (a, b) of ∂C^-, $K_L(a) = (I_1I_3)^\infty$. For (a, b) of S_1, $K_R(a) = (I_1I_3)^\infty$. For (a, b) of S_2, $K_R(-1) \geq K_R(a) \geq K_R(-\frac{1}{6})$.*

Proof. From the graph of $f_{a,b}$ in S_1, by direct calculations, we can conclude that $f_a(a) < 2a < a$ and $f_a^2(a) > -2a > -a$. This means that $K_L(a) = (I_1I_3)^\infty$. On the other hand $f_a(-a) < 2a < a$ and $f_a^2(-a) > -a$. This shows that $K_L(a) = (I_1I_3)^\infty$. From the graph of $f_{a,b}$ in S_2, we can calculate that $f_a(a) = a - 1$ and $f_a^2(a) = -2a$ which means $K_L(a) = (I_1I_3)^\infty$. On the other hand $K_R(-\frac{1}{6}) = (I_1 I_3)^\infty$ and $K_R(-1) = (I_3 I_1)^\infty$. From the definition of the ordering of Σ, $K_R(-\frac{1}{6})$ is the biggest while $K_R(-1)$ is the smallest in S_2.

Lemma 2.2 (see [M-T])
Let a_1, a_2 and a_0 be points of $[-1, -\frac{1}{6}]$ with $a_1 < a_2$ and satisfying $K_R(a_1) > K_R(a_0) > K_R(a_2)$. Then there exists a point b in (a_1, a_2) such that $K_R(b) = K_R(a_0)$.

Proof. Suppose that any $a \in (a_1, a_2)$ does not satisfy $K_R(a) = K_R(a_0)$. Put $M := \{a \in [a_1, a_2] | K_R(a) > K_R(a_0)\}$ and $P := \{a \in [a_1, a_2] | K_R(a) < K_R(a_0)\}$. Then $a_1 \in M$ and $a_2 \in P$. Because $[a_1, a_2]$ is connected, if both of M and P are open, then there exists $b \in (a_1, a_2)$ such that $K_R(b) = K_R(a_0)$, a contradiction. In the following we show the openness of M. For any element $d \in M$, we will show that we can take an open neighborhood U of d in $[a_1, a_2]$ which is contained in M. Since $K_R(d) = d_1, d_2, \cdots > K_R(a_0) = a_1, a_2, \cdots$, there exists the smallest $i \in \mathrm{N}$ with $d_i \neq a_i$. When $d_i \neq C_L$ and $d_i \neq C_R$, we can take U as $U := \{a \in [a_1, a_2] | K_R(a)|_i = K_R(d)|_i\}$. Hence in the following we assume that $d_i = C_L$ or $d_i = C_R$. If $d_i = C_L$, then $K_R(d) = DC_L(I_1 I_3)^\infty$ where D is a finite sequence of symbols I_1, I_2 and I_3. In this case we remark that $K_R(a_0)|_i = DI_1$ if D is even and $K_R(a_0)|_i = DI_2$, DC_R or DI_3 if D is not even. Then there is an open neighborhood U of d such that for any element a of U, $K_R(a) = DI_1(I_1 I_3)^\infty$ or $K_R(a) = DI_2(I_1 I_3)^\infty$ which is bigger than $K_R(a_0)$. If $d_i = C_R$, then $K_R(d) = (DC_R)^\infty$ where D is a finite sequence of symbols I_1, I_2 and I_3. Then by lemma 11.5 in [M-T], there exists an open neighborhood U of d such that for any element a of U, $K_R(a) = (DI_2)^\infty, (DC_R)^\infty$ or $(DI_3)^\infty$. We claim that $K_R(a_0)$ does not satisfy $(DI_2)^\infty < K_R(a_0) < (DC_R)^\infty$ if D is even. If $K_R(a_0)|_i = DI_1$ or $K_R(a_0)|_i = DC_L$, this is obvious. Assume that $K_R(a_0)|_i = DI_2$. Suppose that $(DI_2)^\infty < K_R(a_0)$. Then the i-th shift $\sigma^i(K_R(a_0))$ of $K_R(a_0)$ is bigger than $K_R(a_0)$. On the other hand $K_R(a_0)|_i = DI_2$ implies that $f^i(a_0) < -a_0$ which means that $\sigma^i(K_R(a_0)) \leq K_R(a_0)$, a contradiction. We also claim that $K_R(a_0)$ does not satisfy $(DI_3)^\infty < K_R(a_0) < (DC_R)^\infty$ if D is not even. Assume that $K_R(a_0)|_i = DI_3$. Suppose that $(DI_3)^\infty < K_R(a_0)$. Then $\sigma^i(K_R(a_0)) > K_R(a_0)$. On the other hand $K_R(a_0)|_i = DI_3$ implies that $-a_0 < f^i(a_0)$ which means that $\sigma^i(K_R(a_0)) \leq K_R(a_0)$, a contradiction. Hence there is an open neighborhood U of d in $[a_1, a_2]$ which is contained in M. By using similar arguments we can also prove the openness of P.

Lemma 2.3 (see [D-H])
If a_1 and a_2 of $[-1, -\frac{1}{6}]$ satisfy conditions that $f^n_{a_1}(-a_1) = a_1$, $f^n_{a_2}(-a_2) = a_2$ for some $n \in \mathbf{N}$ and $K_R(a_1) = K_R(a_2)$, then $a_1 = a_2$. Similarly if a_1 and a_2 of $[-1, -\frac{1}{6}]$ satisfy conditions that $f^n_{a_1}(-a_1) = -a_1$, $f^n_{a_2}(-a_2) = -a_2$ for some $n \in \mathbf{N}$ and $K_R(a_1) = K_R(a_2)$, then $a_1 = a_2$.

Proof. Because f_{a_1} and f_{a_2} are elements of S_2, one of their critical values $f_{a_i}(a_i)$ is a 2-periodic point for $i = 1, 2$. Then the assumption shows that f_{a_1} and f_{a_2} are post critically finite maps i.e., critical orbits are finite sets. The condition $K_R(a_1) = K_R(a_2)$ means that $f^j_{a_1}(-a_1) < f^k_{a_1}(-a_1)$ if and only if $f^j_{a_2}(-a_2) < f^k_{a_2}(-a_2)$. Therefore there exists an orientation preserving homeomorphism ψ from $\hat{\mathbf{R}} := \mathbf{R} \cup \{\infty\}$ to itself sending post critical set of f_{a_1} to that of f_{a_2} in order. Moreover there exists an orientation preserving homeomorphism Ψ from $\hat{\mathbf{C}} := \mathbf{C} \cup \{\infty\}$ to itself whose restriction to $\hat{\mathbf{R}}$ is ψ and preserves the upper and lower half planes respectively. Next we define an orientation preserving homeomorphism Φ from $\hat{\mathbf{C}}$ to itself as follows; Considering f_{a_1} and f_{a_2} as rational maps from $\hat{\mathbf{C}}$ to itself, $f^{-1}_{a_i}(\hat{\mathbf{R}})$ decomposes $\hat{\mathbf{C}}$ into 6 cells. We can take a unique branch of $f^{-1}_{a_2} \circ \Psi \circ f_{a_1}$ which preserves this cell decomposition, and denote this map Φ. Then the map Φ satisfies the following conditions;

(1) Φ sends post critical set of f_{a_1} to that of f_{a_2} in order.

(2) $\Psi \circ f_{a_1} = f_{a_2} \circ \Phi$.

(3) Ψ and Φ are isotopic relative to the post critical set of f_{a_1}.

These conditions mean that as rational maps, f_{a_1} and f_{a_2} are equivalent in the sense of Thurston, and they are $PSL_2(\mathbf{C})$-conjugate by theorem 1 of [D-H]. Hence we conclude that $a_1 = a_2$.

Proposition 2.1 *The kneading sequence $K_R(a)$ is monotonely decreasing along the boundary curve ∂C^-.*

Proof. We assume that there exist $a_1 < a_2$ in $[-1, -\frac{1}{6}]$ such that $K_R(a_1) < K_R(a_2)$. If $K_R(a_1) = (AC_R)^\infty$ where A is a finite sequence of symbols I_1, I_2 and I_3, then $K_R(a_2) > K_R(a_1) > K_R(-\frac{1}{6})$ by lemma 2.1. Lemma 2.2 shows that there exists $b \in (a_2, -\frac{1}{6})$ such that $K_R(b) = K_R(a_1)$ which contradicts to lemma 2.3. If $K_R(a_1) = AC_L(I_1I_3)^\infty$ where A is a finite sequence of symbols I_1, I_2 and I_3, similar arguments also hold. Finally if $K_R(a_1)$ is an infinite sequence of I_1, I_2 and I_3,then the condition that $K_R(a_1) < K_R(a_2)$ implies that there exists $b_1 \in (a_1, a_2)$ such that $K_R(b_1) = (AC_R)^\infty$ or $AC_L(I_1I_3)^\infty$ where A is a finite sequence of symbols I_1, I_2 and I_3. Then from lemma 2.1, $K_R(-1) > K_R(b_1) > K_R(a_1)$ and lemma 2.2 shows that there exists $b_2 \in (-1, a_1)$ such that $K_R(b_2) = K_R(b_1)$ which contradicts to lemma 2.3.

Theorem 2.1 *The topological entropy $h(a)$ is monotonely decreasing along the boundary curve ∂C^-.*

Proof. From lemma 2.1 and proposition 2.1, it is enough to show that for a_1 and a_2 of $[-1. - \frac{1}{6}]$ with $a_1 < a_2$, $K_R(a_1) \geq K_R(a_2)$ implies $l(f_{a_1}^n) \geq l(f_{a_2}^n)$ for all $n \in \mathbf{N}$. Moreover by using the fact that $l(f_a^n)$ is equal to the number of finite sequences of symbols I_1, I_2 and I_3 which is equal to $I_{f_a}(x)|_n$ for some point $x \in \mathbf{R}$, it is enough to prove the following; under the assumption that $K_R(a_1) \geq K_R(a_2)$, for $x \in \mathbf{R}$ and a finite sequence A of symbols I_1, I_2 and I_3 of length n with $I_{f_{a_2}}(x)|_n = A$, then there exists $z \in \mathbf{R}$ such that $I_{f_{a_1}}(z)|_n = A$. We prove this claim by induction on n. It is trivial for the case of $n = 1$. We assume that it holds for $n = k$ and there exist $x \in \mathbf{R}$ and a finite sequence A of length $k+1$ with $I_{f_{a_2}}(x)|_{k+1} = A$. We separate our arguments for the cases $A = I_1B, A = I_2B$ and $A = I_3B$ where B is a finite sequence of symbols I_1, I_2 and I_3 of length k. First we consider the case of $A = I_1B$. The induction hypothesis shows that there exists $y \in \mathbf{R}$ such that $I_{f_{a_1}}(y)|_k = B$. From the graph of f_{a_1}, we can assume that $y > a_1 - 1$ and there exists $z < a_1$ such that $f_{a_1}(z) = y$. Therefore $I_{f_{a_1}}(z)|_{k+1} = I_1B = A$. Next we consider the case of

$A = I_3B$. The induction hypothesis shows that there exists $y \in \mathbf{R}$ such that $I_{f_{a_1}}(y)|_k = B$. If $y < f_{a_1}(-a_1)$, then there exists $-a_1 < z$ such that $f_{a_1}(z) = y$. If $y \geq f_{a_1}(-a_1)$, then $I_{f_{a_1}}(y) \geq K_R(a_1)$, and $B \cdots = I_{f_{a_2}}(f_{a_2}(x)) \leq K_R(a_2)$ and $K_R(a_1) \geq K_R(a_2)$ means that $K_R(a_1)|_k = B$ which implies that we may assume that $y < f_{a_1}(-a_1)$. Therefore $I_{f_{a_1}}(z)|_{k+1} = I_3B = A$. Finally we treat the case of $A = I_2B$. The induction hypothesis shows that there exists $y \in \mathbf{R}$ such that $I_{f_{a_1}}(y)|_k = B$. Then $(I_1I_3)^\infty = K_L(a_1) \leq I_{f_{a_1}}(y) = B \cdots$ implies that we may assume that $y > f_{a_1}(a_1)$. If $y \geq f_{a_1}(-a_1)$, then $I_{f_{a_1}}(y) \geq K_R(a_1)$ and $B \cdots = I_{f_{a_2}}(f_{a_2}(x)) \leq K_R(a_2) \leq K_R(a_1)$ means that $K_R(a_1)|_k = B$ which implies that we may assume that $y < f_{a_1}(-a_1)$. Therefore there exists $a_1 < z < -a_1$ such that $f_{a_1}(z) = y$ and $I_{f_{a_1}}(z)|_{k+1} = I_2B = A$.

3 Stretching rays in the escape locus E^-

First of all we briefly review the work of Branner and Hubbard on the structure of the parameter space of complex cubic maps. After complex affine conjugations, every complex cubic map $f : \mathbf{C} \to \mathbf{C}$ can be written as

$$f_{a,b}(z) = z^3 - 3a^2z + b \ (a, b \in \mathbf{C})$$

Therefore we can take $\mathbf{C}^2$ as the parameter space $P(3)$ of complex cubic maps. We decompose $P(3)$ into two complementary subsets *the connectedness locus* $C(3)$ and *the escape locus* $E(3)$. The connectedness locus $C(3)$ consists of cubic maps whose filled-in Julia set K_f is connected and the escape locus $E(3)$ is the complement of $C(3)$. For a cubic map $f \in P(3)$, we define the function $g_f : \mathbf{C} \to \mathbf{R}^+ \cup \{0\}$ by $g_f(z) := \lim_{n\to\infty} \frac{1}{d^n} \log_+(|f^n(z)|)$ where $\log_+(|z|) := max\{0, log(|z|)\}$. Then g_f is the Green function of the filled-in Julia set K_f which measures the escape rate to infinity. In the parameter space we consider a function $G : P(3) \to \mathbf{R}^+ \cup \{0\}$ defined by $G(f) := max\{g_f(-a), g_f(a)\}$. Then G is continuous, $C(3) = G^{-1}(0)$ and for sufficiently large $r > 0$, we can show that $G^{-1}(r)$ is homeomorphic to the three dimensional sphere S^3. Now we have prepared for defining stretching rays; The map $l_s : \mathbf{C} \setminus \overline{D} \to \mathbf{C} \setminus \overline{D}$ $(s \in \mathbf{R}^+)$

(where $\overline{D}$ is the closed disk) given by $l_s(z) := \frac{z}{|z|} \cdot |z|^s$ is a quasi-conformal diffeomorphism commuting with $f_0(z) = z^3$. Every $f \in P(3)$ is conjugate to f_0 on $U_f := \{z \in \mathbf{C} | g_f(z) > G(f)\}$ by the analytic isomorphism φ_f satisfying $\frac{\varphi_f(z)}{z} \to 1$ *as* $z \to \infty$. Let σ_s denote the f-invariant almost complex structure on $\mathbf{C}$ satisfying

$$\sigma_s = \begin{cases} (l_s \circ \varphi_f)^*(\sigma_0) & on\, U_f \\ \sigma_0 & on\, K_f \end{cases}$$

where σ_0 denotes the standard complex structure. Then the Measurable Riemann Mapping Theorem tells us that there exists an analytic isomorphism $F_s : (\mathbf{C}, \sigma_s) \to (\mathbf{C}, \sigma_0)$. We can uniquely choose F_s satisfying $f_s := F_s \circ f_0 \circ F_s^{-1}$ a monic, centered and $l_s \circ \varphi_f \circ F_s^{-1}$ tangent to the identity at ∞. We call $R(f) := \{f_s | s \in \mathbf{R}^+\}$ *the stretching ray through f.* Since $G(f_s) = sG(f)$, the stretching ray intersects $G^{-1}(r)$ in the exactly one point for any $r \in \mathbf{R}^+$. One of the main result of [B-H] is that for any $r \in \mathbf{R}^+$, the map from $\mathbf{R}^+ \times G^{-1}(r)$ to $E(3)$ sending (s, f) to f_s is a homeomorphism. As a corollary of this result, $G^{-1}(r)$ is homeomorphic to S^3 for any $r \in \mathbf{R}^+$.

Now we are ready to prove theorem 1.3. First we consider the purely imaginary locus $\Im P(3)$ of $P(3)$ consisting points (a, b) of $P(3)$ where a and b are both purely imaginary numbers. The restriction of G to $E(3) \cap \Im P(3)$ shows that for sufficiently large $r > 0$, $G^{-1}(r) \cap \Im P(3) \simeq S^1$ and $G^{-1}(r) \cap \Im E^- \simeq (0, 1]$ where $\Im E^- := \{(ia, ib) \in \Im P(3) | (a, b) \in E^-\}$. Because the quasi-conformal diffeomorphism l_s and a cubic map $f_{a,b} \in \Im P(3)$ commute with the map $z \mapsto -\bar{z}$, the stretching ray $R(f)$ through $f \in E(3) \cap \Im P(3)$ is contained in $E(3) \cap \Im P(3)$. In particular for $f \in E^-$ the streching ray $R(f)$ is contained in E^-. Therefore for any $r \in \mathbf{R}^+$, both isomorphisms $\mathbf{R}^+ \times (G^{-1}(r) \cap \Im P(3)) \simeq E(3) \cap \Im P(3)$ and $\mathbf{R}^+ \times (G^{-1}(r) \cap E^-) \simeq E^-$ hold and they imply theorem 1.3.

4 Conclusion

We give a proof of theorem 1.1. Because topological entropy is topological invariant, theorem 1.3 shows that for any $u \in (0, 1]$, topological entropy is constant along the stretching ray $R(\mathbf{R}^+, u)$. Let I_u be the set of accumulation points of $R(\mathbf{R}^+, u)$ on ∂C^-. Then it is easy to check that I_u is closed and connected in ∂C^-. Moreover for any $u, v \in (0,1]$ with $u < v$, the intersection of I_u and I_v is empty or at most one point. Since topological entropy is continuous on P^-, it is constant on $I_u \cup R(\mathbf{R}^+, u)$. Moreover it is monotone decreasing along ∂C^- by theorem 2.1, $h(f_1) \leq h(f_2)$ for any $f_1 \in R(\mathbf{R}^+, u)$ and $f_2 \in R(\mathbf{R}^+, v)$. This shows that in the escape locus E^-, the level set of topological entropy is simply connected.

References

[B] B.Branner, Cubic polynomials: Turning around the connectedness locus. Topological Methods in Modern Mathematics (1993), 391-427.

[B-H] B.Branner and J.H.Hubbard, The iteration of cubic polynomials, Part 1. Acta Math. 160(1988), 143-206.

[D-G-M-T] S.P.Dawson, R.Galeeva, J.Milnor and C.Tresser, A Monotonicity Conjecture for Real Cubic Maps. Real and Complex Dynamical Systems. Edited by B.Branner and P Hjorth NATO ASI Series C464 (1995), 165-183.

[D-H] A.Douady and J.H.Hubbard, A proof of Thurston's topological characterization of rational maps. Acta Math. 171(1993),263-297.

[K] Y.Komori, Monotonicity of the topological entropy on the real escape loci, in preparation.

[M] J.Milnor, Remarks on iterated cubic maps. Experimental Math. 1(1992),5-24.

[M-T] J.Milnor and W.Thurston, On iterated maps of the interval. Springer LNM 1342 (1988),465-563.