The Glimm-Lax Theory via Wave-Front Tracking

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1 Introduction

Conservation laws of physical quantities $U = {}^t(u_1, u_2, \cdots, u_n)$ are expressed in terms of the system of PDE's:

$$U_t + F(U)_x = 0, \quad (x,t) \in R \times R_+. \tag{1}$$

If the conserved quantities $U = U(x,t)$ are differentiable, the conservation laws (1) are equivalent to the system of quasilinear PDE's:

$$U_t + F'(U)U_x = 0, \quad (x,t) \in R \times R_+. \tag{2}$$

We assume that the system of equations (2) is *hyperbolic* : the eigenvalues (characteristic speeds) are real and distinct

$$\lambda_1(U) < \lambda_2(U) < \cdots < \lambda_n(U), \tag{3}$$

and each characteristic direction is *genuinely nonlinear*:

$$R_j(U) \cdot \nabla\lambda_j \neq 0. \tag{4}$$

Here $R_j(U)$ denotes a right eigenvector of $F'(U)$ associated to the eigenvalue $\lambda_j(U)$. We study solutions to the system (1) together with the initial condition at $t = 0$:

$$U(x,0) = U_0(x), \quad x \in R. \tag{5}$$

In particular, if the initial data have the form:

$$U_0(x) = \begin{cases} U_L, & x < 0 \\ U_R, & x > 0, \end{cases} \tag{6}$$

this initial value problem is called the *the Riemann problem*.

We start with Lax's work [15]; he generalized various consepts from fluid dynamics to general hyperbolic conservation laws (4), and showed that the solution to the Riemann problem exists provided each characteristic direction is either genuinely nonlinear or linearly degenerate and two constant states U_R, U_L are sufficiently close. For general initial data with small total variation, Glimm [11] obtained the following remarkable existence theorem (see also Lax [16]).

Theorem 1.1 *Assume that each characteristic direction is either genuinely nonlinear or linearly degenerate. If the total variation of the initial data: $T.V.U_0$ is sufficiently small, then there exists a global in time solution to the initial value problem* (1), (5). *Moreover, if the system admits a convex entropy function, the weak solution obtained satisfies the entropy condition.*

This is a single existence theorem of global solutions for general hyperbolic systems of conservation laws, in particular systems of more than 3 equations; if the system is composed of 2 equations, the method of compensated compactness can be used and L^∞-existence theorem is obtained (DiPerna [10], Tartar [22]). Glimm's approximate solutions involve random sampling points. Liu [18] showed that these approximate solutions converge, if the random sampling points are equi-distributed. Later, DiPerna [9], Bressan [3] and Risebro [20] proposed the method of *wave-front tracking*, an alternative to the random choice method, and obtained the same general existence theorem. Moreover, recent remarkable works Bressan [4], [5] show that the limit of approximate solutions is unique.

The system of conservation laws is invariant under the scale transformation

$$(x,t) \to (\alpha x, \alpha t) \quad (\alpha > 0),$$

which means : if $U(x,t)$ is a solution of the conservation laws (1), then $V(x,t) = U(\alpha x, \alpha t)$ is also a solution. Now assume that $V(x,t)$ has a limit $U_\infty(x,t)$ as $\alpha \to \infty$. Then $U = U_\infty$ is a solution to the Riemann problem with initial data $U_L = U(-\infty, 0), U_R = U(\infty, 0)$ and satisfies

$$U(\alpha x, \alpha t) = U(x,t). \tag{7}$$

A solution satisfying (7) is called a *self-similar* solution, which defines the asymptotic behavior of the weak solution (Liu [19]). Similarly, if the limit $U_O(x,t)$ exists as $\alpha \to 0$, $U_O(x,t)$ is also a solution to the Riemann problem with initial value $U_L = U(-0,0), U_R = U(+0,0)$ and defines its local behavior (DiPerna [8]). These works by Liu and DiPerna are based on the Glimm-Lax theory developed in [12] which is one of the most difficult papers in the theory of conservation laws.

The aim of this short report is to reconstruct the Glimm-Lax theory in the framework of the wave-front tracking method. As Glimm's proof of the existence theorem is considerably simplified by using the front tracking alternative, the Glimm-Lax theory also turns out to be more intuitive. In the next section, we review the wave-front tracking method following mainly Risebro [20]. We shall see how the theory is simplified in section 3. These theories will be applied, in the last section, to study local and global behavior of a phase boundary.

2 Wave-Front Tracking

2.1 Approximation of Solutions the Riemann Problem

We begin with the fundamental existence theorem of solutions to the Riemann problem (Lax [15]).

Theorem 2.1 *Assume that each characteristic direction is either genuinely nonlinear or linearly degenerate. If $|U_L - U_R|$ is sufficiently small, then there exists a self-similar solution which consists of $(n+1)$ constant regions $U_L = U_0, U_1, \ldots, U_n = U_R$ connected by rarefaction waves, contact discontinuities and shock waves. Moreover the solution of this form is unique provided the intermediate constant vectors are restricted to Ω.*

We construct an approximation of the solution. If U_{j-1} and U_j are connected by a contact discontinuity or a shock wave, we leave it as it is. Suppose that these constant state are connected by a rarefaction wave and denote by $U_j(\epsilon; U_0)$ the j-rarefaction curve issuing from U_0. For any positive h, we choose an integer k such that $kh \le \epsilon_j < (k+1)h$ and define constant states $U_j^{(l)}$ $(0 \le l \le k)$ by

$$U_j^{(0)} = U_{j-1}, \quad U_j^{(l)} = U_j(lh; U_{j-1}) = U_j(h; U_j^{(l-1)}). \tag{8}$$

The approximation $U^h(x,t)$ is defined by

$$U^h(x,t) = U_j^{(l)}, \quad \lambda_j(U_j^{(l)})t < x < \lambda_j(U_j^{(l+1)})t \quad (l = 0, 1, \ldots, k) \tag{9}$$

$(U_j^{(k+1)} = U_j)$. This approximation consists of constant states separated by discontinuities in the j-th characteristic direction. These discontinuities are simply called *j-waves* and the approximation U^h is called the *h-approximation*. Obviously, U^h converges to U in uniform convergence. The truncation error as a weak solution is expressed, by assuming the support of ϕ is contained in $[0, T]$,

$$\iint_{R\times R_+} \{U^h \phi_t + F(U^h)\phi_x\}\, dxdt + \int_R U_0(x)\phi(x,0)\, dx = O(hT) \sum_{\epsilon_j > 0} \epsilon_j. \tag{10}$$

2.2 Construction of Approximate Solutions

Now we construct an approximate solution for general initial data by using the h-approxi-mation of the Riemann problem; we follow Risebro [20]. Let us assume for simplicity $U_0(x) - U_\infty \in L^1(R) \cap BV(R)$, $U_\infty = U_0(\pm\infty)$. First, for any positive h, we choose a sequence $\{x_j\}_{j=1}^{N_0}$ in R so that the approximation of the initial data by step functions

$$U_0^h(x) = \begin{cases} U_\infty, & x < x_1, \\ U_0(x_n), & x_n \le x < x_{n+1}, \\ U_\infty, & x \ge x_{N_0} \end{cases}$$

satisfies

$$\sup_{x \neq x_n} |U_0^h(x) - U_0(x)| \leq h.$$

At each point of discontinuity $x = x_n$, we solve the Riemann problem with the initial data $U_L = U_0^h(x_n - 0)$, $U_R = U_0^h(x_n + 0)$. We define the approximate solution by combining the h-approximations (2.1) of these solutions. This approximate solution, denoted by $U^h(x,t)$ and called also the h-approximation, is defined as long as the neighboring waves of the h-approximations collide at $t_1 > 0$. At $t = t_1$, since the approximation $U^h(x, t_1 - 0)$ is also a step function, we can construct a h-approximation by solving the Riemann problem. Then we can extend the approximation to the next collision time $t = t_2 > t_1$. Repeating this construction, we obtain the approximate solution. Let t_m denote the m-th collision time and

$$T = \lim_{m \to \infty} t_m.$$

If $T = \infty$, the approximate solution is defined globally in time. However, this is not always the case. If $T < \infty$, we slightly change the definition of the approximation by neglecting small waves which are produced by the repeated collision. In order to carry out this process, we have to show interaction estimates.

Let $x = x_l$ be the point of collision at $t = t_m$, where p waves collide. We denote these waves by $\alpha_{i_1}^{(1)}, \alpha_{i_2}^{(2)}, \ldots, \alpha_{i_p}^{(p)}$ from left to right. Here the index i_k says that $\alpha_{i_k}^{(k)}$ is in the i_k-th characteristic direction. Since these waves collide, the induces satisfy

$$i_1 \geq i_2 \geq \ldots \geq i_p. \tag{11}$$

We define the amount of interaction at the point of collision $\mathrm{P}_{l,m} = (x_l, t_m)$ by

$$Q(\mathrm{P}_{l,m}) = \sum_{k \neq k'} |\alpha_{i_k}^{(k)} \alpha_{i_{k'}}^{(k')}|. \tag{12}$$

Then we have the following interaction estimates whose proof is carried out in the same manner as Glimm [11].

Lemma 2.1 (Local interaction estimate) *If the amplitude of incoming waves is sufficiently small, then it follows that*

$$\epsilon_i = \sum_k \epsilon_i^{(k)} = \sum_{k : i_k = i} \alpha_{i_k}^{(k)} + O(1) Q(\mathrm{P}_{l,m}), \quad 1 \leq i \leq n. \tag{13}$$

Here the second sum denotes the summation of incoming i-waves.

Lemma 2.2 (Global interaction estimate) *If the total variation of the initial data U_0 is sufficiently small, then it follows that*

$$\sum_{l,m} Q(\mathrm{P}_{l,m}) \leq 2(T.V.U_0)^2. \tag{14}$$

Using the above estimates, we can extend the approximate solution beyond $T = \lim_{m\to\infty} t_m$ in the following way. The estimate (14) implies

$$\sum_l Q(\mathrm{P}_{l,M_1}) < h \quad \text{for some} \quad M_1. \tag{15}$$

At $t = T_{M_1}$, we only change the speeds of the incoming waves in the way that we solve the Riemann problem and select of waves in the characteristic direction of the incoming waves; other waves are removed and the constant state at either side of the wave is extended. Repeating this argument, we obtain a sequence of collision times

$$t_1^{(1)}, \dots, t_{M_1}^{(1)}, t_1^{(2)}, \dots, t_{M_2}^{(2)}, \dots, t_1^{(k)}, \dots, t_{M_k}^{(k)}, \dots \tag{16}$$

which tend to infinity (we set $t_j = t_j^{(1)}, 1 \le j \le M_1$). Since the local interaction estimate (13) holds at these points, the global interaction estimate (14) is true for this approximation. Hence the approximate solutions U^h have a converging subsequence (denoted again by U^h) in L^1_{loc} as $h \to 0$, provided $T.V.U_0$ is sufficiently small. Moreover, denoting by U the limit function, we have $F(U^h) \to F(U)$ in L^1_{loc}. Using the truncation estimate (10), we can see easily that $U(x,t)$ thus obtained is a weak solution of the conservation laws.

3 Glimm-Lax Theory

The Glimm-Lax theory is based on the local interaction estimates (13) and the global estimates (14). Let us define $\alpha^+ = \max\{\alpha, 0\}$, $\alpha^- = \min\{0, \alpha\}$ and the *cancellation* of incoming waves

$$C_i(\mathrm{P}_{l,m}) = \frac{1}{2}\Big(\sum_{k:i_k=i} |\alpha_{i_k}^{(k)}| - |\sum_{k:i_k=i} \alpha_{i_k}^{(k)}|\Big)$$

Then the local interaction estimates (13) are expressed as

$$\epsilon_i^{\pm} = \sum_{k:i_k=i} \alpha_{i_k}^{(k)\pm} \mp C_i(\mathrm{P}_{l,m}) + O(1)Q(\mathrm{P}_{l,m}), \quad 1 \le i \le n. \tag{17}$$

3.1 Generalized Characteristic Curves

Let us define, in an approximate solution, a Lipschitz continuous curve $x = \chi^h(t)$ issuing from an arbitrary point (x_0, t_0) in the j-characteristic direction in the following way: if (x_0, t_0) is in a constant state, χ^h is the j-characteristic curve issuing this point; if it is a point of interaction, χ^h is a j-wave; when the curve χ^h just defined gets to another wave or a point of interaction, we extend this curve according to the following rules.

1. If a j-characteristic curve gets to another wave (a shock wave or an approximation of a rarefaction wave) or enters a j-shock wave, then the curve is continued as a j-characteristic curve or j-shock wave, respectively.
2. If a j-wave (a shock wave or an approximation of a rarefaction wave) gets to a point of interaction with waves of another characteristic family, then the curve is continued as a j-rarefaction wave or j-shock wave; when more than two j-wave are produced as the approximation of the rarefaction wave, we choose the right edge.
3. If a j-wave interacts with another wave of the same family and a j-shock wave is produced, then the curve is continued as a j-shock wave produced.
4. If a j-wave interacts with another wave of the same family and a j-rarefaction wave is produced, the curve is so defined that the rarefaction wave does not cross the curve. More precisely:
   (a) A left rarefaction wave and a right shock wave interact, then the curve is continued as that of the right edge.
   (b) A left shock wave and right rarefaction wave interact, then the curve is continued as the j-characteristic curve of the left edge.

We observe that the case 4-(b) is not important.

Proposition 3.1 *If a left shock wave and right rarefaction wave interact, then the amount of rarefaction wave produced by the interaction is $O(1)Q(\mathrm{P})$.*

Proof. Assume that the state U_L is connected to U_M by a j-shock wave and the state U_M is connected to U_R by an approximation of a rarefaction wave. Since two discontinuities interact, it follows that

$$\lambda_j(U_L) > \lambda_j(U_R). \tag{18}$$

We denote by U_{j-1}, U_j the states connected by the j-rarefaction wave produced, we have

$$\begin{aligned} & \lambda_j(U_j) - \lambda_j(U_{j-1}) \\ = \; & \lambda_j(U_R) - \lambda_j(U_L) + O(1)Q(\mathrm{P}) < O(1)Q(\mathrm{P}), \end{aligned}$$

because of (18). Thus we proved the proposition.

The continuous curve $x = \chi^h(t)$ thus constructed is called a *j-approximate characteristic curve* issuing from (x_0, t_0).

Proposition 3.2 *Let $\chi^h(t)$ be an approximate characteristic curve whose initial point (x_h, t_n) tends to (x_0, t_0). Then we select a subsequence $h' \to 0$ such that $\chi^{h'}(t)$ converges a Lipschitz function $\chi(t)$ uniformly in every bounded time interval. The curve $x = \chi(t)$ is called a generalized characteristic curve issuing from (x_0, t_0).*

Since the Lipschitz constant of $\chi^h(t)$ is estimated by $\max_\Omega |\lambda_j(U)|$. The convergence is obvious by Helly's theorem.

Let Λ be a closed region in R^2 surrounded by a generalized characteristic and segments joining points of interaction. We use the following notation; $L^{\pm}(\Lambda)$: the total amount of rarefactions (+) and shock waves (−) respectively leaving Λ (except for shock waves leaving χ), $E^{\pm}(\Lambda)$: the total amount of rarefactions and shock waves entering Λ (except for shock waves entering χ), $C(\Lambda) = \sum_{P\in\Lambda} C(\mathrm{P})$, $Q(\Lambda) = \sum_{P\in\Lambda} Q(\mathrm{P})$.

Lemma 3.1 (Approximate Conservation Laws)

$$\begin{aligned} L^+(\Lambda) &= E^+(\Lambda) - C(\Lambda) + O(1)Q(\Lambda), & (19)\\ L^-(\Lambda) + L(\chi) &= E^-(\Lambda) + E(\chi) + C(\Lambda) + O(1)Q(\Lambda). & (20) \end{aligned}$$

3.2 Properties of Weak Solutions

The global interaction estimates (14)and (19) imply

$$\sum_P Q(\mathrm{P}) \le O(1)(T.V.U_0)^2, \quad \sum_P C(\mathrm{P}) \le O(1)T.V.U_0 + O(1)(T.V.U_0)^2. \quad (21)$$

Since the space of bounded Radon measure is compact with respect to the w^*-topology, there exist measures dQ, dC such that

$$\lim_{h\to 0}\sum_P Q(\mathrm{P}) = \int dQ, \quad \lim_{h\to 0}\sum_P C(\mathrm{P}) = \int dC. \quad (22)$$

If the above measures are continuous at a point on a generalized characteristic curve, the limit $U(\chi(t)\pm 0, t)$ exists.

Theorem 3.1 *The limit*

$$\lim_{\delta\to\pm 0} U(\chi(t)+\delta, t) = U^{\pm}(t)$$

exists on a generalized j-characteristic curve exept for countable t. Denote $[U(t)] = U^+(t) - U^-(t)$. *If* $[U(t)] \neq 0$, *the Rankine-Hugoniot condition and the Lax entropy condition:*

$$\dot\chi(t)[U(t)] = [F(U(t))], \quad \lambda_j(U^+(t)) < \dot\chi(t) < \lambda_j(U^-(t)) \quad (23)$$

hold; if $[U(t)] = 0$, $\chi(t)$ *satisfies the equation of the characteristic*

$$\dot\chi(t) = \lambda_j(U(t)). \quad (24)$$

4 Phase Boundaries

4.1 The Maxwell Phase Boundary

Let us consider a 2×2-system of conservation laws:

$$u_t - \sigma(v)_x = 0, \quad v_t - u_x = 0. \tag{25}$$

Here $\sigma(v)$ is a C^2- function and there exist $\alpha,\ \beta\ (\alpha < \beta)$ such that

$$\sigma'(v) = \begin{cases} > 0 & \text{for} \quad v < \alpha, \\ < 0 & \text{for} \quad \alpha < v < \beta, \\ > 0 & \text{for} \quad v > \beta \end{cases} \quad \text{and} \quad \sigma''(v) = \begin{cases} < 0 & \text{for} \quad v < \alpha, \\ > 0 & \text{for} \quad v > \beta. \end{cases} \tag{26}$$

$(\sigma'(\alpha) = \sigma'(\beta) = 0)$. A famous example is the equations of van der Waals fluid:

$$\sigma(v) = -\frac{RT}{v-b} + \frac{a}{v^2} \quad (v > b). \tag{27}$$

The system of equations is *hyperbolic* for $v < \alpha,\ v > \beta$ and *elliptic* for $\alpha < v < \beta$; the region $\Omega_\alpha = \{(u,v);\ v < \alpha\}$ is called the *α-phase* and $\Omega_\beta = \{(u,v);\ v > \beta\}$ the *β-phase*. The jump discontinuity of the form:

$$(u(x,t), v(x,t)) = \begin{cases} (u_-, v_-) & \text{for} \quad x < st, \\ (u_+, v_+) & \text{for} \quad x > st, \quad (u_\pm, v_\pm : \text{ constants}). \end{cases} \tag{28}$$

is called a *phase boundary*, if the states $(u_+, v_+),\ (u_-, v_-)$ satisfy the *Rankine-Hugoniot condition* :

$$\begin{cases} s(u_+ - u_-) &= -(\sigma(v_+) - \sigma(v_-)), \\ s(v_+ - v_-) &= -(u_+ - u_-). \end{cases} \tag{29}$$

and belong to the different phases. The *propagating phase boundary* is such that the propagation speed is non-zero. States $(u_m, v_m),\ (u_m^*, v_m^*)$ satisfying

$$\sigma(v_m^*) = \sigma(v_m), \quad u_m^* = u_m, \quad \int_{v_m}^{v_m^*} \sigma(v)dv = \sigma(v_m)(v_m^* - v_m). \tag{30}$$

are called *Maxwell states*; in our case, we have a single pair of such states. These Maxwell states are admissible in the sense that they minimize the entropy rate for v close to v_m (see Dafermos [7])and stationary phase boundaries are not admissible unless $v_- = v_m$ (Hattori's theorem [13]). This admissibility condition is generalized by Abeyaratne-Knowles [1] in ingenious way which we study in the next subsection.

4.2 Admissibility Criteria

The system of conservation laws (25) is endowed with the canonical entropy pair (total mechanical energy [16]):

$$\eta(U) = \frac{1}{2}u^2 + \int_0^v \sigma(w)dw, \quad q(U) = -u\sigma(v). \tag{31}$$

Let the interval $[x_1, x_2]$ contain a finite number of jump discontinuities (28). The local entropy is given by

$$H(t; x_1, x_2) = \int_{x_1}^{x_2} \eta(U(x,t))dx. \tag{32}$$

By direct computation, we find that the rate of decay of local entropy is expressed as

$$\frac{dH}{dt} = -\sum_{\text{jumps}} sf(v_+, v_-) - q(U)|_{x_1}^{x_2} \tag{33}$$

where $q(U)|_{x_1}^{x_2}$ is the entropy flux and

$$f(v_+, v_-) = \int_{v_-}^{v_+} \sigma(v)dv - \frac{1}{2}\{\sigma(v_+) + \sigma(v_-)\}(v_+ - v_-) \tag{34}$$

which is called the *driving traction* in [1]. We assume that the local entropy (32) is decreasing in time:

$$sf(v_+, v_-) \geq 0 \tag{35}$$

holds at every discontinuity. At the shock waves, the Lax entropy condition implies the above inequality. At the phase boundary we adopt Abeyaratne-Knowles' postulate that the speed of the discontinuity is the function of the driving traction:

$$s = \Phi(f), \quad \Phi(0) = 0, \quad \Phi'(f) \geq 0 \tag{36}$$

which they call the *kinetic condition*. They also assumed, and we also adopt, that no new phase occurs from any point in the interior of the α or β-phase, which is called the *nucleation condition*. Under these conditions, we obtain a unique admissible solution to the Riemann problem for (25).

Theorem 4.1 *Assume that Φ in* (36) *is a C^2-function and satisfies $\Phi'(0) > 0$. If $|v_- - v_m|, |v_+ - v_m^*|, |u_\pm - u_m|$ are sufficiently small, then there exists a unique admissible solution which consists of 4 constant regions connected by rarefaction waves, shock waves and a phase boundary. Moreover these constant states are differentiable with respect to the initial data $(v_\pm, u_\pm)$.*

4.3 Global Admissible Solutions

Using the solution to the Riemann problem, we can construct a global in time solution to the initial value problem whose initial data is a perturbation of the Maxwell states

$$(u(x,0),v(x,0)) = \begin{cases} (u_-(x),v_-(x)), & \text{for} \quad x<0, \\ (u_+(x),v_+(x)), & \text{for} \quad x>0. \end{cases} \tag{37}$$

Theorem 4.2 *Assume that the same conditions as Theorem 4.1 hold. If*

$$T.V.|_{x<0}(v_-(x)-v_m), T.V.|_{x>0}(v_+(x)-v_m^*), T.V.|_{x<0 \text{ or } x>0}(u_\pm(x)-u_m)$$

are sufficiently small, then there exists a global in time weak solution . Moreover, there exists a single boundary of two phases which propagates along a Lipschitz curve.

Proof is carried out in the same manner as Chern [6]. In order to prove that the solution obtained is *admissible*, we need the Glimm-Lax theory involving a phase boundary. Since the phase boundary is subsonic and almost stationary, the theory is easier. We obtain

Theorem 4.3 *The limit*

$$\lim_{\delta\to\pm 0} U(\chi(t)+\delta,t) = U^\pm(t)$$

exists on the single boundary $x=\chi(t)$ of two phases exept for countable t. Denote $[U(t)] = U^+(t) - U^-(t)$. Moreover, the Rankine-Hugoniot condition and the following kinetic condition holds.

$$\dot\chi(t)[U(t)] = [F(U(t))], \quad \dot\chi(t) = \Phi(f(v^+(t),v^-(t))). \tag{38}$$

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