SEVERAL CLASSES OF THE INITIAL DATA FOR THE DISCRETE VELOCITIES MODELS OF THE BOLTZMANN EQUATIONS WITH LINEAR AND QUADRATIC TERMS

MITSURU YAMAZAKI

Institute of Mathematics , University of Tsukuba
Tsukuba-shi, Ibaraki 305, Japan
e-mail: yamazaki@ math.tsukuba.ac.jp

ABSTRACT. For the discrete models of the Boltzmann equations which take into account not only the binary collisions but also the linear reflection at the inner wall of an infinite thin tube, we discuss several classes of the initial data for which we have the time-global existence of solutions under some suitable conditions.

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0. Introduction.

We study the discrete velocities models of the Boltzmann equations which take into account not only the binary collisions of molecules, but also the linear reflection of molecules at the inner wall of an infinite thin tube [7]. We denote by $u_i(x,t)$ ($i \in I$: finite set) the distribution function of the molecules moving at the velocity $C_i \in \mathbf{R}^3$ where the $x-axis$ is the direction of the infinite thin tube and $c_i \in \mathbf{R}$ is the x-component of $C_i \in \mathbf{R}^3$.

Remark : We always have $C_i \neq C_j$ for $i \neq j$ but we don't have $c_i \neq c_j$ for $i \neq j$ in general. Nevertheless we sometimes assume :

$$\text{(A1)} \qquad i \neq j \Rightarrow c_i \neq c_j \ .$$

We investigate the following the initial value problem :

$$\text{(0.1)} \qquad \begin{cases} \dfrac{\partial u_i}{\partial t} + c_i \dfrac{\partial u_i}{\partial x} = Q_i(u) + L_i(u) \ , \\ u_i|_{t=0} = u_i^0(x) \ , \quad i \in I, \end{cases}$$

where

$$\text{(0.2)} \qquad Q_i(u) = \sum_{j,k,\ell \in I} (A_{ij}^{k\ell} u_k u_\ell - A_{k\ell}^{ij} u_i u_j) \ ,$$

$$(0.3) \qquad L_i(u) = \sum_{k \in I} (\alpha_i^k u_k - \alpha_k^i u_i) .$$

The natural observation imposes

$$(0.4) \qquad \begin{cases} A_{k\ell}^{ij} = A_{k\ell}^{ji} = A_{\ell k}^{ij} \geq 0 \ , \\ A_{k\ell}^{ij} = 0 \quad \text{if} \quad i = j \ , \\ \qquad \alpha_k^i \geq 0 \ . \end{cases}$$

We consider, in this paper, two classes of the initial data : the first one is the class of the positive, summable and bounded functions, and the second one is the class of the functions with their entropy locally finite. We have already considered other classes of the initial data [8].

For the first class of the initial data, we introduce the following conditions :

$$(\text{A2}) \qquad \begin{cases} A_{k\ell}^{ij} \neq 0 \Rightarrow c_i + c_j = c_k + c_\ell, \\ \forall i \in I, \sum_{k \in I} \alpha_k^i \, (c_k - c_i) = 0. \end{cases}$$

These conditions assure the conservation law of the momentum through the collisions and the reflection, respectively :

Proposition 1.— *Suppose the condition* (A2) *is satisfied. Let* $u = u(x,t) \in C^1(\mathbf{R}_+, \mathcal{S}(\mathbf{R}))$ *be a solution of* (0.1)*. Then, for any* $t \in \mathbf{R}_+$*, we have*

$$(0.5) \qquad \int_{\mathbf{R}} \sum_{i \in I} u_i(x,t) dx = \int_{\mathbf{R}} \sum_{i \in I} u_i^0(x) dx$$

(conservation law of the mass) ,

$$(0.6) \qquad \int_{\mathbf{R}} \sum_{i \in I} c_i u_i(x,t) dx = \int_{\mathbf{R}} \sum_{i \in I} c_i u_i^0(x) dx$$

(conservation law of the momentum) .

For the second class of the initial data, we define the notion of (local-) entropy.

Definition.— *Let* $f = (f_i)_{i \in I}$ *be a set of positive measurable functions on* $\mathbf{R}$*. We call*

$$(0.7) \qquad \int_{\mathbf{R}} \sum_{i \in I} f_i(x) \log f_i(x) dx$$

their entropy. We say $f = (f_i)_{i \in I}$ *is a set of functions with their entropy finite if and only if their entropy is finite.*

Furthermore, we say $f = (f_i)_{i \in I}$ *is a set of functions with their entropy locally finite if and only if* $f \cdot 1_K = (f_i \cdot 1_K)_{i \in I}$ *is a set of functions with their entropy finite for all compact set* K *in* $\mathbf{R}$ *, where* 1_K *is the characteristic function of* K *.*

To make good use of the entropy, we recall the weak type of the microreversibility condition or Stückelberg condition [3] :

$$\text{(A3)} \qquad \sum_{k,\ell\in I} A^{ij}_{k\ell} = \sum_{k,\ell\in I} A^{k\ell}_{ij} \quad \text{for} \quad \forall i,j \in I \ .$$

Remark : It is clear to see that this weak type of the microreversibility condition is weaker than the (usual) microreversibility condition :

$$\text{(A4)} \qquad A^{ij}_{k\ell} = A^{k\ell}_{ij} \quad \text{for} \quad \forall i,j,k,\ell \in I \ .$$

Remark : We should pay attention to the fact that this condition (A3) is not only a mathematical generalization of (A4), but also a product demanded by the physical experiments. In fact, Hamilton and Peng [4] showed that the mesonic process $h_\nu + P \to N + \psi^+$ doesn't fulfill the condition (A3) due to the breaking effect, but the condition (A3).

To define the notion of the solutions for the initial data with their entropy locally finite, we introduce a Banach space and the notion of "weak" solutions on the assumption of (A1). We emphasize that this (A1) is crucial for the definition of weak solutions.

Definition.— *We denote $\mathcal{B}(\mathbf{R}\times[0,T])$ the Banach space of the class of the measurable functions $u = (u_i(x,t))_{i\in I}$ defined on $\mathbf{R}\times[0,T]$ $(T<\infty)$ such that the following norm is finite :*

$$\text{(0.8)} \qquad \|u\|_{\mathcal{B}} = \sum_{i\in I} \int_{\mathbf{R}} \operatorname*{ess\,sup}_{t\in[0,T]} \left| u^\sharp_i(x,t) \right| dx \ ,$$

where $u^\sharp_i(x,t) = u_i(x+c_it,t)$.

Definition.— *Suppose the condition (A1) is satisfied. Let be $u^0_i \in L^1_{\ell oc}(\mathbf{R})$. We say $u \in \mathcal{B}(\mathbf{R}\times[0,T])$ is a weak solution (or simply solution) of the initial value problem* (0.1) *if and only if*

$$\text{(0.9)} \qquad u^\sharp_i(x,t) = u^0_i(x) + \int_0^t \{Q^\sharp_i(u)(x,s) + L^\sharp_i(u)(x,s)\} ds \ .$$

Remark : For $u \in \mathcal{B}(\mathbf{R}\times[0,T])$, we can show that the right-hand side of the above equation belongs to $\mathcal{B}(\mathbf{R}\times[0,T])$ if we assume the condition (A1).

Remark : For the bounded functions u_i, $u = (u_i)$ is a weak solution of (0.1) if and only if u is a solution (0.1) in the sense of distributions.

1. Main Results.

We now state the results on the time-global existence of solutions for two class of the initial data.

Theorem 1([9]).— *Suppose the conditions* (A2) *is satisfied. For the initial data* $u^0 = (u_i^0)$ *positive, summable and bounded, there exists an unique global bounded solution* $u = (u_i(x,t)) \in L^\infty(\mathbf{R} \times \mathbf{R}_+)$ *and we obtain the estimate*

$$\|u\|_{L^\infty(\mathbf{R}\times\mathbf{R}_+)} \leq C_1 \cdot \exp(C_2 \cdot \mu) \ , \tag{1.1}$$

where $C_1 = C_1(\|u^0\|_{L^\infty}) \leq C_3(1 + \|u^0\|_{L^\infty})$, C_2 *and* C_3 *depends only on the equations and* μ *is the total mass*

$$\text{i.e.} \qquad \mu = \int_{\mathbf{R}} \sum_{i\in I} u_i^0(x)dx \ . \tag{1.2}$$

Theorem 2([10]).— *Suppose the conditions* (A1) *and* (A3) *are satisfied. For the initial data* $u^0 = (u_i^0)$ *with their entropy locally finite, there exists an unique weak solution of* (0.1) *defined on* $\mathbf{R} \times [0,\infty)$.

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