Existence of the local solution of the fluid equation in non-uniform space-time

By

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1 Introduction

If we consider, together, time t and the physics phenomenon (x, y, z) in the following way $(ct, x, y, z) = (x^0, x^1, x^2, x^3) = x$, we are able to handle both the time and the the phenomenon geometrically. This space is referred to as the space-time, where c denotes the speed of light. By the general theory of relativity, space-time depends on material. The metric is given by

$$ds^2 = \sum_{i=0}^{3} \sum_{k=0}^{3} g_{ik}(x)\, dx^i\, dx^k, \tag{1}$$

where $g_{ik}(x), i, k = 0, 1, 2, 3$ is a metric tensor which depends on x, and this shows the nature of the gravity place. The metric tensor $g_{ik}(x)$ is governed by the Einstein equation.

Simultaneity space-time is given by

$$g_{0\alpha} = g_{\alpha 0} = 0, \quad \alpha = 1, 2, 3. \tag{2}$$

It is known that the optional gravity place can be changed into the simultaneity space-time.

When $g_{ik}(x)$ is known, we considered the motion of the macroscopic fluid. Though it was stated first, we may suppose $g_{ik}(x)$ to satisfy (2). The velocity of the fluid particle v^* is repersented as,

$$v^* = {}^t(v^1, v^2, v^3) = {}^t(\frac{\partial x^1}{\partial t}, \frac{\partial x^2}{\partial t}, \frac{\partial x^3}{\partial t}), \tag{3}$$

the mass-energy density of the fluid as ρ, and the speed of sound as a. The motion of a macroscopic fluid in the simultaneity space-time is governed by

$$\frac{\partial w}{\partial t} + \sum_{\gamma=1}^{3} \frac{\partial}{\partial x^\gamma} f^\gamma(w) = S(\rho, v^*). \tag{4}$$

Where

$$\left\{\begin{array}{l}
w = \begin{pmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} M_1\,\rho \\ -M_2\,\rho v_* \end{pmatrix}, \\[2ex]
f^\gamma(w) = \begin{pmatrix} f_0^\gamma \\ f_1^\gamma \\ f_2^\gamma \\ f_3^\gamma \end{pmatrix} = \begin{pmatrix} M_2\,g_{00}\rho v^\gamma \\ -M_2\,\rho v^\gamma v_* + a^2 \rho \delta^\gamma \end{pmatrix}, \\[2ex]
v_\gamma = \sum_{\beta=1}^{3} g_{\beta\gamma} v^\beta, \ \delta^\gamma = {}^t(\delta_{1\gamma}, \delta_{2\gamma}, \delta_{3\gamma}), \quad \gamma = 1, 2, 3, \\
v_* = {}^t(v_1, v_2, v_3), \ y = {}^t v_* v^* = \sum_{\alpha=1}^{3}\sum_{\beta=1}^{3} g_{\alpha\beta} v^\alpha v^\beta, \\[2ex]
M_1 = \dfrac{c^4 g_{00} - a^2 y}{c^2(c^2 g_{00} + y)}, \ M_2 = \dfrac{a^2 + c^2}{c^2 g_{00} + y}
\end{array}\right. \tag{5}$$

so that $S(\rho, v^*)$ doesn't contain the differential term of ρ and v. Thus, it can be changed to $S = S(w)$ by the implicit function theorem.

Putting $A^\gamma(w) = \dfrac{\partial}{\partial w} f^\gamma(w), \gamma = 1, 2, 3$, system (4) can then be written as

$$\frac{\partial w}{\partial t} + \sum_{\gamma=1}^{3} A^\gamma(w) \frac{\partial w}{\partial x^\gamma} = S(w, x, t). \tag{6}$$

It is well known that this system can be transformed to a symmetric hyperbolic system to which the Friedrichs-Lax-Kato existence theory of local smooth solutions is applicable, see, for example, Majda [2]. $\tilde{A}^0(w)$ and $\tilde{A}^\gamma(w) = \tilde{A}^0(w)A^\gamma(w), \gamma = 1,2,3$, exists when the following conditions are satisfied

$$(7) \quad \begin{cases} \text{(i)} \quad \tilde{A}^0(w) \text{ and } \tilde{A}^\gamma(w), \text{ are all real symmetric and smooth } w. \\ \text{(ii)} \quad \tilde{A}^0(w) \text{ is positive definite.} \end{cases}$$

A scalar function $\eta = \eta(w)$ is called an entropy function to (6) if there exist scalar functions, $q^\gamma(w), \gamma = 1,2,3$, satisfying

$$(8) \qquad D_w\eta(w)D_wf^\gamma(w) = D_wq^\gamma(w).$$

Here and in the sequal, D_wh is taken as a row vector in case h is a scalar function and is the Jacobi matrix case h is a vector valued function.

According to Godunov [5] ,(see also Kawashima-Shizuta [6]), if a strictly convex entropy function exists, then the transformation

$$(9) \qquad w \to u = D_w\eta(w),$$

is well-defined and reduces the system (4) to a symmetric hyperbolic system of the form

$$(10) \qquad \tilde{A}^0(u)\frac{\partial u}{\partial t} + \sum_{\gamma=1}^{3} \tilde{A}^\gamma(u)\frac{\partial u}{\partial x^\gamma} = \tilde{S}(u,x,t),$$

whose coefficients

$$(11) \qquad \begin{cases} \tilde{A}^0(u) = D_uw = (D_w^2\eta)^{-1}, \\ \tilde{A}^\gamma(u) = D_uf^\gamma = D_wf^\gamma D_uw, \end{cases}$$

satisty the condition (7).

The space-time whose Metric tensor is given as

$$(12) \qquad g_{00} = -g_{11} = -g_{22} = -g_{33} = 1, g_{ik} = 0 \ (i \neq k)$$

and is referred to as the Minkowski space-time. The existence of entropy function with in Minkowski space-time is known by Makino-Ukai [1]. In this paper, the existence of entropy function in simultaneity space-time could be shown. The main result is given in the following,

Theorem 1 *System* (4) *has an entropy pair* $(\eta, q^\gamma), \gamma = 1,2,3,$ *and thus we can make the symmetrizer* $\tilde{A}^0(w)$ *when condition* (7) *is satisfied.*

2 Existence of entropy function

Makino-Ukai [1] assumed that entropy pair (η, q^γ) is of the Minkowski space-time form of

$$\eta = H(\rho, y)\ ,\ q^\gamma = Q(\rho, y)v^\gamma, \quad \gamma = 1,2,3. \tag{13}$$

When I considered the entropy pair as the simulated space-time, the above form also resulted. Remembering that $z = (\rho, v)$ can be solved in the way of system (4), then it can be understood that,

$$D_w z = \begin{pmatrix} M_3 & M_4\,{}^t v_* \\ M_5 \dfrac{v^*}{\rho} & -M_6 \dfrac{v^*\,{}^t v^*}{\rho} + M_7 \dfrac{G^*}{\rho} \end{pmatrix}, \tag{14}$$

$$D_z f^\gamma = \begin{pmatrix} g_{00} M_2 v^\gamma & g_{00} M_2 \rho \delta^\gamma - M_8 \rho\,{}^t v_* v^\gamma \\ -M_2 v_* v^\gamma + a^2 \delta^\gamma & M_8 \rho\,{}^t v_* v^\gamma - M_2 \rho (v^\gamma G_* - {}^t v_* \delta^\gamma) \end{pmatrix}. \tag{15}$$

Here

$$\begin{cases} M_3 = \dfrac{c^2(c^2 g_{00} - y)}{c^4 g_{00} + a^2 y}, M_4 = -\dfrac{2c^2 g_{00}}{c^4 g_{00} + a^2 y}, \\ M_5 = -\dfrac{c^2(c^2 g_{00} + y)}{c^4 g_{00} + a^2 y}, M_6 = \dfrac{2\theta(c^2 g_{00} + y)}{c^4 g_{00} + a^2 y}, \\ M_7 = -\dfrac{c^2 g_{00} + y}{a^2 + c^2}, M_8 = \dfrac{2 g_{00}(a^2 + c^2)}{(c^2 g_{00} + y)^2}, \\ G^* = \begin{pmatrix} g^{11} & g^{12} & g^{13} \\ g^{21} & g^{22} & g^{23} \\ g^{31} & g^{32} & g^{33} \end{pmatrix}, G_* = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix}, \end{cases} \tag{16}$$

and

$$\theta = \frac{a^2}{a^2 + c^2}. \tag{17}$$

Then by putting

$$(b_{ij}^{\gamma}) = D_w z D_z f^{\gamma}, \quad i,j = 0,1,2,3, \tag{18}$$

we obtain

$$\begin{cases} b_{00}^{\gamma} = B_1 v^{\gamma}, \\ b_{0\beta}^{\gamma} = B_2 \rho \delta_{\beta}^{\gamma}, \\ b_{\alpha 0}^{\gamma} = B_3 \rho^{-1} v^{\alpha} v^{\gamma} + B_4 \rho^{-1} g^{\alpha\gamma}, \\ b_{\alpha\beta}^{\gamma} = B_5 v^{\alpha} \delta_{\beta}^{\gamma} + v^{\gamma} \delta_{\beta}^{\alpha}, \end{cases} \tag{19}$$

with

$$\begin{cases} B_1 = \dfrac{c^2(c^2-a^2)g_{00}}{c^4 g_{00} + a^2 y}, & B_2 = \dfrac{c^2 g_{00}(a^2+c^2)}{c^4 g_{00} + a^2 y}, \\ B_3 = -\dfrac{a^2(c^2-a^2)(c^2 g_{00}+y)}{(a^2+c^2)(c^4 g_{00}+a^2 y)}, & B_4 = -\dfrac{a^2(c^2 g_{00}+y)}{a^2+c^2}, \\ B_5 = -\dfrac{a^2(c^2 g_{00}+y)}{c^4 g_{00}+a^2 y}. \end{cases} \tag{20}$$

We shall solve (8) assuming that our entropy pair (η, q^{γ}) is of the form (13). The condition (13) reduces to the following set of equations for the functions H and Q.

$$H_y = Q_y, \tag{21}$$

$$B_1 H_\rho + 2(B_3 y + B_4)\frac{1}{\rho} H_y = Q_\rho, \tag{22}$$

$$B_2 \rho H_\rho + 2 B_5 y H_y = Q. \tag{23}$$

From (21), there should exist a function $U = U(\rho)$ of ρ only such that

$$H(\rho, y) = Q(\rho, y) + U(\rho). \tag{24}$$

On the other hand, eliminating ρH_ρ from (22) and (21) and using (21), we have

$$\rho H_\rho = {B_1}^{-1} \rho Q_\rho - 2{B_1}^{-1}(B_3 y + B_4) Q_y. \tag{25}$$

This and (22) then yield

$$\rho U_\rho = -\frac{c^2 g_{00} + y}{c^2 g_{00}} \rho Q_\rho + \frac{c^2 g_{00} + y}{(a^2 + c^2) g_{00}} Q, \tag{26}$$

or putting $q(\rho, y) = (c^2 g_{00} + y) Q(\rho, y)$,

$$c^2 g_{00} \rho U_\rho = (1 - \theta) q - \rho q_\rho. \tag{27}$$

Since the left hand side is a function of ρ only, q must be the form

$$q(\rho, y) = \rho^{1-\theta} [g(\rho) + h(y)], \tag{28}$$

where g and h are arbitrary functions. Substituting (28) into (25) or

$$\rho q_\rho - q = -2\theta (c^2 g_{00} + y) q_y, \tag{29}$$

we get, with a constant K_0,

$$\rho g'(\rho) - \theta g(\rho) = \theta \{h(y) - 2(c^2 g_{00} + y) h'(y)\} = -\theta K_0, \tag{30}$$

K_j's being arbitrary constants. Now, substitution of (30) into (28) and then into (27) yields

$$U = -\frac{\theta}{c^2 g_{00}} K_2 \rho + K_3. \tag{31}$$

Thus we get

$$\eta = H(\rho, y) = \frac{K_1}{\sqrt{c^2 g_{00} + y}} \rho^{1-\theta} + K_2 \left(\frac{1}{c^2 g_{00} + y} - \frac{\theta}{c^2 g_{00}} \right) \rho + K_3, \tag{32}$$

and

$$Q(\rho, y) = \frac{K_2}{c^2 g_{00} + y} \rho + \frac{K_1}{\sqrt{c^2 g_{00} + y}} \rho^{1-\theta}. \tag{33}$$

3 Symmetrization

The change of variable, $w \to u$ is given by

$$u = D_w \eta(w), \tag{34}$$

and $\tilde{A}_0(u)$ becomes

(35) $$\tilde{A}_0(u) = (D_{w^2}\eta(w))^{-1}.$$

System (4) is changed by arranging the equation with the right side containing no differential terms of u,

(36) $$\tilde{A}_0(u)u_t + \sum_{k=1}^{3} \tilde{A}_k(u)u_{x_k} = \tilde{S}(u,x,t).$$

Thus, we can compute

(37) $$D_u z = \begin{pmatrix} -\dfrac{g_{00}(a^2+c^2)}{a^2K_1\sqrt{c^2g_{00}+y}}\rho^{1+\theta} & \dfrac{(a^2+c^2)^2}{a^2K_1\sqrt{c^2g_{00}+y}}\rho^{1+\theta}\,{}^t v_* \\ \dfrac{(a^2+c^2)^2}{a^2K_1\sqrt{c^2g_{00}+y}}\rho^{1+\theta}v_* & -\dfrac{(a^2+c^2)\sqrt{c^2g_{00}+y}}{K_1}\rho^{\theta}I \end{pmatrix},$$

and

(38) $$\begin{cases} \tilde{A}^0(u) = D_w u = \begin{pmatrix} M_9 & M_{10}\,{}^t v_* \\ M_{10}v_* & M_{11}v_*\,{}^t v_* + M_{12}G_* \end{pmatrix}, \\ M_9 = \dfrac{g_{00}(a^2+c^2)^2(3a^2y - c^4g_{00})\rho^{1+\theta}}{a^2c^2K_1(c^2g_{00}+y)^{\frac{3}{2}}}, \\ M_{10} = \dfrac{(a^2+c^2)^2(c^4g_{00} - a^2y + 2a^2c^2g_{00})\rho^{1+\theta}}{a^2c^2K_1(c^2g_{00}+y)^{\frac{3}{2}}}, \\ M_{11} = \dfrac{(a^2+c^2)^2(3a^2+c^2)\rho^{1+\theta}}{a^2K_1(c^2g_{00}+y)^{\frac{3}{2}}}, \\ M_{12} = -\dfrac{(a^2+c^2)\rho^{1+\theta}}{K_1\sqrt{c^2g_{00}+y}}. \end{cases}$$

If $K_1 < 0$, then $\tilde{A}_0(u)$ becomes a positive definite. When the same calculation is done, it is understood easily that it becomes

$$
(39)\begin{cases}
\tilde{A}^\gamma(u) = \begin{pmatrix} M_{13} v^\gamma & M_{14}\, {}^t v_* v^\gamma + M_{15}\, {}^t\delta^\gamma \\ M_{14} v_* v^\gamma + M_{15}\delta^\gamma & M_{16} v_*\, {}^t v_* v^\gamma + M_{17} V \end{pmatrix}, \\
V = \delta^\gamma\, {}^t v_* + v_*\, {}^t\delta^\gamma + G_* v^\gamma, \\
M_{13} = \dfrac{(a^2+c^2)^2 g_{00}(-2a^2c^2 g_{00} - c^4 g_{00} + a^2 y)\rho^{1+\theta}}{a^2c^2K_1(c^2g_{00}+y)^{\frac{3}{2}}}, \\
M_{14} = \dfrac{(a^2+c^2)^2 g_{00}(3a^2+c^2)\rho^{1+\theta}}{a^2K_1(c^2g_{00}+y)^{\frac{3}{2}}}, \quad M_{15} = -\dfrac{(a^2+c^2)^2 g_{00}\rho^{1+\theta}}{K_1\sqrt{c^2g_{00}+y}}, \\
M_{16} = -\dfrac{(a^2+c^2)^2\rho^{1+\theta}(3a^2+c^2)}{a^2K_1(c^2g_{00}+y)^{\frac{3}{2}}}, \quad M_{17} = \dfrac{(a^2+c^2)^2\rho^{1+\theta}}{K_1\sqrt{c^2g_{00}+y}}.
\end{cases}
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