The Fractal Dimension of the Attractor of the Logisitic Map With Diffusion

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1 Introduction

The logistic map is a very famous one-dimensional system as a chaotic dynamical system. It can be obtained by discretization of the differential equation

$$u_t = (E - hu)u, \tag{1.1}$$

which is called logistic equation and was proposed to describe the evolution of a biological density. Equation (1.1) itself is simple enough to deduce an exact solution. But, the behavior of its discretization

$$x_{n+1} = \mu(1 - x_n)x_n, \tag{1.2}$$

is quite different from that of Eq.(1.1), i.e., its generates a chaos as changing the value of μ. The properties of such one-dimensional chaotic dynamical systems are well described in Devaney[2].

The existence of a chaos for the logistic map was published by biologist R.May in 1974(May[10]), and by mathematician T.Li and J.A.Yorke in 1975(Li and Yorke[9]). Since then, many researches have been done and are still in progress.

The phenomena 'chaos' is induced by the non-linear terms, even if those terms are simple. For example, in the Henon map and the Lorenz equation, a simple non-linear term provides chaos. The complex bifurcation is well known for the logistic map, and the strange attractors are well known for the Henon map and Lorenz equation.

We can find the same chaotic behavior in an extension of (1.2) to multi-dimension. From (1.1), by the addition of the Laplacian and by the difference scheme, we get a system which shows a similar bifurcation to that of (1.2)

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(Itaya[8], Noda[11], Saito[13]). More interesting, in the range of the parameter which generates chaos, this system has one-dimensional attractor, namely, straight-line-like attractor. This paper is aimed at show the existence of such attractors of an extension of (1.2).

We will state some details about this extension in section 2.

For the study of the geometrical properties of such chaotic attractor, few tools are available. We think, the concept of the fractal dimension will be one of the strong methods for analyzing chaos and fractal. The fractal dimension can characterize quantitatively a complex geometry of the strange attractor. There are several definitions of the fractal dimension. Only one of them will be studied here, that is, the Hausdorff dimension (Falconer[3], Federer[6]).

At this moment, however, no way is known to estimate the fractal dimension directly from the equations or mappings. So, we shall use the concept of the Lyapunov exponents (or the Lyapunov numbers) though it gives only an upper bound of the fractal dimension. This method is comparatively studied well. Expressing in a simple way, the Lyapunov exponents are the average of the eigenvalues of the locally linearized matrix of the dynamical systems, along its orbits. We will say more intuitively. The unit ball in the phase space is contracted in some directions and expanded in other directions. The Lyapunov exponents describe such distortion through the associated eigenvalue problem.

More concretely, let X be an invariant set of a dynamical system, and $L = L(u), u \in X$ be the Jacobian matrix of the that system. Then, L^*L is a positive, self-adjoint, and continuous operator. We denote by $e_i, i \in I$ the eigenvectors of $(L^*L)^{1/2}$, with the corresponding eigenvalues $\alpha_i = \alpha_i(L)$,

$$\begin{cases} \alpha_1 \geq \alpha_2 \geq \ldots \geq 0 \\ (L^*L)^{1/2} e_i = \alpha_i e_i, \quad \forall i, \end{cases} \tag{1.3}$$

and by B the unit ball in the phase space. Then, the set $L(B)$ becomes the ellipsoid whose axis are directed along the vectors Le_i, with lengths $\alpha_i(L)$. Next, we define $\omega_m = \omega_m(L)$ by

$$\omega_m = \alpha_1 \ldots \alpha_m. \tag{1.4}$$

For a non-integer $d = n + s$, n is an integer, $0 < s < 1$, we set

$$\omega_d = \omega_n^{1-s} \omega_{n+1}^s. \tag{1.5}$$

Let $\bar{\alpha}_i$, $\bar{\omega}_i$ be defined as follows.

$$\begin{aligned} \bar{\alpha}_i &= \sup_{u \in X} \alpha_i, \\ \bar{\omega}_i &= \sup_{u \in X} \omega_i. \end{aligned} \tag{1.6}$$

And, we call

$$\Lambda_1 = \bar{\omega}_1, \qquad \Lambda_m = \frac{\bar{\omega}_m}{\bar{\omega}_{m-1}}, \quad m \geq 2, \tag{1.7}$$

Lyapunov numbers, and call

$$\mu_m = \log \Lambda_m, \qquad m \geq 1, \tag{1.8}$$

Lyapunov exponents. If there exists $d > 0$ such that

$$\bar{\omega}_d < 1, \tag{1.9}$$

then,this d is an upper bound of Hausdorff dimension of X. The relation between the Hausdorff dimension and the Lyapunov exponents is as follows. Further details are described in Temam[14].

Let H be a Hilbert space with the norm $(\cdot)$, and $X \subset H$ be a compact set. Let S be a continuous mapping from X into H satisfying

$$SX = X \tag{1.10}$$

and

$$\text{uniformly differentiable on X}^1 \tag{1.11}$$

We denote by L the this differential operator. And we assume

$$\sup_{u \in X} |L(u)| < +\infty, \tag{1.12}$$

$$\sup_{u \in X} \omega_d(L(u)) < 1, \quad (for\ some\ \ d > 0). \tag{1.13}$$

Then, we have

Theorem 1 *Under the above assumptions (1.10)-(1.13) the Hausdorff dimension of X is finite and is less than or equal to d.*

An alternative form of Theorem 1 using the Lyapunov exponents is,

Theorem 2 *Under the assumptions (1.10)-(1.12) and, if for some $n \geq 1$,*

$$\mu_1 + \cdots + \mu_{n+1} < 0, \tag{1.14}$$

then

$$\mu_{n+1} < 0, \quad \frac{\mu_1 + \cdots + \mu_n}{|\mu_{n+1}|} < 1, \tag{1.15}$$

and the Hausdorff dimension of X is less than or equal to

$$n + \frac{(\mu_1 + \cdots + \mu_n)_+}{|\mu_{n+1}|}. \tag{1.16}$$

[1] for $\forall u \in X$, there exist a linear operator $L(u)$ satisfing

$$\sup_{u,v \in X,\ 0 \leq |u-v| \leq \epsilon} \frac{|Su - Sv - L(u) \cdot (v-u)|}{|v-u|} \to 0,$$

when $\epsilon \to 0$.

Note that, as mentioned above, this methods of estimation of the fractal dimension provides only the upper bound, so that we are eager for some lower estimation. There is no way to estimate the lower bound of the fractal dimension of the attractor. Without the lower bound, we can't say how the upper one is reliable. However, since this method is one of the most promising means for analyzing the chaos and fractal, we tried to apply it to the logistic map with diffusion.

In order to adapt this method, first, we must find a bounded invariant set. Finding strict invariant set is so difficult that we show the positively invariant set. In section 3, we will prove the existence of such positively invariant set, and lead it to the form suitable for our aim.

Finally, in section 4, we will calculate the Lyapunov exponents and estimate the fractal dimension.

2 Logistic Map with Diffusion

This section is intended to introduce an extension of the logistic map, the well-known one-dimensional chaotic dynamical system, to a multi-dimensional system.

Equation (1.1) was first proposed in 1838. Although the name of the discoverer, P.F.Verhulst, had been forgotten, till Pearl and Reed refound his paper in 1920, the equation itself had been used by many researchers. The equation was proposed originally to describe the change of a population of flies, and at one time, it was expected that this system would also describe the human population. But soon, it became clear that the equation was not sufficient to explain the human population. However many examples imply that, the populations of the most of the lives in the laboratory obey this equation.

As was mentioned above, this equation is so simple mathematically, that we can get the exact solution easily, that is,

$$u = \frac{c_2 e^{\epsilon t}}{1 - c_1 e^{\epsilon t}} \tag{2.1}$$

where $c_1 = hc$, $c_2 = \epsilon c$, c being an arbitrary constant.

From the view point of the chaotic dynamical systems, there is more interest in equation (1.2), the Euler discritization of (1.1), than in (1.1) itself. Although (1.2) looks much simpler than (1.1), in 1974 biologist R.May showed that its orbits of this mapping change drastically with the change of parameter μ. We show, in Figure 1, their behavior for μ from 0 to 4. His study invoked many important researches, for example, Li and Yorke's famous paper "Period Three implies Chaos"(1975, May[10]). And, Sarkovskii's theorem(1964) which include more important results was refound.

In this paper, we don't discuss the bifurcation of the logistic map, but the geometrical properties of its attractor, namely, fractal geometry.

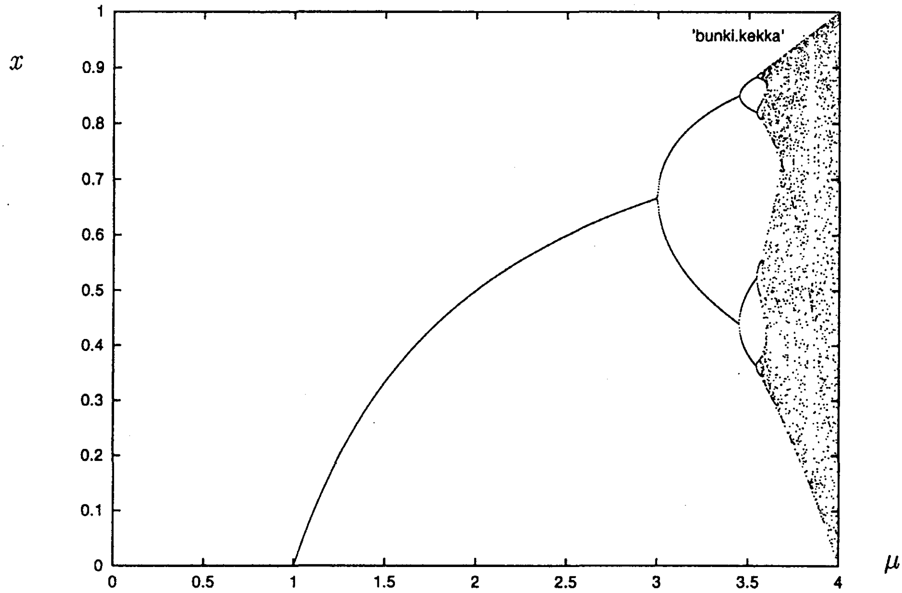

Figure 2.1: bifurcation diagram of logistic mapping

Adding the Laplacian to (1.1). Equation (1.2) becomes a partial differential equation which describes the growth of a spatially distributed population in a region.

Let $\alpha > 0$ be a diffusion coefficient. Logistic equation with diffusion is,

$$u_t = \alpha \Delta u + u(\epsilon - hu), \quad u \in \Omega, t \geq 0, \tag{2.2}$$

and consider with the Dirichlet boundary condition,

$$u|_{\partial\Omega} = 0. \tag{2.3}$$

Here $\Omega \subset \mathcal{R}^m$ is a domain with border $\partial\Omega$. With the Neumann condition, we can reduce (2.2) to a one-dimensional mapping, but with the Dirichlet condition, we cannot do such reduction.

Now, we induce a discrete model for (2.2); By central difference for Δu and backward one for t,

$$\frac{u_{i+1,j} - u_{i,j}}{\Delta t} = \alpha \frac{u_{i+1,j+1} - 2u_{i+1,j} + u_{i+1,j-1}}{(\Delta x)^2} + (\epsilon - hu_{i,j})u_{i,j}.$$

and by transformation,

$$-\frac{\alpha\Delta t}{(\Delta x)^2}u_{i+1,j+1} + \left(1 + \frac{2\alpha\Delta t}{(\Delta x)^2}\right)u_{i+1,j} - \frac{\alpha\Delta t}{(\Delta x)^2}u_{i+1,j-1} = (\epsilon - hu_{i,j})u_{i,j} + u_{i,j}$$

$$= (1 + \Delta t\epsilon)\left(1 - \frac{\Delta h}{1 + \Delta t\epsilon} u_{i,j}\right) u_{i,j}.$$

If we set

$$\frac{\alpha \Delta t}{(\Delta x)^2} = r, \qquad 1 + \Delta t\epsilon = \mu, \qquad \frac{\Delta t h}{1 + \Delta t\epsilon} u_{i,j} = v_{i,j},$$

then, we obtain

$$-r v_{i+1,j+1} + (1 + 2r) v_{i+1,j} - r v_{i+1,j-1} = \mu v_{i,j}(1 - v_{i,j}), \tag{2.4}$$

where, from (2.3),

$$v_{i,0} = v_{i,n} = 0, \qquad \forall i. \tag{2.5}$$

Here, we set $v_n = (v_{n,1}, v_{n,2}, \cdots, v_{n,i-1})^t$, and,

$$A = \begin{pmatrix} 1+2r & -r & & & \mathbf{0} \\ -r & 1+2r & -r & & \\ & -r & \ddots & \ddots & \\ & & \ddots & 1+2r & -r \\ \mathbf{0} & & & -r & 1+2r \end{pmatrix},$$

and,

$$f(v) = \mu \begin{pmatrix} v_1(1 - v_1) \\ v_2(1 - v_2) \\ \vdots \\ v_{i-1}(1 - v_{i-1}) \end{pmatrix}.$$

Then, we get

$$A v_{i+1} = f(v_i). \tag{2.6}$$

So finally, we can write

$$v_{n+1} = A^{-1} f(v_n). \tag{2.7}$$

In this paper ,we consider only simplest case $i = 4$. That is, by the boundary condition $v_0 = 0, v_4 = 0$, we obtain the system which is practically 3 dimensional. Furthermore, the symmetry assumption makes it 2 dimensional. Parameter r is concerned with the diffusion coefficient μ with proportional relation. So, it is obvious that logistic mapping with diffusion is completely reduced to (1.2) when each v_i have no relation, i.e., $r = 0$.

The 2 dimensional logistic mapping with diffusion is,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{1 + 4r + 2r^2} \begin{pmatrix} 1+2r & r \\ 2r & 1+2r \end{pmatrix} \begin{pmatrix} f(x) \\ f(y) \end{pmatrix}, \tag{2.8}$$

$$f(x) = \mu x(1 - x).$$

The bifurcation diagram of (2.8) is shown in Figure(2.2). The periodic points for $r = 10$,and $0 \leq \mu \leq 4$ are plotted in this figure. Then even with diffusion term, we can see the bifurcation similar to the original one. For $\mu \geq 3.5$, by the unboundedness(non-compactness) of the attractor, orbits are diverging to $-\infty$. Numerically, we have got 3.442 as the value of μ which causes such divergence. Of course, we treat the values less than this value as μ.

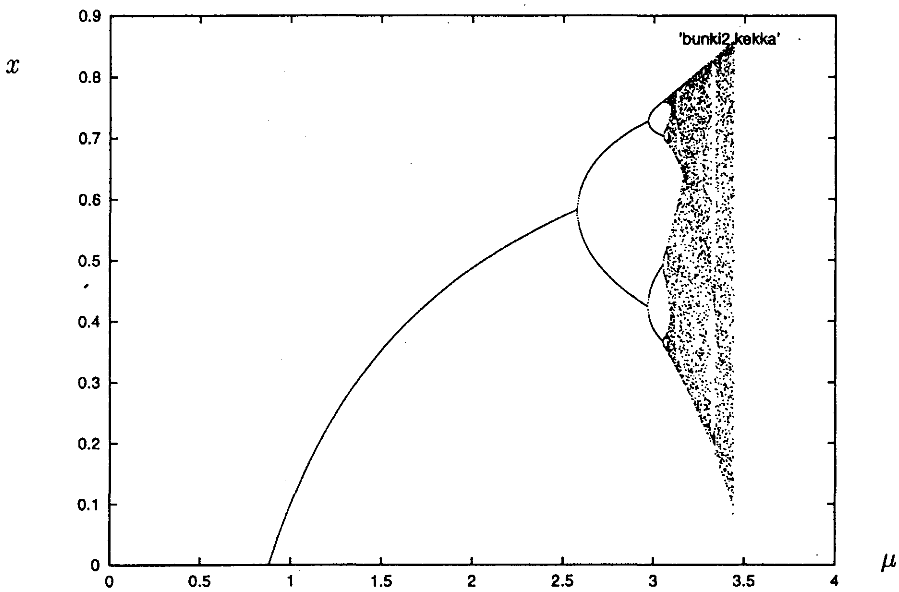

Figure 2.2: bifurcation diagram of logistic map with diffusion

By applying the scaling for more sensitiveness to the parameter r, i.e., by following transformation,

$$
\begin{aligned}
\frac{1}{1+4r+2r^2}\begin{pmatrix} 1+2r & r \\ 2r & 1+2r \end{pmatrix} &= \frac{1+3r}{1+4r+2r^2}\begin{pmatrix} \frac{1+2r}{1+3r} & \frac{r}{1+3r} \\ \frac{2r}{1+3r} & \frac{1+2r}{1+3r} \end{pmatrix} \\
&= \frac{1+3r}{1+4r+2r^2}\begin{pmatrix} \frac{1+2r}{1+3r} & \frac{r}{1+3r} \\ \frac{1+4r}{1+3r}\frac{2r}{1+3r} & \frac{1+4r}{1+3r}\frac{1+2r}{1+3r} \end{pmatrix}.
\end{aligned}
$$

and by embedding $\dfrac{1+3r}{1+4r+2r^2}$ in the parameter μ, we get

$$
\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a_{11}f(x) & + & a12f(y) \\ ca_{21}f(x) & + & ca_{22}f(y) \end{pmatrix}, \quad f(x) = \mu x(1-x), \tag{2.9}
$$

where,

$$
\begin{aligned}
a_{11} &= \frac{1+2r}{1+3r}, \quad a_{12} = \frac{r}{1+3r}, \\
a_{21} &= \frac{2r}{1+4r}, \quad a_{22} = \frac{1+2r}{1+4r}, \\
c &= \frac{1+4r}{1+3r},
\end{aligned}
\tag{2.10}
$$

Further, to make analysis easy, we apply the translation of x,y to $x - \frac{1}{2}$, $y - \frac{1}{2}$. Then, (2.8) is reduced to

$$
\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} - \mu a_{11}x^2 - \mu a_{12}y^2 + \frac{1}{4}(\mu - 2) \\ - \mu c a_{21}x^2 - \mu c a_{22}y^2 + \frac{1}{4}(c\mu - 2) \end{pmatrix}. \tag{2.11}
$$

From now on, we call (2.11) simply the "logistic map". We assume μ is larger than 2.5, because the chaotic behavior seems to occur for such μ.

3 The Invariant Set of the Logistic Map with Diffusion

In this section, we show the existence of an invariant set of an logistic map (2.11). Before that, we recall the definitions of the invariant set and the attractor. Let H be a Hilbert space and S be an operator from H into H itself.

Definition 1 *A set $X \in H$ is a positively invariant set of S if*

$$SX \subset X.$$

Definition 2 *A set $X \in H$ is an invariant set of S if*

$$SX = X.$$

Definition 3 *An attractor of S is a set $\mathcal{A} \subset H$ that enjoys the following properties;*

(1) *$\mathcal{A}$ is an invariant set of S,*

(2) *$\mathcal{A}$ possesses an open neighborhood $\mathcal{U}$ such that, for every u_0 in $\mathcal{U}$, Su_0 converges to $\mathcal{A}$ as $t \to \infty$.*

We write (2.11) again as

$$
\begin{pmatrix} x' \\ y' \end{pmatrix} = S \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\mu a_{11}x^2 - \mu a_{12}y^2 + s \\ -\mu c a_{21}x^2 - \mu c a_{22}y^2 + t \end{pmatrix}, \tag{3.1}
$$

where $s = \frac{1}{4}(\mu - 2)$, $t = \frac{1}{4}(c\mu - 2)$.

By the numerical computation, it was observed that as time increasing all points of the phase space diverge to ∞ or converge to one or more lines. Figure 3.1 shows an attractor of (3.1) for $\mu = 3.4$ and $r = 16$. Thus, we can guess that the logistic map (3.1) has a straight-line-like attractor, and this attractor include a segment which terminated with the points $S\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $S^2\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, i.e., denoting by u and u' these points, we have

$$u = S\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 14(\mu - 2) \\ 14(c\mu - 2) \end{pmatrix} = \begin{pmatrix} s \\ t \end{pmatrix}, \tag{3.2}$$

and

$$u' = S^2\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -\mu a_{11}s^2 - \mu a_{12}t^2 + s \\ -\mu c a_{21}s^2 - \mu c a_{22}t^2 + t \end{pmatrix}. \tag{3.3}$$

The line which connects u and u' is

$$l : \alpha x - \beta y + \gamma = 0 \tag{3.4}$$

where,

$$\left\{ \begin{array}{rcl} \alpha &=& t' - t \;=\; -\mu c a_{21}s^2 - \mu c a_{22}t^2 \\ \beta &=& s' - s \;=\; -\mu a_{11}s^2 - \mu a_{12}t^2 \\ \gamma &=& s't - st' \\ &=& -\mu a_{11}s^2 t - \mu a_{12}t^3 + \mu c a_{21}s^3 - \mu c a_{22}st^2 \end{array} \right. \tag{3.5}$$

Now, we consider a point (x, y) and its image (x', y') by S The condition for a point (x, y) to be

$$dist(l, (x, y)) < \delta,$$

for some $\delta > 0$ is

$$\frac{|\alpha x - \beta y + \gamma|}{D} < \delta, \tag{3.6}$$

where $D = \sqrt{\alpha^2 + \beta^2}$. Similarly, the condition to be

$$dist(l, (x', y')) < \delta$$

is

$$\frac{|\alpha x' - \beta' y + \gamma|}{D} = \frac{A}{D}\left|t^2 x^2 - s^2 y^2\right| < \delta \tag{3.7}$$

where $A = \mu^2 \frac{1 + 4r + 2r^2}{(1 + 3r)^2}$. It is easy to see that u is a point on the asymptote of the hyperbola

$$\frac{A}{D}\left|t^2 x^2 - s^2 y^2\right| = \delta. \tag{3.8}$$

The upper boundedness of the orbits of the map is also easy to see, because quadratic term of the map is negative, but the lower boundedness is not always

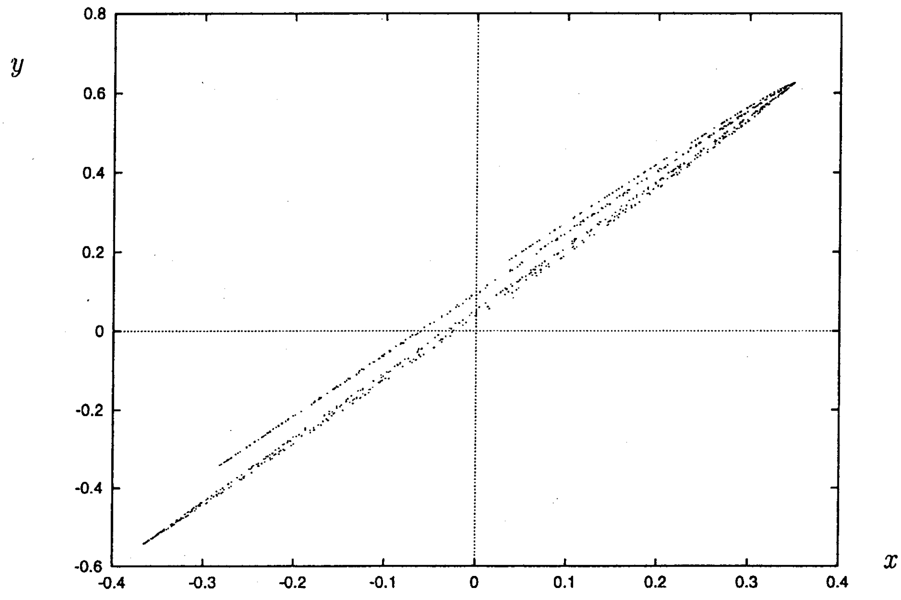

Figure 3.1: the attractor of the logistic map($\mu = 3.4, r = 16$)

held. Indeed, for some μ and r, the orbits of the map is not bounded. We need $s > |s'|$ and $t > |t'|$ for the boundedness of the orbits of the map. It follows from the symmetry of the logistic map for x and y, i.e. (3.1) have only quadratic terms of x and y. So, we consider μ and r satisfying

$$\begin{cases} s > |-\mu a_{11}s^2 - \mu a_{12}t^2 + s| \\ t > |-\mu c a_{21}s^2 - \mu c a_{22}t^2 + t| \end{cases},$$

that is,

$$\begin{cases} \mu a_{11}s^2 + \mu a_{12}t^2 < 2s \\ \mu c a_{21}s^2 + \mu c a_{22}t^2 < 2t \end{cases}. \tag{3.9}$$

For μ and r not satisfying (3.9), the orbits of the logistic map diverge to $-\infty$ as seen in Figure 2.2. We plot μ and r satisfying (3.9) in Figure 3.2. For every r, there exist μ satisfying the above conditions.

Now, we consider the particular case that u' is also the points on the asymptote of the hyperbola (3.8). In this case, $\gamma = 0$ and

$$\frac{t}{s} = \frac{\alpha}{\beta} = \frac{ca_{21} + ca_{22}\left(\frac{t}{s}\right)^2}{a_{11} + a_{12}\left(\frac{t}{s}\right)^2}.$$

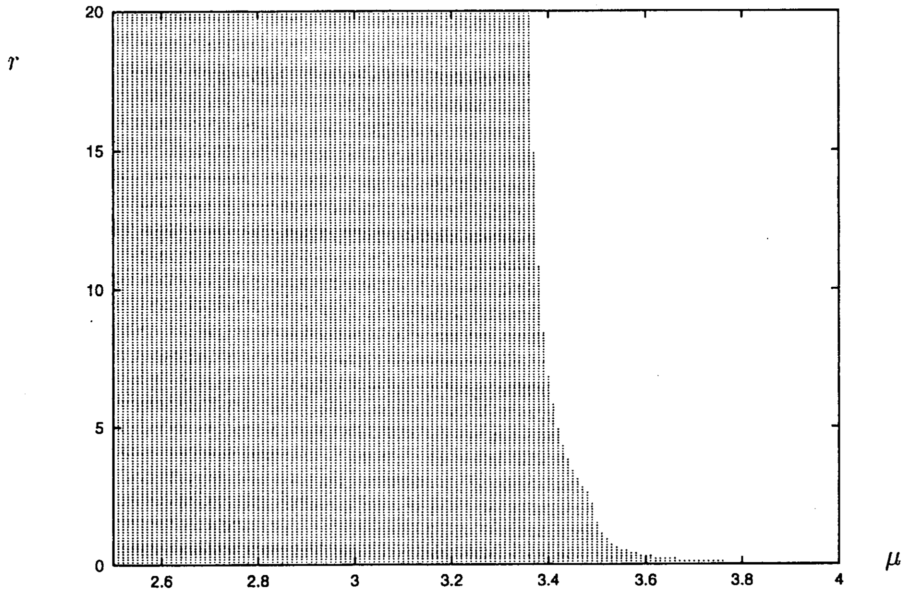

Figure 3.2: The range of the parameter satisfying (3.9)

must be held. We set $g(\omega) = \dfrac{ca_{21} + ca_{22}(t/s)^2}{a_{11} + a_{12}(t/s)^2}$, then, $\omega = \dfrac{t}{s}$ is the fixed point of g. Since

$$g(x) = \frac{ca_{21} + ca_{22}x^2}{a_{11} + a_{12}} = a + \frac{b}{a_{11} + a_{12}x^2},$$

with

$$a = \frac{1+3r}{r} > 0, \qquad b = \frac{-1-6r-7r^2}{r} < 0,$$

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with

$$a = \frac{1+3r}{r} > 0, \qquad b = \frac{-1-6r-7r^2}{r} < 0,$$

$g(x)$ is decreasing function and converges to $a > 0$. Hence, the curves $y = g(x)$ and $y = x$ intersect necessarily, and there exists such fixed points. Since in this case, $\dfrac{t}{s}$ depends on only r, because $\dfrac{\alpha}{\beta}$ doesn't contain μ except in s

and t, we can set $\omega_0(r) = \frac{t}{s}$. Furthermore, c also depends on only r, with $\omega_0(r) = \frac{c\mu - 2}{\mu - 2}$, thus, we can write

$$\mu = 2\frac{\omega_0(r) - 1}{\omega_0(r) - c(r)}.$$

Therefore, for every r, there exists μ, for which (3.3) coincides with the asymptote of the hyperbola (3.8). But μ must satisfy (3.9), i.e., the attractor must be bounded, consequently r is smaller than approximately 4 (Figure 3.3).

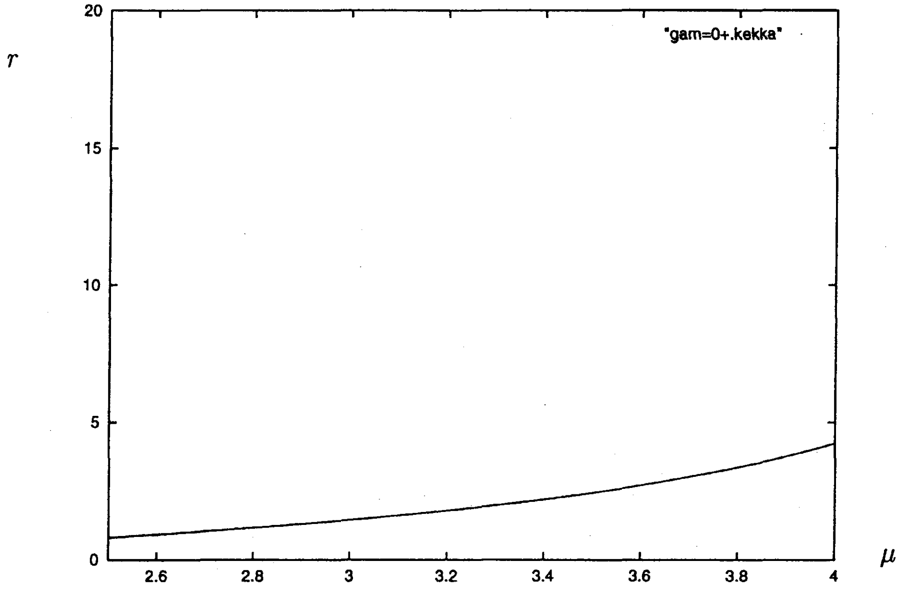

Figure 3.3: The values of parameter which make $\gamma = 0$ in (3.4)

We reconsider the distance appearing in (3.6) and (3.7). Setting $z = y - \frac{t}{s}x$, (3.5) is rewritten as

$$|z| = \left|y - \frac{t}{s}x\right| = \left|y - \frac{\alpha}{\beta}\right| = \frac{|\beta y - \alpha x|}{|\beta|} < \frac{\delta D}{|\beta|}.$$

Therefore,

$$
\begin{aligned}
\left|t^2x^2 - s^2y^2\right| &= |tx - sy|\,|tx + sy| = s^2\left|\frac{t}{s}x - y\right|\left|\frac{t}{s}x + y\right| \\
&\leq s^2\,|z|\,|2y - z| \\
&\leq s^2\,|z|\,(2\,|y| + |z|) \\
&\leq s^2\frac{\delta D}{|\beta|}\left(2t + \frac{\delta D}{|\beta|}\right)
\end{aligned}
\tag{3.10}
$$

If the left hand side of (3.9) is bounded by $\dfrac{\delta D}{A}$, i.e.,

$$
s^2\frac{\delta D}{|\beta|}\left(2t + \frac{\delta D}{|\beta|}\right) < \frac{\delta D}{A},
$$

then,

$$
\delta < \frac{|\beta|^2}{s^2AD} - \frac{2t\,|\beta|}{D}. \tag{3.11}
$$

Thus, if

$$
\frac{|\beta|^2}{s^2AD} - \frac{2t\,|\beta|}{D} > 0 \tag{3.12}
$$

is satisfied, there exist a rectangle-like positively invariant set along the line (3.4) whose width are smaller than δ. Figure 3.4 plots μ and r satisfying (3.12).

Thus, we can say that the logistic map has a rectangle-like absorbing set for the above values of μ and r, and by the continuity, we can also say that the same is true for the values of μ and r which are sufficiently close to the above values.

Next, we consider the general case. As in above case, we transform (3.6) into

$$
\left|y - \frac{\alpha}{\beta} - \frac{\gamma}{\beta}\right| \leq \frac{\delta D}{|\beta|}
$$

and set $z = y - \dfrac{\alpha}{\beta} - \dfrac{\gamma}{\beta}$. Then, (3.7) is changed to

$$
\begin{aligned}
\left|t^2x^2 - s^2y^2\right| &= \left|(tx - sy)(tx + sy)\right| \\
&= \left|\left(tx - s\left(z + \frac{\alpha}{\beta} + \frac{\gamma}{\beta}\right)\right)\left(tx + s\left(z + \frac{\alpha}{\beta} + \frac{\gamma}{\beta}\right)\right)\right| \\
&= s^2\left|\left(\left(\frac{t}{s} - \frac{\alpha}{\beta}\right)x - z - \frac{\gamma}{\beta}\right)\left(\left(\frac{t}{s} + \frac{\alpha}{\beta}\right)x + z + \frac{\gamma}{\beta}\right)\right| \\
&\leq s^2\left(\left|\frac{t}{s} - \frac{\alpha}{\beta}\right|x + |z| + \left|\frac{\gamma}{\beta}\right|\right)\left(\left|\frac{t}{s} + \frac{\alpha}{\beta}\right|x + |z| + \left|\frac{\gamma}{\beta}\right|\right) \\
&\leq s^2\left(\left|\frac{t}{s} - \frac{\alpha}{\beta}\right|x + \left|\frac{\delta D}{|\beta|}\right| + \left|\frac{\gamma}{\beta}\right|\right)\left(\left|\frac{t}{s} + \frac{\alpha}{\beta}\right|x + \left|\frac{\delta D}{|\beta|}\right| + \left|\frac{\gamma}{\beta}\right|\right)
\end{aligned}
$$

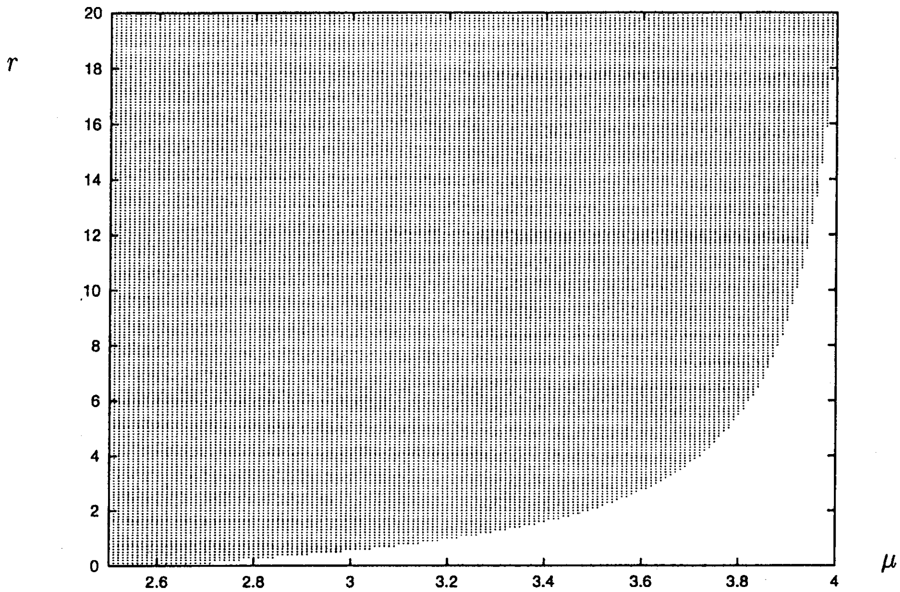

Figure 3.4: The values of μ and r satisfying (3.12) with the values satisfying γ=0 in (3.4)

If this right hand side is bounded by $\frac{\delta D}{A}$, and if we think $\frac{\beta}{\alpha} \cong \frac{t}{s}$, then

$$\begin{aligned} &\delta^2 D^2 + \left(2|\gamma| + |\beta t - \alpha s| + |\beta t + \alpha s| - \frac{|\beta|^2}{s^2 A}\right)\delta D \\ &\quad + |\gamma|^2 + (|\beta t - \alpha s| + |\beta t + \alpha s|)|\gamma| + |\beta t - \alpha s||\beta t + \alpha s| \\ &\quad < 0. \end{aligned} \tag{3.13}$$

Since the conditions of existence of δ satisfying (3.13) is the same to the conditions for equation

$$\begin{aligned} &\delta^2 D^2 + \left(2|\gamma| + |\beta t - \alpha s| + |\beta t + \alpha s| - \frac{|\beta|^2}{s^2 A}\right)\delta D \\ &\quad + |\gamma|^2 + (|\beta t - \alpha s| + |\beta t + \alpha s|)|\gamma| + |\beta t - \alpha s||\beta t + \alpha s| \\ &\quad = 0. \end{aligned}$$

to have two different real solutions and at least one of them is positive, that is,

$$\begin{aligned} &|\beta t - \alpha s|^2 + |\beta t + \alpha s|^2 - 2|\beta t - \alpha s||\beta t + \alpha s|^2 \\ &\quad - \frac{|\beta|^2}{s^2 A}\left(4|\gamma| + 2|\beta t - \alpha s| + 2|\beta t + \alpha s| - \frac{|\beta|^2}{s^2 A}\right) \\ &\quad > 0, \end{aligned} \tag{3.14}$$

and

$$2|\gamma| + |\beta t - \alpha s| + |\beta t + \alpha s| - \frac{|\beta|^2}{s^2 A} < 0. \tag{3.15}$$

Figure 3.5 shows the μ and r satisfying (3.14) and (3.15).

We find that the range of parameters μ and r satisfying (3.14), (3.15) is little different from that satisfying (3.9).

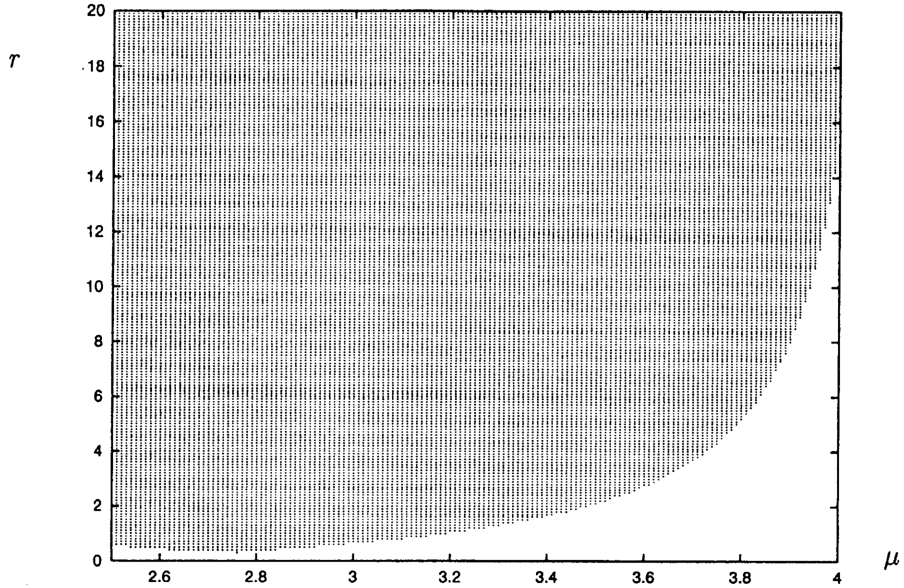

Figure 3.5: The range of μ and r satisfying (3.14) and (3.15)

4 Fractal Dimension of Attractor

Now, we calculate the Lyapunov exponents of the logistic map, and estimate the fractal dimension of its attractor. We use the theorems mentioned in the section 1. S means the logistic map. In the section 3, we saw that the logistic map satisfies (1.10), and (1.11) would be obvious. Let us derive L. It is ordinary Jacobian matrix of the logistic map.

$$L = -2\mu \begin{pmatrix} a_{11}x & a_{12}y \\ ca_{21}x & ca_{22}y \end{pmatrix} \tag{4.1}$$

By the boundedness of the X, we can say that (4.12) is held. So, L^*L is

$$\begin{aligned} L^*L &= (-2\mu)^2 \begin{pmatrix} a_{11} & ca_{21}x \\ a_{12}x & ca_{22}y \end{pmatrix} \begin{pmatrix} a_{11} & a_{12}x \\ ca_{21}x & ca_{22}y \end{pmatrix} \\ &= 4\mu^2 \begin{pmatrix} (a_{11}^2 + c^2a_{21}^2)x^2 & (a_{11}a_{12} + c^2a_{21}a_{22})xy \\ (a_{11}a_{12} + c^2a_{21}a_{22})xy & (a_{12}^2 + c^2a_{22}^2)y^2 \end{pmatrix} \end{aligned} \tag{4.2}$$

Here, if we set

$$\begin{aligned} a_1 &= 4\mu(a_{11}^2 + c^2a_{21}^2) = 4\mu\left(\frac{1+4r+8r^2}{(1+3r)^2}\right) \\ a_2 &= 4\mu(a_{12}^2 + c^2a_{22}^2) = 4\mu\left(\frac{1+4r+5r^2}{(1+3r)^2}\right) \\ b &= 4\mu(a_{11}a_{12} + c^2a_{21}a_{22}) = 4\mu\left(\frac{3r+6r^2}{(1+3r)^2}\right) \end{aligned},$$

then

$$L^*L = \begin{pmatrix} a_1x^2 & bxy \\ bxy & a_2y^2 \end{pmatrix} \tag{4.3}$$

Denoting by λ the eigenvalues of the matrix (4.3) and by I the unit matrix, since

$$\lambda I - L^*L = 0,$$

is satisfied, we solve $|\lambda I - L^*L| = 0$ by using

$$\begin{aligned} |\lambda I - L^*L| &= (\lambda - a_1x^2)(\lambda - a_2y^2) - b^2x^2y^2 \\ &= \lambda^2 - (a_1x^2 + a_2y^2)\lambda + (a_1a_2 - b^2)x^2y^2 \end{aligned}.$$

Thus, we obtain

$$\lambda = \frac{1}{2}\left\{(a_1x^2 + a_2y^2) \pm \sqrt{(a_1x^2 + a_2y^2)^2 - 4(a_1a_2 - b^2)x^2y^2}\right\}. \tag{4.4}$$

$|\lambda I - L^*L| = 0$ by using

$$\begin{aligned} |\lambda I - L^*L| &= (\lambda - a_1x^2)(\lambda - a_2y^2) - b^2x^2y^2 \\ &= \lambda^2 - (a_1x^2 + a_2y^2)\lambda + (a_1a_2 - b^2)x^2y^2 \end{aligned}.$$

Thus, we obtain

$$\lambda = \frac{1}{2}\left\{(a_1x^2 + a_2y^2) \pm \sqrt{(a_1x^2 + a_2y^2)^2 - 4(a_1a_2 - b^2)x^2y^2}\right\}. \tag{4.5}$$

Let each of them be λ_+ and λ_-, then α_1, α_2 are

$$\alpha_1 = \sqrt{\lambda_+}, \quad \alpha_2 = \sqrt{\lambda_-}. \tag{4.6}$$

From (1.4), we get

$$
\begin{aligned}
\omega_1 &= \alpha_1 \\
&= \sqrt{\frac{1}{2}\left\{(a_1x^2+a_2y^2)+\sqrt{(a_1x^2+a_2y^2)^2-4(a_1a_2-b^2)x^2y^2}\right\}} \\
\omega_2 &= \alpha_1\alpha_2 \\
&= \sqrt{\lambda_+}\sqrt{\lambda_-} \\
&= \sqrt{(a_1a_2-b^2)x^2y^2}
\end{aligned} \tag{4.7}
$$

and can calculate $\bar{\omega}_1$, and $\bar{\omega}_2$. Recalling (1.7), (1.8),

$$\Lambda_1 = \bar{\omega}_1, \qquad \Lambda_2 = \frac{\bar{\omega}_2}{\bar{\omega}_1}, \tag{4.8}$$

and

$$\mu_i = \log \Lambda_i, \qquad i = 1, 2 \tag{4.9}$$

Now, we apply Theorem 2 for the logistic map.

First, we consider the case in which the line (3.4) and the asymptote of hyperbola (3.8) are overlapping. We calculate for the values of μ and r satisfying (3.9),(3.12) and $\gamma = 0$ in (3.4). For example, for $\mu = 3.020$ and $r = 1.486$

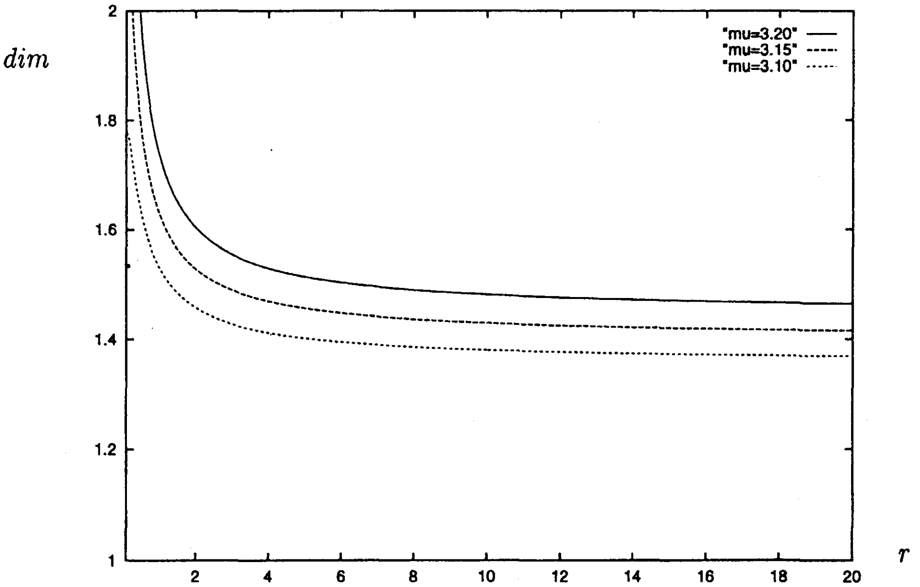

Figure 4.1: The Hausdorff dimension for some values of μ

which approximately satisfy the above conditions, we got the value 1.37 as the upper bound of the Hausdorff dimension. But, if the value of μ is larger than about 3.4, where the value of r is larger than approximately 2.2, then the logistic map is about to diverge to $-\infty$, and the upper bound of Hausdorff dimension is close to 2. Although the Hausdorff dimension is supposed to decrease as the increase of r, the divergence of the system seems to have more dependence on μ.

That tendency is shown in the general case. Figure 4.1 plots the values of parameter r and the Hausdorff dimension. Although the dimension of the attractor of the logistic map is not so larger than 1, we cannot say that this attractor is one-dimensional. Of course, this estimation is upper one, and our invariant set seems to be too large. We are anxious for the lower estimations and the proper invariant sets.

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