Mallat Transformation with Periodic Boundary Condition and Application to Tribology

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Abstract

The relation between Mallat transformation and wavelet transformation is investigated. It is shown that in case of band limited function, if sampling density is 50 percent larger than that is required by Shannon's sampling theorem, Mallat transformation with Meyer basis gives exact wavelet transformation. Mallat transformation for periodic signal is also formulated. This formulation is applied to the tribology. It is shown that the separation of roughness from error of form can be done efficiently by using Meyer basis.

Keyword : Wavelet, Meyer Basis, Mallat transformation,
Sampling theorem

1 Introduction

When one apply wavelet analysis to discrete sampled data, there are mainly two methods. One is so called FFT(Fast Fourier Transformation) aided wavelet transformation. In this method, wavelet coefficients are calculated by constructing interpolated function from sampled data with aid of FFT.

The other method is called Mallat transformation. [1] Mallat transformation directly treats sampled data with linear transformation.

As some information may drooped out due to discrete sampling, one need to know the conditions that these methods gives exact results. In case of FFT aided wavelet transformation, the condition that the exact results can be obtained from sampled data is known as Shannon's sampling theorem.[2]

The aim of this study are to know the condition that Mallat transformation gives exact results and to investigate the relation among Mallat transformation, FFT aided wavelet transformation and wavelet transformation of original function.

In section 2, Mallat transformation is introduced. In section 3, it is shown that reproduction of bandlimited data by Meyer basis can be made if appropriate sampling data was chosen and Mallat transformation gives exact wavelet transformation.

Coefficients of MRA (Multi resolution analysis) are calculated in section 4.

Mallat transformation for periodic signal is formulated without cut off of coefficients in section 5. Section 6 is discussion about extension of Meyer basis. This discrete wavelet transformation is applied to characterization of surface profile in Section 7.

2. Mallat Transformation

In this section, outline of Mallat transformation is explained .Scaling function $\varphi_{j,k}(x)$ and corresponding wavelet $,\psi_{j,k}(x)$ are

$$\varphi_{j,k}(x) = 2^{-j/2}\varphi(2^{-j}x-k) \qquad (1.1a)$$

$$\psi_{j,k}(x) = 2^{-j/2}\psi(2^{-j}x-k) \qquad (1.1b)$$

Let f(t) is original continuous data. Then discrete sampling data with sampling density 1 are expressed as f(n) (n: integer) .

In first Mallat transformation, sampled data are regarded as the coefficients of scaling function, i.e. the function C(x), we analyze in Mallat transformation is

$$\mathrm{C}(x) = C^{(0)}(x) \equiv \sum_{n=-\infty}^{\infty} \mathrm{f}(n)\varphi_{0,n}(x) = \sum_{n=-\infty}^{\infty} \mathrm{f}(n)\varphi(x-n) \quad (1.2)$$

In second step of Mallat transformation, C(x) is decomposed into two parts, coarser scaling function part $C^{(1)}(x)$ and wavelet part $D^{(1)}(x)$ and this procedure continue for positive integer j, as

$$C^{(j-1)}(x) = \mathrm{C}^{(j)}(x) + \mathrm{D}^{(j)}(x) \qquad (1.3a)$$

$$\mathrm{C}^{(j)}(x) = \sum_n \mathrm{c}_n^{(j)}\varphi_{j,n}(x) \qquad (1.3b)$$

$$\mathrm{D}^{(j)}(x) = \sum_n \mathrm{d}_n^{(j)}\psi_{j,k}(x) \qquad (1.3c)$$

Mallat showed that these coefficients $c_k^{(j)}$, $d_k^{(j)}$ can be obtained by following linear transformation.[1)]

$$c_k^{(j)} = \sum_n h_{n-2k} c_n^{(j-1)} \qquad (1.4a)$$

$$d_k^{(j)} = \sum_n g_{n-2k} c_n^{(j-1)} \qquad (1.4b)$$

$$\left(j = 1,2....,\ c_n^{(0)} = f(n)\right)$$

where h_n, g_n are coefficients of Multi-resolution analysis (MRA) as

$$\varphi_{j,k}(x) = \sum_n h_{n-2k} \varphi_{j-1,n}(x) \qquad (1.5a)$$

$$\psi_{j,k}(x) = \sum_n g_{n-2k} \varphi_{j-1,n}(x) \qquad (1.5b)$$

$$g_n = (-1)^n h_{1-n} \qquad (1.5c)$$

Coefficients $c_n^{(j)}, d_n^{(j)}$ are also calculated from equation (1.2) by using orthogonality of wavelets and scaling function as

$$c_k^{(j)} = \int_{-\infty}^{\infty} C(x) \varphi_{j,n}(x) dx \qquad (1.6a)$$

$$d_k^{(j)} = \int_{-\infty}^{\infty} C(x) \psi_{j,n}(x) dx \qquad (1.6b)$$

From equation (1.6a,b), it is known that if we can take appropriate scaling function that makes C(x) coincides with original function f(x), Mallat transformation gives exact wavelet transformation. In next section, the condition that original function can be reproduced from sampled data is investigated.

2. Conditions for Reproduction of Original Function from Discrete Sampled Data

In this section we investigate the property of C(x) which is constructed from sampled data f(n) (n: integer) by scaling function φ(x) as

$$C(x) = \sum_n \varphi(x-n) f(n) \qquad (2.1)$$

The answer can be obtained by applying the procedure to obtain Shannon's sampling theorem. By Fourier transformation of equation (2.1), it is shown

$$\hat{C}(\xi) = \sum_n \int \varphi(x-n) f(n) \exp(-i\xi x)$$
$$= \hat{\varphi}(\xi) \sum_n f(n) \exp(-i\xi n)$$
$$= \hat{\varphi}(\xi) \sum_n \hat{f}(\xi + 2\pi n) \quad (2.2)$$

For obtaining equation (2.2), Poisson's summation formula (2.3) is used.

$$\sum_n \hat{f}(x + 2\pi n) = \sum_n f(n) \exp(-inx) \quad (2.3)$$

In general $\hat{C}(\xi)$ does not coincide with $\hat{f}(\xi)$. What kind of conditions make $\hat{C}(\xi)$ coincide with $\hat{f}(\xi)$.

To know the answer we use the procedure to obtain Shannon's sampling theorem i.e. f(x) and $\varphi(x)$ are band limited,

$$\text{support}[\hat{f}(\bullet)] = [-\Omega, \Omega] \qquad (2.4a)$$
$$\text{support}[\hat{\varphi}(\bullet)] = [-\omega, \omega] \qquad (2.4b)$$

and

$$\hat{\varphi}(\xi) = 1, \quad \xi \in [-\omega_0, \omega_0] \quad (\omega_0 \le \omega) \ (2.5)$$

From equations (2.3)-(2.5), we see that C(x) reproduce f(x) if following conditions are satisfied.

$$\Omega < \omega_0 \qquad (2.6a)$$
$$\Omega + \omega < 2\pi \quad (2.6b)$$

Shannon's sampling theorem is special case that corresponds to $\omega_0 = \omega$. In this case f(x) is reproduced from f(n) as

$$f(\mathrm{x}) = \sum_n \mathrm{sinc}(\mathrm{x-n}) f(\mathrm{n}) \quad (2.7a)$$
$$\mathrm{sinc}(\mathrm{x}) = \frac{\sin \pi(\mathrm{x-n})}{\pi(\mathrm{x-n})} \qquad (2.7b)$$

As sinc(x) is not L^1, we can not use sinc(x) as scaling function to generate wavelet. Is there any scaling function to reproduce C(x) ? In the following, we show that Meyer basis can satisfy above condition by taking appropriate sampling density.

Meyer base M(x) is defined in Fourier space as

$$\hat{M}(\xi) = 1 \qquad \left(\frac{2}{3}\pi \le |\xi|\right)$$

$$= \cos\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}|\xi| - 1\right)\right] \quad \left(\frac{2\pi}{3} \le |\xi| \le \frac{4\pi}{3}\right)$$

$$= 0 \qquad \text{(otherwise)} \qquad (2.8)$$

Function ν(x) is smooth function satisfying

$$\nu(x) = 0 \text{ if } x \le 0$$
$$= 1 \text{ if } x \ge 1 \quad (2.9)$$

From equations (2.4b) and (2.5), we see that equation (2.8) means $\omega_0 = 2\pi/3$ and $\omega = 4\pi/3$.

By using equation (2.4a,b)-(2.6a,b), we conclude that Meyer basis can reproduce original function if

$$\text{support}[\hat{f}(\bullet)] = [-\Omega, \Omega] \quad (\Omega < \frac{2\pi}{3}) \qquad (2.10)$$

In this case the original function is reproduced from sampled data as

$$f(x) = \sum f(n)M(x-n) \qquad (2.11)$$

According to Shannon's sampling theorem, sampling density of this band limited signal is $\Omega/\pi = 2/3$. This fact means that sampling density that is required in case of Meyer basis is 50 percent larger than that is required by Shannon's theorem.

3. Relation between Mallat transformation and wavelet transformation

In previous section, we obtained the conditions that Meyer basis can reproduce original function from sampled data. By using these results, it is known that Mallat transformation gives exact wavelet transformation as

$$c_k^{(j)} = \int_{-\infty}^{\infty} f(x)M_{j,k}(x)dx \quad (j \ge 0) \qquad (3.1a)$$

$$d_k^{(j)} = \int_{-\infty}^{\infty} f(x)\mu_{j,k}(x)dx \quad (j \ge 1) \qquad (3.1b)$$

where $M_{j,k}(x)$Mj,k(x) is Meyer basis

$$M_{j,k}(x) \equiv 2^{-\frac{j}{2}} M(2^{-j}x - k) \qquad (3.2)$$

and $\mu_{j,k}(x)$ is corresponding wavelet generated from Meyer basis and defined as

$$\mu_{j,k}(x) = 2^{-j/2}\mu(2^{-j}x - k) \qquad (3.3a)$$

$$\hat{\mu}(\xi) = e^{i\frac{\xi}{2}}\left[\hat{M}(\xi + 2\pi) + \hat{M}(\xi - 2\pi)\right]\hat{M}\left(\frac{\xi}{2}\right) \qquad (3.3b)$$

Next, we investigate the relation between Mallat transformation and wavelet transformation for general signal.

In this case Shannon's sampling theorem can not be applied but we can still construct two functions C(x), $f_I(x)$ from sampled data.

$$C(x) = \sum M(x-n)f(n) \qquad (3.4a)$$

$$f_I(x) = \sum_{n=-\infty}^{\infty} \mathrm{sinc}(x-n)f(n) \qquad (3,4b)$$

C(x) is used for Mallat transformation and $f_I(x)$ is for FFT aided wavelet transformation. Of course, neither C(x) nor $f_I(x)$ (defined below) reproduce original function f(x).

In the followings of this section, we compare the results obtained by using these two functions .For later discussion we define two kinds of coefficients $W_{j,k}$,and $S_{j,k}$ as

$$S_{j,k} \equiv \int_{-\infty}^{\infty} f_I(x)M_{j,k}(x)dx \qquad (3.5a)$$

$$W_{j,k} \equiv \int_{-\infty}^{\infty} f_I(x)\mu_{j,k}(x)dx \qquad (3.5b)$$

where $W_{j,k}$ is wavelet transformation of $f_I(x)$ and $S_{j,k}$ inner product of scaling function with function $f_I(x)$. These two coefficients are calculated as (See appendix A)

$$S^I_{j,k} = \int_{-\infty}^{\infty}\sum_n f(n)\frac{\sin(\pi(x-n))}{\pi(x-n)}M_{j,k}(x)dx$$

$$= \sum_n f(n)\int_{-\pi}^{\pi} 2^{\frac{j}{2}}\hat{M}(2^j\xi)e^{in\xi}d\xi \qquad (3.6a)$$

$$W^I_{j,k} = \int_{-\infty}^{\infty}\sum_n f(n)\frac{\sin(\pi(x-n))}{\pi(x-n)}\mu_{j,k}(x)dx$$

$$= \sum_n f(n)\int_{-\pi}^{\pi} 2^{\frac{j}{2}}\hat{\mu}(2^j\xi)e^{in\xi}d\xi \qquad (3.6b)$$

Coefficients of Mallat transformation can be calculated directly by equation.(1.4a.b) (see appendix B)

$$c_k^{(j)} = \int_{-\infty}^{\infty} C(x) M_{j,k}(x) dx$$

$$= \sum_n f(n) \int_{-\infty}^{\infty} M(x-n)\, M_{j,k}(x) dx$$

$$= \sum_n f(n) \int_{-\frac{4\pi}{3}}^{\frac{4\pi}{3}} \hat{M}(\xi)\, 2^{\frac{j}{2}} \hat{M}(2^j \xi) e^{in\xi} dx \qquad (3.7a)$$

$$d_k^{(j)} = \int_{-\infty}^{\infty} C(x) \mu_{j,k}(x) dx$$

$$= \sum_n f(n) \int_{-\frac{4\pi}{3}}^{\frac{4\pi}{3}} \hat{M}(\xi)\, 2^{\frac{j}{2}} \hat{\mu}(2^j \xi) e^{in\xi} dx \qquad (3.7b)$$

By considering support of Meyer basis and wavelet,

$$\text{support}[M(2^j \bullet)] = [-\frac{2^{2-j}\pi}{3}, \frac{2^{2-j}\pi}{3}] \qquad (3.8c)$$

$$\text{support}[\mu(2^j \bullet)] = [-\frac{2^{3-j}\pi}{3}, \frac{2^{3-j}\pi}{3}] \qquad (3.8c)$$

we see that for $j \geq 1$ (see also appendix A,B)

$$c_k^{(j)} = S_{j,k} = \sum_n f(n)\, M_{j,k}(n) \qquad (j \geq 1) \qquad (3.9a)$$

$$d_k^{(j)} = W_{j,k} = \sum_n f(n)\, \mu_{j,k}(n) \qquad (j \geq 2) \qquad (3.9b)$$

Note the difference of region of j between equations (3.9a,b) and reproducible case (3.1a,b).

$$c_k^{(j)} = S_{j,k} = \sum_n f(n)\, M_{j,k}(n) = \int_{-\infty}^{\infty} f(x) \varphi_{j,n}(x) dx \quad (j \geq 0) \qquad (3.1a)$$

$$d_k^{(j)} = W_{j,k} = \sum_n f(n)\, \mu_{j,k}(n) = \int_{-\infty}^{\infty} f(x) \psi_{j,n}(x) dx \quad (j \geq 1) \qquad (3.1b)$$

4. Calculation of MRA coefficients of Meyer basis

In this section the MRA coefficients of Meyer basis are calculated.

Values of coefficients, h_n for Mallat transformation are defined as coefficients of multi-resolution analysis (MRA) of Meyer basis as

$$\mathrm{M}(x) = \sqrt{2}\sum_n h_n \mathrm{M}(2x-n) = \sum_n h_n \mathrm{M}_{-1,n}(x) \qquad (4.1a)$$

or

$$\hat{\mathrm{M}}(\xi) = \frac{1}{\sqrt{2}}\sum_n h_n \exp(-in\xi/2)\,\hat{\mathrm{M}}(\xi/2) \qquad (4.1b)$$

$$h_n = \sqrt{2}\int \mathrm{M}(x)\mathrm{M}(2x-n)dx \qquad (4.2)$$

By using Parsevals equality

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}dx = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi \qquad (4.3)$$

Coefficients of multi-resolution analysis, h_n are expressed by Meyer basis as

$$\begin{aligned} h_n &= \frac{1}{2\pi\sqrt{2}}\int_{-\infty}^{\infty}\hat{\mathrm{M}}(\xi)\hat{\mathrm{M}}\left(\frac{\xi}{2}\right)\exp\left(i\frac{\xi n}{2}\right)\mathrm{dx} \\ &= \frac{1}{2\pi\sqrt{2}}\int_{-4\pi/3}^{4\pi/3}\hat{\mathrm{M}}(\xi)\exp\left(i\frac{\xi n}{2}\right)\mathrm{dx} \\ &= \frac{1}{2\pi\sqrt{2}}\int_{-\infty}^{\infty}\hat{\mathrm{M}}(\xi)\exp\left(i\frac{\xi n}{2}\right)\mathrm{dx} \\ &= \frac{1}{\sqrt{2}}\mathrm{M}\left(\frac{n}{2}\right) \\ &= \mathrm{M}_{1,0}(\mathrm{n}) \end{aligned} \qquad (4.4)$$

For deriving equation (4.4), we use the facts

$$\mathrm{M}\left(\frac{\xi}{2}\right) = 1\,, \quad \xi \in \mathrm{support}[\hat{\mathrm{M}}(\bullet)] = [-\frac{4\pi}{3}, \frac{4\pi}{3}] \qquad (4.5)$$

First 32 term of h_n with $v(x) = x^4(35-84x+70x^2-20x^3)$ are shown in Table 1.

5. Mallat transformation for periodic signal

In this section periodic signal of period N, i.e. f(x+N) =f(x) is treated. For convenience we assume $N=2^J$ (J: positive integer). Periodic version of wavelet $M_{j,k}^{per}(x)$, and scaling function $\mu_{j,k}^{per}(x)$ are defined as [2)]

$$M_{j,k}^{per}(x)=\sum_{l=-\infty}^{\infty}M_{j,k}(x+lN) \qquad (5.1a)$$

$$\mu_{j,k}^{per}(x)=\sum_{l=-\infty}^{\infty}\mu_{j,k}(x+lN) \qquad (5.1b)$$

Note that for $j\geq J$

$$M_{j,k}^{per}(x)=1 \qquad (5.2a)$$

$$\mu_{j,k}^{per}(x)=0 \qquad (5.2b)$$

and only finite number of coefficients

$$c_k^{(j)}(per)\quad (0\leq j\leq J,\ 0\leq k\leq 2^{(J-j)}-1) \qquad (5.3a)$$

$$d_k^{(j)}(per)\quad (1\leq j\leq J-1,\ 0\leq k\leq 2^{(J-j)}-1) \qquad (5.3b)$$

are needed. C(x) becomes

$$C^{per}(x)=\sum_{n=0}^{N-1}f(n)M_{0,n}^{per}(x) \qquad (5.3)$$

and Mallat transformation becomes

$$c_k^{(j)}(per)=\sum_{n=0}^{2^{(J-j)}-1}h_{n-2k}^{(j)}c_n^{(j)}(per) \qquad (5.6a)$$

$$d_k^{(j)}(per)=\sum_{n=0}^{2^{(J-j)}-1}g_{n-2k}^{(j)}d_n^{(j)}(per) \qquad (5.6b)$$

$$g_n^{(j)}=(-1)^n h_{1-n}^{(j)} \qquad (5.6c)$$

In this case, coefficients MRA, $_{hn}{}^{(j)}$ depends on the scaling parameter, j. The coefficients are calculated as (See appendix C)

$$h_n^{(j)} = \int_0^N M_{j,0}^{per}(x) M_{j-1,n}^{per}(x) dx$$
$$= \sum_{l=-\infty}^{\infty} h_{n+2^{(J-j)}l}$$
$$= \sum_{|k| \le \frac{2^{(J-j+1)}}{3}} 2^{\left(-J+j-\frac{1}{2}\right)} M\left(2\pi 2^{(-J+j)} k\right) e^{i\pi 2^{(-j+j)} kN} \qquad (5.7)$$

It is also interesting to note that equation (5.7) can be used to calculate the value of h_n (not $h_n^{(j)}$) because the h_n decrease as order of n^{-4}. For example, if we take for example $N=2^{10}$ then only k=0 term is appreciable.

Next, consider the periodic version of equations (3.1a,b) and (3.9a,b). Interpolated function $f_I(x)$ becomes

$$f_I(x) = \sum_{n=0}^{n=N-1} f(n) \sin \pi(x-n) \cot\left(\frac{\pi(x-n)}{N}\right) \qquad (5.8)$$

In deriving equation(5.1), following summation formulae is used.

$$\lim_{m\to\infty} \sum_{n=-m}^{m} \frac{1}{x+n} = \pi \cot \pi x \qquad (5.9)$$

we see that for reproducible case

$$c_k^{(j)}(per) = S_{j,k} = \sum_{n=0}^{N-1} f(n) M_{j,k}^{per}(n) = \int_0^{\infty} f(x) M_{j,k}^{per}(x) dx \quad (j \ge 0) \qquad (5.10a)$$

$$d_k^{(j)}(per) = W_{j,k} = \sum_{N=0}^{N-1} f(n) \mu_{j,k}^{per}(n) = \int_0^{N} f(x) \mu_{j,k}^{per}(x) dx \quad (j \ge 1) \qquad (5.10b)$$

and for general signal

$$c_k^{(j)}(per) = \int_0^{\infty} f_I(x) M_{j,k}^{per}(x) dx = \sum_n f(n) M_{j,k}^{per}(n) \qquad (j \ge 1) \qquad (5.11a)$$

$$d_k^{(j)}(per) = \int_0^{N} f_I(x) \mu_{j,k}^{per}(x) dx = \sum_n f(n) \mu_{j,k}^{per}(n) \qquad (j \ge 2) \qquad (5.11b)$$

6. Extension of Meyer basis

In this section we consider extension of Meyer basis. Meyer basis is generalized in case of $0<a<\pi$ as

$$\hat{M}(\xi) = 1 \qquad (a\pi \geq |\xi|)$$
$$= \cos\left[\frac{\pi}{2} v\left(\frac{1}{a}|\xi| - 1\right)\right] \qquad (a\pi \leq |\xi| \leq 2\pi - a)$$
$$= 0 \qquad \text{(otherwise)} \qquad (6.1)$$

If the sampling density predicted by Shannon's sampling theorem is 1 then sampling density by Generalized Meyer basis is π/a. In the limit of $a \to \pi - 0$, we have Shannon's sampling theorem.

7. Characterization of Surface Roughness by Mallat Transformation

In this section, Mallat transformation is applied to the characterization of surface profile. In tribology (or lubrication engineering) , the surface profile has important roles on friction and wear. To estimate the tribological behavior of engineering surface, it is necessary to use the proper method of separation of surface roughness from error of form. As the standard of roughness and error of form depend on precision for machine desired performance, the flexible mathematical tools are needed for this characterization.[3-5)]

Figure 1 shows example of such engineering surface. In machine engineering, the surface profile is required to decompose three part, (1)longer wavelength part called waviness, (2) short small amplitude variation called roughness and (3) short wavelength large amplitude variation called scar.

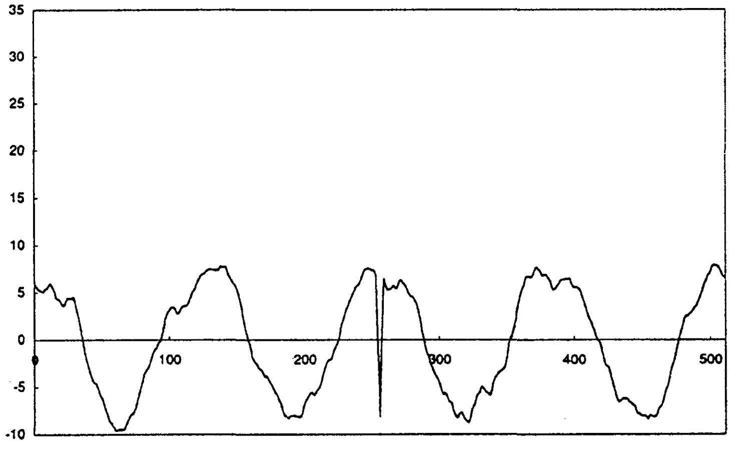

Fig.1 Example of Surface Profile

The standard tool of surface characterization in mechanical engineering is Gaussian filter. As the Gaussian filter separate variation of surface profile by scale, it is very difficult to separate roughness from scar.

On the other hand, wavelet can extract local information. Especially Meyer basis has compact support in Fourier space and fast decaying character in real space and thought to have good performance to characterization. Figure 2 shows the result of separation by Mallat transformation with Meyer basis.

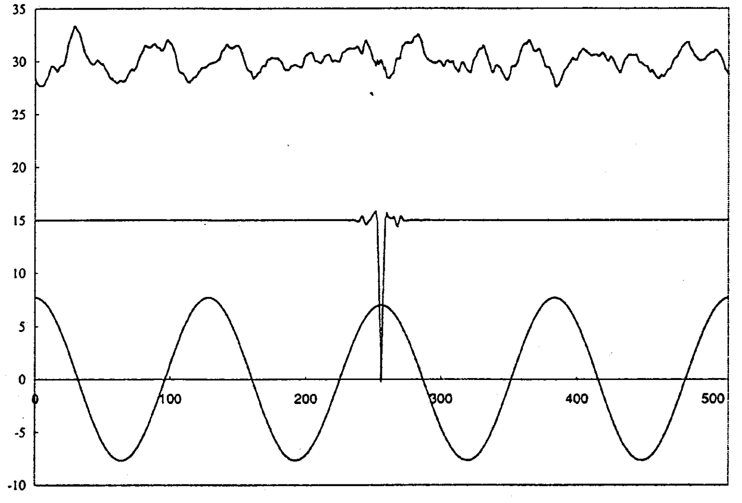

Fig. 2 Filter data by Mallat Transformation with Meyer basis

From this results, it is known that discrete wavelet transformation is powerful tool for the characterization of surface profile and tribology.

Appendix A Calculation of coefficients of Mallat transformation

$$S^{I}_{j,k} = \int_{-\infty}^{\infty} \sum_{n} f(n) \mathrm{sinc}(\pi(x-n)) M_{j,k}(x) dx$$

$$= \sum_{n} f(n) \int_{-\pi}^{\pi} 2^{\frac{j}{2}} \hat{M}(2^{j}\xi) e^{in\xi} d\xi \qquad \text{(A.1)}$$

using Parseval's identity

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi \qquad \text{(A.2)}$$

and fact that Fourier transformation of sinc(x-n) is

$$\int_{-\infty}^{\infty} \text{sinc}(x-n)e^{-i\xi x}dx = \chi_{[-\pi,\pi]}(x)e^{-in\xi} \qquad \text{(A.3)}$$

where χ is characteristic function., we obtain

$$\begin{aligned} S_{j,k} &= \frac{1}{2\pi}\sum_n f(n)\int_{-\infty}^{\infty} \chi_{[-\pi,\pi]}(\xi)e^{i\xi n}2^{\frac{l}{2}}M(2^j\xi)d\xi \\ &= \frac{1}{2\pi}\sum_n f(n)\int_{-\pi}^{\pi} e^{i\xi n}2^{\frac{l}{2}}M(2^j\xi)d\xi \qquad \text{(A.4)} \end{aligned}$$

Coefficients $W_{j,k}$ can be obtained be same procedure.

$$\begin{aligned} W_{j,k} &= \frac{1}{2\pi}\sum_n f(n)\int_{-\infty}^{\infty} \chi_{[-\pi,\pi]}(\xi)e^{i\xi n}2^{\frac{l}{2}}\mu(2^j\xi)d\xi \\ &= \frac{1}{2\pi}\sum_n f(n)\int_{-\pi}^{\pi} e^{i\xi n}2^{\frac{l}{2}}\mu(2^j\xi)d\xi \qquad \text{(A.4)} \end{aligned}$$

By considering the fact that

$$\text{support}[M(2^j \bullet)] = [-\frac{2^{2-j}\pi}{3},\frac{2^{2-j}\pi}{3}] \subset [-\pi,\pi] \quad (j \geq 1) \qquad \text{(A.5a)}$$

$$\text{support}[\mu(2^j \bullet)] = [-\frac{2^{3-j}\pi}{3},\frac{2^{3-j}\pi}{3}] \subset [-\pi,\pi] \quad (j \geq 2) \qquad \text{(A.5b)}$$

we see that

$$\begin{aligned} S_{j,k} &= \frac{1}{2\pi}\sum_n f(n)\int_{-\infty}^{\infty} e^{i\xi n}2^{\frac{l}{2}}M(2^j\xi)d\xi \\ &= \sum_n f(n)\,2^{\frac{-j}{2}}M(2^{-j}x-n) \\ &= \sum_n f(n)\,M_{j,n}(x) \qquad (j \geq 1) \qquad \text{(A.6a)} \end{aligned}$$

and

$$W_{j,k} = \frac{1}{2\pi}\sum_n f(n)\int_{-\infty}^{\infty} e^{i\xi n} 2^{\frac{l}{2}} \mu(2^j\xi) d\xi$$

$$= \sum_n f(n)\, 2^{\frac{-j}{2}} \mu(2^{-j}x - n)$$

$$= \sum_n f(n)\, \mu_{j,n}(x) \qquad (j \geq 2) \qquad \text{(A.6b)}$$

Appendix B Calculation of coefficients of Mallat transformation

$$c_k^{(j)} = \int_{-\infty}^{\infty} C(x)\varphi_{j,k}(x)dx$$

$$= \sum_n \int_{-\infty}^{\infty} f(n)M(x-n)2^{\frac{j}{2}}\, M(2^j x - k)dx$$

$$= \sum_n f(n) \int_{-\infty}^{\infty} \hat{M}(\xi)\, 2^{-\frac{j}{2}}\, \hat{M}(2^{-j}\xi)\, e^{i2^j k\xi + in\xi} d\xi$$

Using equation (A.5a,b) and

$$M(\xi) = 1 \quad \xi \in \text{support}\left[M(2^j \bullet)\right] = \left[-\frac{2^{2-j}\pi}{3}, \frac{2^{2-j}\pi}{3}\right] \quad (j \geq 1) \qquad \text{(A.7a)}$$

$$c_k^{(j)} = \sum_n f(n) \int_{-\infty}^{\infty} \hat{M}(2^j\xi)\, 2^{-\frac{j}{2}}\; e^{i2^j k\xi + in\xi} d\xi$$

$$= \sum_n f(n) 2^{\frac{j}{2}}\, M(n + 2^{-j}k)$$

$$= \sum_n f(n)\, M_{j,k}(n)$$

$$d_k^{(j)} = \int_{-\infty}^{\infty} C(x)\mu_{j,k}(x)dx$$

$$= \sum_n \int_{-\infty}^{\infty} f(n)M(x-n)2^{\frac{j}{2}}\, \mu(2^{-j}x - k)dx$$

$$= \sum_n f(n) \int_{-\infty}^{\infty} \hat{M}(\xi)\, 2^{-\frac{j}{2}}\, \hat{\mu}(2^{-j}\xi)\, e^{i2^j k\xi + in\xi} d\xi$$

also using the fact that

$$M(\xi)=1 \quad \xi \in \text{support}\left[\mu(2^{j}\bullet)\right] \ (j \geq 2) \qquad (A.7a)$$

$$\begin{aligned} d_k^{(j)} &= \sum_n f(n) \int_{-\infty}^{\infty} \hat{\mu}(2^j\xi)\, 2^{-\frac{j}{2}}\, e^{i2^j k\xi + in\xi} d\xi \\ &= \sum_n f(n) 2^{\frac{j}{2}}\, \mu(n+2^{-j}k) \\ &= \sum_n f(n)\, \mu_{j,k}(n) \end{aligned}$$

Appendix C Calculation of Mallat coefficients in periodic case

$$\begin{aligned} h_n^{(j)} &= \int_0^N M_{j,0}^{per}(x) M_{j-1,n}^{per}(x)dx \\ &= \int_0^N M_{j,0}^{per}(x) \sum_{l=-\infty}^{\infty} M_{j-1,n}(x+lN)dx \\ &= \int_{-\infty}^{\infty} M_{j,0}^{per}(x) M_{j-1,n}(x)dx \\ &= \int_{-\infty}^{\infty} \sum_{l=-\infty}^{\infty} M_{j,0}(x+lN) M_{j-1,n}(x)dx \end{aligned}$$

$$\begin{aligned} &= \int_{-\infty}^{\infty} \sum_{l=-\infty}^{\infty} 2^{-\frac{j}{2}} M(2^{-j}x+2^{-j}lN) 2^{-\frac{j-1}{2}} M\left(2^{-(j-1)}x-n\right)dx \\ &= \int_{-\infty}^{\infty} \sum_{l=-\infty}^{\infty} 2^{\frac{1}{2}} M(y) M\left(2y-n+2^{(J-j+1)}lN\right)dx \qquad (N=2^J) \\ &= \sum_{l=-\infty}^{\infty} h_{n+2^{(J-j+1)}l} \\ &= \sum_{|k|\leq\frac{2^{(J-j+1)}}{3}} 2^{\left(-J+j-\frac{1}{2}\right)} M\left(2\pi 2^{(-J+j)}k\right) e^{i\pi 2^{(-J+j)}kN} \end{aligned}$$

$$\sum_{l=-\infty}^{\infty} h_{n+2^{(J-j+1)}l} = \sum_{l=-\infty}^{\infty} 2^{\frac{1}{2}} \int_{-\infty}^{\infty} M(x) M\left(2x - n - 2^{(J-j+1)} l\right) dx$$

$$= \frac{1}{2\pi} \sum_{l=-\infty}^{\infty} 2^{-\frac{1}{2}} \int_{-\infty}^{\infty} \hat{M}(\xi) \hat{M}\left(\frac{\xi}{2}\right) e^{i\frac{n+2^{(J-j+1)} l}{2}\xi} d\xi$$

$$= \frac{1}{2\pi} \sum_{l=-\infty}^{\infty} 2^{-\frac{1}{2}} \int_{-\infty}^{\infty} \hat{M}(\xi) e^{i\frac{n+2^{(J-j+1)} l}{2}\xi} d\xi$$

$$= \sum_{l=-\infty}^{\infty} 2^{-\frac{1}{2}} M\left(\frac{n}{2} + 2^{(J-j)} l\right)$$

$$= \sum_{k=-\infty}^{\infty} 2^{-\frac{1}{2}-(J-j)} \hat{M}\left(2\pi 2^{-(J-j)} k\right) e^{i\pi 2^{-(J-j)} kn}$$

$$= \sum_{|k|<k_m} 2^{-\frac{1}{2}-(J-j)} \hat{M}\left(2\pi 2^{-(J-j)} k\right) e^{i\pi 2^{-(J-j)} kn} \qquad (k_m = \frac{2^{(J-j+1)}}{3})$$

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Wavelet transforms on spheres — a brief survey

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In this brief survey, some recent constructions of the spherical wavelet transforms are mentioned. The following are the constructions I presented at "Symposium on Applied Mathematics". This survey is far from complete.

1) The spherical wavelet transforms introduced by Rubin are associated with appropriate analytic operator families ([4]).

2) The wavelet transforms on $L_2(\mathbb{R})$ and $L_2(\mathbb{R}^2)$ are generalized to tangent bundles TS^1 and TS^2 by Dahlke and Maass ([2]). Specific groups that admit square-integrable representations in $L_2(TS^1)$ and $L_2(TS^2)$ are considered.

3) The concept of a multiresolution analysis on $\mathbb{R}^n$ is generalized to specific Riemannian manifolds by Dahlke ([1]).

The wavelet transforms on $\mathbb{R}^n$ are defined with microlocal views in mind by myself ([3]).

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