Chaos Arising from Euler's Discretization and to Measure Asymmetry of Figures

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1 Introduction

We can show the great distinction between differential equation and difference equation from the point of view of chaos. Let us consider Euler's discretization of autonomous ordinary differential equation with one dimension. Some kinds of differential equations which have strongly stable equilibrium point are apt to turn into chaos in its difference equation, especially with any mesh size [1]. One of necessary conditions for chaos is the break of point symmetry in the neighborhood of stable equilibrium point [2]. If differential equation is symmetric, chaos never happens. Inspired in this asymmetry, we also tried to measure of asymmetry of figures. As the first trial, we will investigate the envelope which is given by bisectors of area.

2 Chaos arising from discretization

Let us consider one dimensional autonomous ordinary differential equation

$$\frac{du}{dt} = f(u) \quad u \in \mathrm{R}^1 \ , \tag{1}$$

under the following assumptions:

$$\begin{cases} f(u) \text{ is continuous in } \mathrm{R}^1 \\ f(u) > 0 \quad (u < 0) \\ f(0) = 0 \\ f(u) < 0 \quad (0 < u) \ . \end{cases}$$

This simple equation has a unique stable equilibrium point $u = 0$. Defined Δt as mesh size, Euler's discretization of (1) is

$$\frac{x_{n+1} - x_n}{\Delta t} = f(x_n) \ ,$$

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$$x_{n+1} = x_n + \Delta t \cdot f(x_n) \quad .$$

Now, define difference equation $F_{\Delta t}(x)$ as

$$F_{\Delta t}(x) = x + \Delta t \cdot f(x) \quad , \tag{2}$$

and we investigate this dynamical system $F_{\Delta t}$.

It is well known that differential equation which has more than 2 equilibrium points and if one of these is stable, its difference equation is chaotic in the sense of Li-Yorke ([3],[4]). In this case, however, we have to assume sufficiently large mesh size. The next example is a remarkable one, because $F_{\Delta t}$ is always chaotic with any size of mesh [1].

$$\frac{du}{dt} = \begin{cases} \sqrt{-u} & (u < 0) \\ -3.41\sqrt{u} & (u \geq 0) \end{cases} \quad .$$

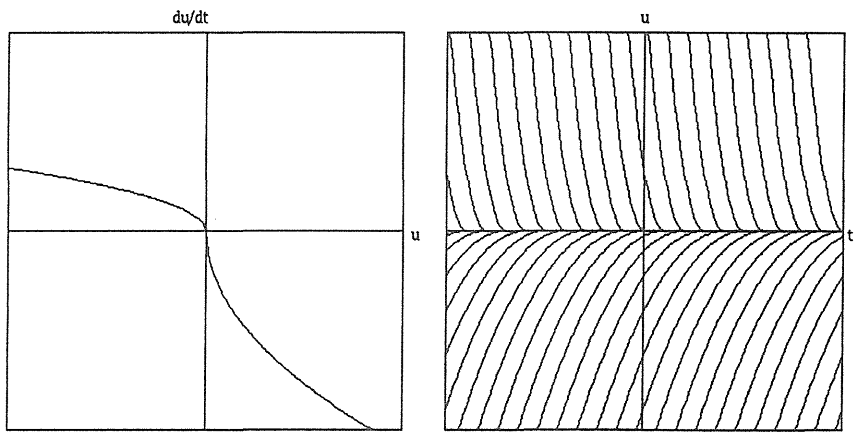

Figure 1: differential equation and its solution

We can easily solve this differential equation (see, figure 1). Equilibrium point is so strongly stable that uniqueness of solution is broken at $u = 0$. On the other hand, in difference equation $F_{\Delta t}$, the fixed point $x = 0$ is strongly unstable (see, figure 2). This example shows the great distinction between differential equation and difference equation. We found out the following two necessary conditions for chaos in the case of sufficiently small mesh size.

Theorem 1. ([5])
(i) $F^n_{\Delta t}(x)$ *converges to the equilibrium point in its neighborhood with a sufficiently small* Δt.

$$\Longleftrightarrow \overline{\lim_{u\to -0}} \frac{f(u)}{-u} < +\infty \quad or \quad \overline{\lim_{u\to +0}} \frac{-f(u)}{u} < +\infty \ .$$

(ii) $F^n_{\Delta t}(x)$ *never converges to the equilibrium point with any* Δt.

$$\Longleftrightarrow \lim_{u\to 0} \frac{f(u)}{u} = -\infty \ .$$

Theorem1 shows that non-Lipcshitz continuity is one of necessary condition for chaos. The next theorem provides another necessary condition, that is to say, if $f(u)$ is point symmetric around $u = 0$, chaos never occurs.

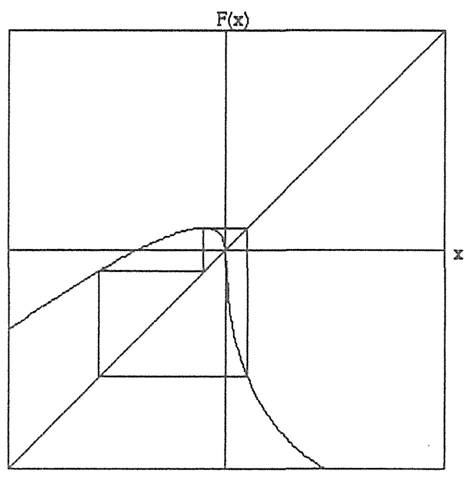

Figure 2: difference equation and 3-periodic orbit

Theorem 2. ([2])
Assume that f *holds the next conditions* $(*)$.

$$(*)\begin{cases} \textit{(i) } f(u) \textit{ is continuous and strictly decreasing in } \mathbf{R}^1 \\ \textit{(ii) } f(0) = 0 \\ \textit{(iii) } f(u) \textit{ is } \mathrm{C}^1\textit{-class except } u = 0 \\ \textit{(iv) } \lim_{u\to 0} f(u)/u = -\infty \\ \textit{(v) } f(-u) = -f(u) \quad (f(u) \textit{ is an odd function}). \end{cases}$$

Then,
(I) for any $\Delta t > 0$, $F_{\Delta t}$ doesn't have any periodic orbits with period $n > 2$,
(II) there exists $\Delta T > 0$, such that for any Δt $(0 < \Delta t < \Delta T)$, there is a periodic orbit with period 2, and any 2-periodic point x_0 satisfies $F_{\Delta t}(x_0) = -x_0$.

3 Measurement of asymmetry of figure

We have already seen that there is some relation between chaos and asymmetry. Now, how can we measure a distance from point symmetry of figures? Here, we will discuss about 2-dimensional figures which have finite areas.

Let S be a convex 2-dimensional figure with finite area and whose boundary ∂S is in piecewise C^1-class. $\{l_\theta\}$ ($\theta \in [0, \pi)$) is a one parameter family of lines which cut S into halves, and E is a envelope given by $\{l_\theta\}$.

Proposition 1.
Envelope E is a geometric locus of middle point of line segment l_θ. Thus,

$$S \text{ is point symmetric } \iff E = \{G\} \ ,$$

where G is a center of gravity of S.

The above proposition indicates that E has some spread unless S is point symmetric.

Proposition 2. (figure 3)
If S is asymmetric and ∂S is piecewise linear, E is given as a piecewise hyperbolic closed curve.

Remark: If both ends of segment l_θ are on the same ellipse (respectively, parabola, hyperbola), E is also given as ellipse (respectively, parabola, hyperbola) in its neighborhood.

Proposition 3.
Unless E degenerates into $\{G\}$, the number of cusps of E is odd and greater than 3. In particular, if ∂S is in C^1-class, any S has at least three l_θ such that tangents of ∂S are parallel at the cross points of ∂S and l_θ.

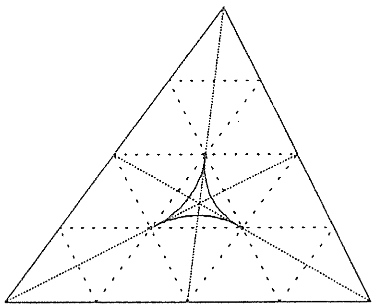

Figure 3: envelope of triangle

Now we define a function $N(x) : \mathbf{R}^2 \to \mathbf{N}$ as follows:

$$N(x) \stackrel{\text{def}}{=} \#\{l_\theta \text{ passes through } x\} \quad .$$

With this function, we can characterize E.

Proposition 4.
E is given as a set of discontinuous points of function $N(x)$.

Proposition 5.

$$N(x) = 2 \cdot n(E; x) + 1 \quad (\, x \in R^2/E \,) \; ,$$

where $n(E;x)$ denotes a winding number of E around x. Therefore almost everywhere, the value of $N(x)$ is odd.

From the above facts, let us define a measure of asymmetry $D(S)$ as

$$D(S) \stackrel{\text{def}}{=} \frac{1}{|S|} \iint n(E;x) \; dxdy \quad .$$

Example: Any triangle has the same value $D = \frac{3}{4}\log 2 - \frac{1}{2}$, and of course, $D = 0$ if S is point symmetric.

As for quadrangle, using invariant property of D under affine transformation, without loss of generality, we can set 4 vertexes of quadrangle

$(0,0),(1,0),(0,1),(a,b)$ where $0 \le a \le b \le 1$.

$$\begin{aligned}
D(\forall\text{triangle}) &= \frac{3}{4}\log 2 - \frac{1}{2} \approx 0.01986 \\
D(\text{rhombus } (b=1)) &= \frac{1}{2}\log\frac{2}{1+a} + \frac{1+a^2}{4(1-a^2)}\log\frac{2}{1+a^2} - \frac{1}{2(a+1)} \\
D(\text{kite } (b=a)) &= \frac{1+a}{4(1-a)}\log\frac{1}{a} - \frac{1}{2} \\
D(\text{convex quadrangle}) &= \frac{1}{2}\log\frac{2}{a+b} + \frac{2b-(a+b)(1-a)}{4(a+b)(1-a)}\log\frac{2b}{2b-(a+b)(1-a)} \\
&\quad + \frac{(a+b-1)b}{2(a+b)(1-b)}\log\frac{1}{b} - \frac{1}{2} \\
D(\text{regular pentagon}) &= \frac{1}{4\tan^2\theta}\log\frac{1+\tan^2\theta}{1-\tan^2\theta} - \frac{1}{2} \approx 0.001870
\end{aligned}$$

where θ is $\pi/10$.

Theorem 3.
The distortion of convex quadrangle is less than that of triangle.

Remark: D of concave quadrangle may be greater than that of triangle.

Conjecture

$$D(\forall\text{convex figure}) \le D(\text{triangle}) = \frac{3}{4}\log 2 - \frac{1}{2} \quad .$$

4 Envelope of 3-dimensional solid

In the case of 3-dimensional solid, envelope is given as a geometric locus of the center of gravity of half-cut plane domain. Regular tetrahedron is an only asymmetric solid among Platonic regular polyhedrons (figure 4). This E looks like modified octahedron but it is composed of 7 surfaces.

E of general polyhedron has self-intersection and its Eulerian number is 1. Winding number of 2-dimensional figure is correspond to number of loops which is given by points of tangency in 3-dimensional figure.

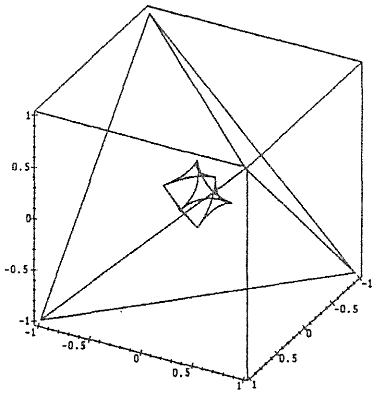

Figure 4: tetrahedron and its envelope

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