Josai Mathematical Monographs
vol 5 (2012), pp. 19-37

THE GROWTH FUNCTIONS OF NONCOMPACT 3-DIMENSIONAL HYPERBOLIC COXETER GROUPS WITH 4 AND 5 GENERATORS

YOHEI KOMORI AND YURIKO UMEMOTO

ABSTRACT. We calculate the growth functions of noncompact 3-dimensional hyperbolic Coxeter groups with 4 and 5 generators.

1. INTRODUCTION: THE GROWTH FUNCTION $f_S(t)$ OF (Γ, S)

Let (Γ, S) be the pair of an infinite group Γ and a finite set of generators S satisfying $S = S^{-1}$. Then we can define the *word length* $\ell_S(x)$ *of* $x \in \Gamma$ *with respect to* S by the smallest integer $n \geq 0$ for which there exist $s_1, s_2, \cdots, s_n \in S$ such that $x = s_1 s_2 \cdots s_n$. For $x, y \in \Gamma$, set $d_S(x,y) := \ell_S(x^{-1}y)$. Then d_S becomes a metric on Γ and the left multiplication of an element of Γ on Γ itself becomes an isometry of the metric space (Γ, d_S).

The *growth function* $f_S(t)$ of (Γ, S) is the formal power series $\sum_{k=0}^{\infty} a_k t^k$ where a_k is the number of elements $g \in \Gamma$ satisfying $\ell_S(g) = k$.

Since the cardinality of Γ is infinite and that of S is finite, the *growth rate* of (Γ, S) $\omega := \limsup_{k\to\infty} \sqrt[k]{a_k}$ is bigger than or equal to 1 while it is less than or equal to the cardinality $|S|$ of S, since $a_k \leq |S|^k$, i.e. $1 \leq \omega \leq |S|$. By means of Cauchy-Hadamard formula, the *radius of convergence* R of $f_S(t)$ is the reciprocal of ω, i.e. $1/|S| \leq R \leq 1$. Therefore $f_S(t)$ is not only a formal power series but also an analytic function of $t \in \mathbb{C}$ on the disk $|t| < R$.

Γ is said to be of *exponential growth* if $\omega > 1$. The notion of exponential growth is independent from the choice of S ([5]). Milnor proved the following fundamental result for the growth rate: the fundamental group $\pi_1(X)$ of a compact Riemannian manifold X is of exponential growth if all sectional curvatures of X are strictly negative ([10]).

We remark that the growth function of the direct porduct $(G_1 \times G_2, S_1 \cup S_2)$ of (G_1, S_1) and (G_2, S_2) is the product of growth functions of (G_1, S_1) and (G_2, S_2), i.e. $f_{S_1 \cup S_2}(t) = f_{S_1}(t) \cdot f_{S_2}(t)$.

2. HYPERBOLIC COXETER GROUPS

2.1. Hyperbolic Coxeter groups and their growth functions.

A convex polyhedron P of finite volume in the n-dimensional hyperbolic space $\mathbb{H}^n$ is called a *Coxeter polyhedron* if its dihedral angles are the integer parts of

2010 *Mathematics Subject Classification.* Primary 20F55, Secondary 20F65.
Key words and phrases. growth function; Coxeter group; Perron number.

π. Any Coxeter polyhedron is a fundamental domain of the discrete group Γ generated by S the reflections with respects to its facets. We call (Γ, S) a *n-dimensional hyperbolic Coxeter group*. In particular when P is a (generalized) simplex of $\mathbb{H}^n$, (Γ, S) is also called a *(generalized) simplex reflection group* ([12]). From now on we assume that (Γ, S) is a hyperbolic Coxeter group. In this case the growth function $f_S(t)$ which is analytic on $|t| < R$ extends to a rational function $P(t)/Q(t)$ on $\mathbb{C}$ by analytic continuation where $P(t), Q(t) \in \mathbb{Z}[t]$ are relatively prime: in practice there are formulas due to Solomon and Steinberg to calculate the rational function $P(t)/Q(t)$ from the Coxeter diagram of (Γ, S) ([14, 15]. See also [6]).

Theorem 1. *(Solomon's formula:)*
The growth function $f_S(t)$ of an irreducible spherical Coxeter group (Γ, S) can be written as $f_S(t) = \prod_{i=1}^{k}[m_i + 1]$ where $[n] := 1 + t + \cdots + t^{n-1}$ and $\{m_1, m_2, \cdots, m_k\}$ is the set of exponents of (Γ, S).

We give explicitly the growth functions of irreducible spherical Coxeter groups in Table 1 where we use the notation $[n, m] = [n][m]$ (See [7] p.59).

Graph	Exponents	$f_S(t)$
A_n	$1, 2, \cdots, n$	$[2, 3, \cdots, n+1]$
B_n	$1, 3, \cdots, 2n-1$	$[2, 4, \cdots, 2n]$
D_n	$1, 3, \cdots, 2n-3, n-1$	$[2, 4, \cdots, 2n-2][n]$
E_6	$1, 4, 5, 7, 8, 11$	$[2, 5, 6, 8, 9, 12]$
E_7	$1, 5, 7, 9, 11, 13, 17$	$[2, 6, 8, 10, 12, 14, 18]$
E_8	$1, 7, 11, 13, 17, 19, 23, 29$	$[2, 8, 12, 14, 18, 20, 24, 30]$
F_4	$1, 5, 7, 11$	$[2, 6, 8, 12]$
H_3	$1, 5, 9$	$[2, 6, 10]$
H_4	$1, 11, 19, 29$	$[2, 12, 20, 30]$
$I_2(m)$	1, m-1	$[2, m]$

TABLE 1. The growth functions of irreducible spherical Coxeter groups

Theorem 2. *(Steinberg's formula)*
Let (Γ, S) be a hyperbolic Coxeter group. Let us denote the Coxeter subgroup of (Γ, S) generated by the subset $T \subseteq S$ by (Γ_T, T), and denote its growth function by $f_T(t)$. Set $\mathcal{F} = \{T \subseteq S \,:\, \Gamma_T$ is finite $\}$. Then

$$\frac{1}{f_S(t^{-1})} = \sum_{T \in \mathcal{F}} \frac{(-1)^{|T|}}{f_T(t)}.$$

In this case, $t = R$ is a pole of $f_S(t)$ on the circle $|t| = R$. Hence R is a real zero of the denominator $Q(t)$ closest to the origin $0 \in \mathbb{C}$ of all zeros of $Q(t)$. Solomon's formula implies that $P(0) = 1$. Hence $a_0 = 1$ implies that $Q(0) = 1$. Moreover de la Harpe proved that Γ is of exponential growth, i.e.

$\omega > 1$ ([4]). Therefore $\omega > 1$, the reciprocal of R, becomes a real algebraic integer whose conjugates have moduli less than or equal to the modulus of ω. If $t = R$ is a unique zero of $Q(t)$ with the smallest modulus, then $\omega > 1$ is a real algebraic integer whose conjugates have moduli less than the modulus of ω: such an algebraic integer is called a *Perron number.*

The growth function $f_S(t)$ of a *cocompact* hyperbolic Coxeter group with Coxeter generators has special symmetries: Serre and Charney-Davis proved that for cocompact n-dimensional hyperbolic Coxeter groups, $f_S(t)$ is *reciprocal,* i.e. $f_S(1/t) = f_S(t)$ when n is even, and *anti-reciprocal,* i.e. $f_S(1/t) = -f_S(t)$ when n is odd ([13, 2]). But in general we can expect neither reciprocal nor anti-reciprocal properties for noncompact hyperbolic Coxeter groups.

2.2. **Low dimensional cases.** For low dimensional cocompact hyperbolic Coxeter groups, the denominator $Q(t)$ of $f_S(t)$ has more special properties: a *Salem number* is a real algebraic integer $\tau > 1$ such that τ^{-1} is an algebraic conjugate of τ and all algebraic conjugates of τ other than τ and τ^{-1} lie on the unit circle. The monic irreducible polynomial over $\mathbb{Z}$ of a Salem number is called a *Salem polynomial.* For two and three dimensional cocompact hyperbolic Coxeter groups, Cannon-Wagraich and Parry showed that $Q(t)$ is a product of distinct irreducible cyclotomic polynomials (possibly none) and exactly one Salem polynomial. In particular the growth rate ω is a Salem number. ([1, 11]). From the definition, a Salem number is a Perron number.

Kellerhals and Perren calculated growth functions of four dimensional cocompact hyperbolic Coxeter groups with five and six generators and showed that Salem polynomials do not appear as factors of denominators. They also checked that ω is a Perron number numerically. ([8]).

For noncompact case, Floyd proved that the growth rate ω of two dimensional noncompact hyperbolic Coxeter groups is a *Pisot-Vijayaraghavan number*, where a real algebraic integer $\tau > 1$ is called a Pisot-Vijayaraghavan number if algebraic conjugates of τ other than τ lie in the unit disk ([3]). From the definition, a Pisot-Vijayaraghavan number is also a Perron number. In practice Floyd showed that for any two dimensional noncompact hyperbolic Coxeter group (Γ, S), there exists a sequence of two dimensional cocompact hyperbolic Coxeter groups whose growth rates, which are Salem numbers, converges the growth rate ω of (Γ, S) to conclude that ω is a Pisot-Vijayaraghavan number.

In this paper we start to go to the next stage: we calculate growth functions of three dimensional noncompact hyperbolic Coxeter groups with four and five generators. The three dimensional noncompact hyperbolic Coxeter groups with four generators were classified by Lannér ([9]) (see Table 2) and with five generators by Tumarkin ([16]) (see Table 3). We represent their classifications in terms of Coxeter diagram since any Coxeter polyhedron P can be represented by its Coxeter diagram: the nodes of a Coxeter diagram correspond to the facets of P. Two nodes are joined by a m-labeled edge if the corresponding dihedral angle is equal to π/m. It is a custom to omit labeling for $m = 3$ and omit putting an edge for $m = 2$. If the corresponding facets are parallel the nodes are joined by a bold edge.

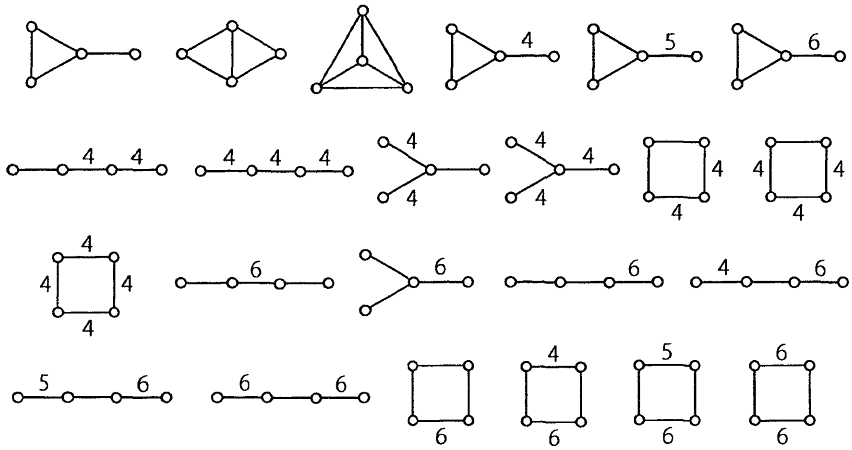
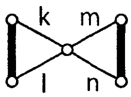
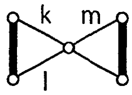


TABLE 2. 4-generators



k = 2, 3, 4; m = 2, 3, 4;
l = 3, 4; n = 3, 4.

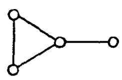


k = 5, 6; m = 2, 3;
l = 2, 3, 4, 5, 6.

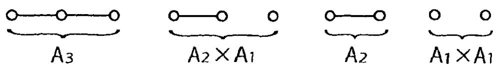
TABLE 3. 5-generators

3. GROWTH FUNCTIONS

Now we shall show the way to calculate the growth functions. This is the first diagram in Table 2.



There are two non-isomorphic Coxeter subgroups with three generators and two non-isomorphic Coxeter subgroups with two generators. Their corresponding growth functions are given below.



type of subgroup	growth function	number
A_3	$[2,3,4]$	2
$A_2 \times A_1$	$[2,2,3]$	1
A_2	$[2,3]$	4
$A_1 \times A_1$	$[2,2]$	2

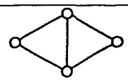
Also there are 4 Coxeter subgroups with one generator, i.e. of type A_1. By Theorem 2, we have that

$$\frac{1}{f_S(t^{-1})} = \frac{-2}{[2,3,4]} + \frac{-1}{[2,2,3]} + \frac{4}{[2,3]} + \frac{2}{[2,2]} + \frac{-4}{[2]} + 1.$$

This yields the growth function

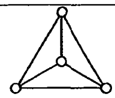
$$f_S(t) = \frac{(t+1)(t^2+1)(t^2+t+1)}{(t-1)(t^3+t-1)}.$$

The growth functions for the other three dimensional noncompact hyperbolic Coxeter groups with four generators are given below. We remark that every Coxeter group has 4 Coxeter subgroups of type A_1.



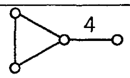
type of subgroup	number
A_3	2
A_2	5
$A_1 \times A_1$	1

$$f_S(t) = \frac{(t+1)(t^2+1)(t^2+t+1)}{(t-1)(t^4+t^3+t^2+t-1)}$$



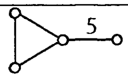
type of subgroup	number
A_2	6

$$f_S(t) = \frac{(t+1)(t^2+t+1)}{(t-1)(3t^2+t-1)}$$



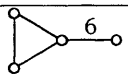
type of subgroup	number
B_3	2
$A_2 \times A_1$	1
B_2	1
A_2	3
$A_1 \times A_1$	2

$$f_S(t) = \frac{(t+1)^3(t^2+1)(t^2-t+1)(t^2+t+1)}{(t-1)(t^7+t^6+2t^5+2t^4+t^3+t^2-1)}$$



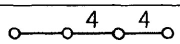
type of subgroup	number
H_3	2
$A_2 \times A_1$	1
$I_2(5)$	1
A_2	3
$A_1 \times A_1$	2

$$f_S(t) = \frac{(t+1)^3(t^2-t+1)(t^2+t+1)(t^4-t^3+t^2-t+1)}{(t-1)(t^9+t^7+t^6+t^4+t^2+t-1)}$$



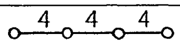
type of subgroup	number
$A_2 \times A_1$	1
$I_2(6)$	1
A_2	3
$A_1 \times A_1$	2

$$f_S(t) = \frac{(t+1)^2(t^2-t+1)(t^2+t+1)}{(t-1)(2t^5+t^4+t^2+t-1)}$$



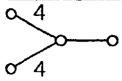
type of subgroup	number
B_3	1
$B_2 \times A_1$	1
$A_2 \times A_1$	1
B_2	2
A_2	1
$A_1 \times A_1$	3

$$f_S(t) = \frac{(t+1)^3(t^2+1)(t^2-t+1)(t^2+t+1)}{(t-1)(t^7+t^6+t^5+t^4+t^3-1)}$$



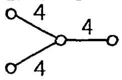
type of subgroup	number
$B_2 \times A_1$	2
B_2	3
$A_1 \times A_1$	3

$$f_S(t) = \frac{(1+t)^3(1+t^2)}{(t-1)(t^2+t-1)(t^2+t+1)}$$



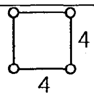
type of subgroup	number
B_3	2
$A_1 \times A_1 \times A_1$	1
B_2	2
A_2	1
$A_1 \times A_1$	3

$$f_S(t) = \frac{(t+1)^3(t^2+1)(t^2-t+1)}{(t-1)(t^5+t^3+t-1)}$$



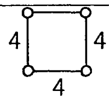
type of subgroup	number
$A_1 \times A_1 \times A_1$	1
B_2	3
$A_1 \times A_1$	3

$$f_S(t) = \frac{(t+1)^3(t^2+1)}{(t-1)(2t^4+3t^3+t^2-1)}$$



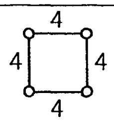
type of subgroup	number
B_3	2
A_3	1
B_2	2
A_2	2
$A_1 \times A_1$	2

$$f_S(t) = \frac{(t+1)^3(t^2+1)(t^2-t+1)}{(t-1)(t^5+t^4+t^2+t-1)}$$



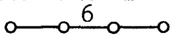
type of subgroup	number
B_3	2
B_2	3
A_2	1
$A_1 \times A_1$	2

$$f_S(t) = \frac{(t+1)^3(t^2+1)(t^2-t+1)}{(t-1)(t^6+t^5+t^4+t^3+t^2+t-1)}$$



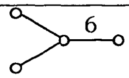
type of subgroup	number
B_2	4
$A_1 \times A_1$	2

$$f_S(t) = \frac{(t+1)^2(t^2+1)}{(t-1)(3t^3+t^2+t-1)}$$



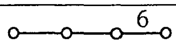
type of subgroup	number
$A_2 \times A_1$	2
$I_2(6)$	1
A_2	2
$A_1 \times A_1$	3

$$f_S(t) = \frac{(t+1)^2(t^2-t+1)(t^2+t+1)}{(t-1)(t^5+t^4+t-1)}$$



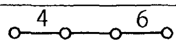
type of subgroup	number
A_3	1
$A_1 \times A_1 \times A_1$	1
$I_2(6)$	1
A_2	2
$A_1 \times A_1$	3

$$f_S(t) = \frac{(t+1)^3(t^2+1)(t^2-t+1)(t^2+t+1)}{(t-1)(t^8+2t^7+2t^6+3t^5+t^4+t^3-1)}$$

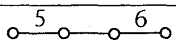


type of subgroup	number
A_3	1
$I_2(6) \times A_1$	1
$A_2 \times A_1$	1
$I_2(6)$	1
A_2	2
$A_1 \times A_1$	3

$$f_S(t) = \frac{(t+1)^3(t^2+1)(t^2-t+1)(t^2+t+1)}{(t-1)(t^7+t^6+t^5+t^4-1)}$$

	
type of subgroup	number
B_3	1
$I_2(6) \times A_1$	1
$B_2 \times A_1$	1
$I_2(6)$	1
B_2	1
A_2	1
$A_1 \times A_1$	3

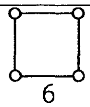
$$f_S(t) = \frac{(t+1)^3(t^2+1)(t^2-t+1)(t^2+t+1)}{(t-1)(t^3+t-1)(t^4+t^3+t^2+t+1)}$$

	
type of subgroup	number
H_3	1
$I_2(6) \times A_1$	1
$I_2(5) \times A_1$	1
$I_2(6)$	1
$I_2(5)$	1
A_2	1
$A_1 \times A_1$	3

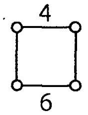
$$f_S(t) = \frac{(t+1)^3(t^2-t+1)(t^2+t+1)(t^4-t^3+t^2-t+1)(t^4+t^3+t^2+t+1)}{(t-1)(t^{13}+t^{12}+2t^{11}+2t^{10}+2t^9+2t^8+2t^7+2t^6+2t^5+t^4+t^3-1)}$$

	
type of subgroup	number
$I_2(6) \times A_1$	2
$I_2(6)$	2
A_2	1
$A_1 \times A_1$	3

$$f_S(t) = \frac{(t+1)^3(t^2-t+1)(t^2+t+1)}{(t-1)(t^2+t-1)(t^4+t^3+t^2+t+1)}$$

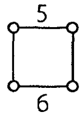
	
type of subgroup	number
A_3	2
$I_2(6)$	1
A_2	3
$A_1 \times A_1$	2

$$f_S(t) = \frac{(t+1)^2(t^2+1)(t^2-t+1)(t^2+t+1)}{(t-1)(t^7+t^6+2t^5+t^4+t^3+t-1)}$$



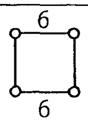
type of subgroup	number
B_3	2
$I_2(6)$	1
B_2	1
A_2	2
$A_1 \times A_1$	2

$$f_S(t) = \frac{(t+1)^3(t^2+1)(t^2-t+1)(t^2+t+1)}{(t-1)(t^4+t^3+t^2+t-1)(t^4+t^3+t^2+t+1)}$$



type of subgroup	number
H_3	2
$I_2(6)$	1
$I_2(5)$	1
A_2	2
$A_1 \times A_1$	2

$$f_S(t) = \frac{(t+1)^3(t^2-t+1)(t^2+t+1)(t^4-t^3+t^2-t+1)}{(t-1)(t^{10}+t^9+t^8+t^7+t^6+t^5+t^4+t^3+t^2+t-1)}$$



type of subgroup	number
$I_2(6)$	2
A_2	2
$A_1 \times A_1$	2

$$f_S(t) = \frac{(t+1)^2(t^2-t+1)(t^2+t+1)}{(t-1)(3t^5+t^4+t^3+t^2+t-1)}$$

$$f_S(t) = \frac{(t+1)^3(t^2+1)(t^2-t+1)(t^2+t+1)}{(t-1)(t^7+2t^6+2t^5+2t^4+2t^3+t^2+t-1)}$$

(2,3,4,4)	
type of subgroup	number
B_3	2
$B_2 \times A_1$	2
B_2	2
A_2	1
$A_1 \times A_1$	5

$$f_S(t) = \frac{(t+1)^3(t^2+1)(t^2-t+1)}{(t-1)(t^5+t^4+t^3+2t-1)}$$

(2,4,2,4)	
type of subgroup	number
$B_2 \times A_1$	2
$A_1 \times A_1 \times A_1$	1
B_2	2
$A_1 \times A_1$	6

$$f_S(t) = \frac{(t+1)^3(t^2+1)}{(t-1)(t^4+2t^3+t^2+t-1)}$$

(2,4,3,3)	
type of subgroup	number
B_3	2
$A_2 \times A_1$	2
B_2	1
A_2	2
$A_1 \times A_1$	5

$$f_S(t) = \frac{(t+1)^3(t^2+1)(t^2-t+1)(t^2+t+1)}{(t-1)(t^7+2t^6+2t^5+3t^4+2t^3+t^2+t-1)}$$

(2,4,3,4)	
type of subgroup	number
B_3	1
$B_2 \times A_1$	1
$A_2 \times A_1$	1
B_2	2
A_2	1
$A_1 \times A_1$	5

$$f_S(t) = \frac{(t+1)^3(t^2+1)(t^2-t+1)(t^2+t+1)}{(t-1)(t^8+2t^7+3t^6+3t^5+3t^4+3t^3+t^2+t-1)}$$

(2,4,4,4)	
type of subgroup	number
$B_2 \times A_1$	2
B_2	3
$A_1 \times A_1$	5

$$f_S(t) = \frac{(t+1)^3(t^2+1)}{(t-1)(2t^4+3t^3+2t^2+t-1)}$$

(3,3,3,3)	
type of subgroup	number
A_3	4
A_2	4
$A_1 \times A_1$	4

$$f_S(t) = \frac{(t+1)^2(t^2+1)}{(t-1)(t^2+2t-1)}$$

(3,3,3,4)	
type of subgroup	number
B_3	2
A_3	2
B_2	1
A_2	3
$A_1 \times A_1$	4

$$f_S(t) = \frac{(t+1)^3(t^2+1)(t^2-t+1)}{(t-1)(t^5+2t^4+t^2+2t-1)}$$

(3,3,4,4)	
type of subgroup	number
B_3	4
B_2	2
A_2	2
$A_1 \times A_1$	4

$$f_S(t) = \frac{(t+1)^3(t^2+1)(t^2-t+1)}{(t-1)(t^5+2t^4+t^3+t^2+2t-1)}$$

(3,4,3,4)	
type of subgroup	number
B_3	2
A_3	1
B_2	2
A_2	2
$A_1 \times A_1$	4

$$f_S(t) = \frac{(t+1)^3(t^2+1)(t^2-t+1)}{(t-1)(t^6+t^5+2t^4+t^3+t^2+2t-1)}$$

(3,4,4,4)	
type of subgroup	number
B_3	2
B_2	3
A_2	1
$A_1 \times A_1$	4

$$f_S(t) = \frac{(t+1)^3(t^2+1)(t^2-t+1)}{(t-1)(2t^6+t^5+2t^4+2t^3+t^2+2t-1)}$$

(4,4,4,4)	
type of subgroup	number
B_2	4
$A_1 \times A_1$	4

$$f_S(t) = \frac{(t+1)^2(t^2+1)}{(t-1)(4t^3+t^2+2t-1)}$$

(k,l,m)=(5,2,2)	
type of subgroup	number
H_3	1
$I_2(5) \times A_1$	1
$A_2 \times A_1$	1
$A_1 \times A_1 \times A_1$	1
$I_2(5)$	1
A_2	1
$A_1 \times A_1$	6

$$f_S(t) = \frac{(t+1)^3(t^2-t+1)(t^2+t+1)(t^4-t^3+t^2-t+1)(t^4+t^3+t^2+t+1)}{(t-1)(t^{13}+t^{12}+2t^{11}+2t^{10}+3t^9+2t^8+3t^7+2t^6+3t^5+t^4+2t^3+t-1)}$$

(5,2,3)	
type of subgroup	number
H_3	2
$A_2 \times A_1$	2
$I_2(5)$	1
A_2	2
$A_1 \times A_1$	5

$$f_S(t) = \frac{(t+1)^3(t^2-t+1)(t^2+t+1)(t^4-t^3+t^2-t+1)}{(t-1)(t^9+t^8+2t^6+t^4+t^3+2t-1)}$$

(5,3,2)	
type of subgroup	number
H_3	1
A_3	1
$I_2(5) \times A_1$	1
$A_2 \times A_1$	1
$I_2(5)$	1
A_2	2
$A_1 \times A_1$	5

$$f_S(t) = \frac{(t+1)^3(t^2+1)(t^2-t+1)(t^2+t+1)(t^4-t^3+t^2-t+1)(t^4+t^3+t^2+t+1)}{(t-1)(t^{15}+2t^{14}+3t^{13}+5t^{12}+5t^{11}+7t^{10}+6t^9+7t^8+6t^7+6t^6+5t^5+3t^4+3t^3+t-1)}$$

(5,3,3)	
type of subgroup	number
H_3	2
A_3	2
$I_2(5)$	1
A_2	3
$A_1 \times A_1$	4

$$f_S(t) = \frac{(t+1)^3(t^2+1)(t^2-t+1)(t^4-t^3+t^2-t+1)}{(t-1)(t^9+t^8-t^7+3t^6-t^5+t^4+2t^3-2t^2+3t-1)}$$

(5,4,2)	
type of subgroup	number
H_3	1
B_3	1
$I_2(5) \times A_1$	1
$B_2 \times A_1$	1
$I_2(5)$	1
B_2	1
A_2	1
$A_1 \times A_1$	5

$$f_S(t) = \frac{(t+1)^3(t^2+1)(t^2-t+1)(t^4-t^3+t^2-t+1)(t^4+t^3+t^2+t+1)}{(t-1)(t^{13}+t^{12}+2t^{11}+2t^{10}+3t^9+2t^8+3t^7+2t^6+3t^5+t^4+3t^3-t^2+2t-1)}$$

(5,4,3)	
type of subgroup	number
H_3	2
B_3	2
$I_2(5)$	1
B_2	1
A_2	2
$A_1 \times A_1$	4

The growth functions for three dimensional noncompact hyperbolic Coxeter groups with five generators are given below, where the first line (k, l, m, n) or (k, l, m) of each table represents the Coxeter diagram in Table 3. We remark that every Coxeter group has 5 Coxeter subgroups of type A_1.

(k,l,m,n)=(2,3,2,3)	
type of subgroup	number
A_3	1
$A_2 \times A_1$	2
$A_1 \times A_1 \times A_1$	1
A_2	2
$A_1 \times A_1$	6

$$f_S(t) = \frac{(t+1)^3(t^2+1)(t^2+t+1)}{(t-1)(t^5+2t^4+2t^3+t^2-1)}$$

(2,3,2,4)	
type of subgroup	number
B_3	1
$B_2 \times A_1$	1
$A_2 \times A_1$	1
$A_1 \times A_1 \times A_1$	1
B_2	1
A_2	1
$A_1 \times A_1$	6

$$f_S(t) = \frac{(t+1)^3(t^2+1)(t^2-t+1)(t^2+t+1)}{(t-1)(t^7+t^6+2t^5+t^4+2t^3+t-1)}$$

(2,3,3,3)	
type of subgroup	number
A_3	2
$A_2 \times A_1$	2
A_2	3
$A_1 \times A_1$	5

$$f_S(t) = \frac{(t+1)^2(t^2+1)(t^2+t+1)}{(t-1)(t^4+2t^3+t^2+t-1)}$$

(2,3,3,4)	
type of subgroup	number
B_3	1
A_3	1
$B_2 \times A_1$	1
$A_2 \times A_1$	1
B_2	1
A_2	2
$A_1 \times A_1$	5

$$f_S(t) = \frac{(t+1)^3(t^2+1)(t^2-t+1)(t^4-t^3+t^2-t+1)}{(t-1)(t^9+t^8+2t^6+3t^3-2t^2+3t-1)}$$

(5,5,2)	
type of subgroup	number
H_3	2
$I_2(5) \times A_1$	2
$I_2(5)$	2
A_2	1
$A_1 \times A_1$	5

$$f_S(t) = \frac{(t+1)^3(t^2-t+1)(t^4-t^3+t^2-t+1)(t^4+t^3+t^2+t+1)}{(t-1)(t^{11}+t^{10}+t^9+2t^8+t^7+2t^6+t^5+2t^4+t^3+2t-1)}$$

(5,5,3)	
type of subgroup	number
H_3	4
$I_2(5)$	2
A_2	2
$A_1 \times A_1$	4

$$f_S(t) = \frac{(t+1)^3(t^2-t+1)(t^4-t^3+t^2-t+1)}{(t-1)(t^7+t^6-t^5+2t^4-t^2+3t-1)}$$

(5,6,2)	
type of subgroup	number
H_3	1
$I_2(6) \times A_1$	1
$I_2(5) \times A_1$	1
$I_2(6)$	1
$I_2(5)$	1
A_2	1
$A_1 \times A_1$	5

$$f_S(t) = \frac{(t+1)^3(t^2-t+1)(t^2+t+1)(t^4-t^3+t^2-t+1)(t^4+t^3+t^2+t+1)}{(t-1)(t^{14}+2t^{13}+3t^{12}+4t^{11}+5t^{10}+5t^9+5t^8+5t^7+5t^6+5t^5+3t^4+3t^3+t^2+t-1)}$$

(5,6,3)	
type of subgroup	number
H_3	2
$I_2(6)$	1
$I_2(5)$	1
A_2	2
$A_1 \times A_1$	4

$$f_S(t) = \frac{(t+1)^3(t^2-t+1)(t^2+t+1)(t^4-t^3+t^2-t+1)}{(t-1)(2t^{10}+t^9+2t^8+t^7+2t^6+2t^5+t^4+2t^3+t^2+2t-1)}$$

(6,2,2)	
type of subgroup	number
$I_2(6) \times A_1$	1
$A_2 \times A_1$	1
$A_1 \times A_1 \times A_1$	1
$I_2(6)$	1
A_2	1
$A_1 \times A_1$	6

$$f_S(t) = \frac{(t+1)^3(t^2-t+1)(t^2+t+1)}{(t-1)(t^6+2t^5+t^4+t^3+t^2+t-1)}$$

(6,2,3)	
type of subgroup	number
$A_2 \times A_1$	2
$I_2(6)$	1
A_2	2
$A_1 \times A_1$	5

$$f_S(t) = \frac{(t+1)^2(t^2-t+1)(t^2+t+1)}{(t-1)(2t^5+t^4+t^3+2t-1)}$$

(6,3,2)	
type of subgroup	number
A_3	1
$I_2(6) \times A_1$	1
$A_2 \times A_1$	1
$I_2(6)$	1
A_2	2
$A_1 \times A_1$	5

$$f_S(t) = \frac{(t+1)^3(t^2+1)(t^2-t+1)(t^2+t+1)}{(t-1)(t^8+2t^7+3t^6+3t^5+3t^4+2t^3+t^2+t-1)}$$

(6,3,3)	
type of subgroup	number
A_3	2
$I_2(6)$	1
A_2	3
$A_1 \times A_1$	4

$$f_S(t) = \frac{(t+1)^2(t^2+1)(t^2-t+1)(t^2+t+1)}{(t-1)(2t^7+t^6+4t^5+t^4+3t^3+2t-1)}$$

(6,4,2)	
type of subgroup	number
B_3	1
$I_2(6) \times A_1$	1
$B_2 \times A_1$	1
$I_2(6)$	1
B_2	1
A_2	1
$A_1 \times A_1$	5

$$f_S(t) = \frac{(t+1)^3(t^2+1)(t^2-t+1)(t^2+t+1)}{(t-1)(t^8+2t^7+3t^6+4t^5+3t^4+3t^3+t^2+t-1)}$$

(6,4,3)	
type of subgroup	number
B_3	2
$I_2(6)$	1
B_2	1
A_2	2
$A_1 \times A_1$	4

$$f_S(t) = \frac{(t+1)^3(t^2+1)(t^2-t+1)(t^2+t+1)}{(t-1)(2t^8+3t^7+5t^6+6t^5+5t^4+4t^3+2t^2+t-1)}$$

(6,6,2)	
type of subgroup	number
$I_2(6) \times A_1$	2
$I_2(6)$	2
A_2	1
$A_1 \times A_1$	5

$$f_S(t) = \frac{(t+1)^3(t^2-t+1)(t^2+t+1)}{(t-1)(2t^6+3t^5+2t^4+2t^3+2t^2+t-1)}$$

(6,6,3)	
type of subgroup	number
$I_2(6)$	2
A_2	2
$A_1 \times A_1$	4

$$f_S(t) = \frac{(t+1)^2(t^2-t+1)(t^2+t+1)}{(t-1)(4t^5+t^4+2t^3+t^2+2t-1)}$$

4. ACKNOWLEDGEMENT

The first author was partially supported by Grant-in-Aid for Scientific Research(C) (19540194), Ministry of Education, Science and Culture of Japan. The second author was partially supported by the JSPS Institutional Program for Young Researcher Overseas Visits " Promoting international young researchers in mathematics and mathematical sciences led by OCAMI ".

REFERENCES

[1] J. W. Cannon, P. Wagreich, Growth functions of surface groups, Math. Ann. 293 (1992), 239-257.
[2] R. Charney, M. W. Davis, Reciprocity of growth functions of Coxeter groups, Geom. Dedicata 39 (1991), 373-378.
[3] W. Floyd, Growth of planar Coxeter groups, P.V. numbers, and Salem numbers, Math. Ann. 293 (1992), 475-43.
[4] P. de la Harpe, Groupes de Coxeter infinis non affines, Exposition. Math 5 (1987), 91-96.
[5] P. de la Harpe, Topics in geometric group theory, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 2000.
[6] G. J. Heckman, The volume of hyperbolic Coxeter polytopes of even dimension, Indag. Math. (N.S.) 6 (1995), 189-196.
[7] J. Humphreys, Reflection groups and Coxeter groups, Cambridge University Press, 1990.
[8] R. Kellerhals, G. Perren, On the growth of cocompact hyperbolic Coxeter groups, European J. Combin. 32 (2011), 1299-1316.
[9] F. Lannér, On complexes with transitive groups of automorphisms. Comm. Sem. Math. Univ. Lund 11 (1950), 1-71.
[10] J. Milnor, A note on curvature and fundamental group, J. Deiff. Geom. 2 (1968), 1-7.
[11] W. Parry, Growth series of Coxeter groups and Salem numbers, J. Algebra 154 (1993), 406-415.
[12] J. Ratcliffe, Foundations of hyperbolic manifolds, Graduate Texts in Mathematics 149, 1994.
[13] J.-P. Serre, Cohomologie des groupes discrets, Prospects Math., Ann. Math. Stud. 70, 77-169, 1971.
[14] L. Solomon, The orders of the finite Chevalley groups, J. Algebra 3(1966), 376-393.
[15] R. Steinberg, Endomorphisms of linear algebraic groups, Mem. Amer. Math. Soc. 80(1968).
[16] P. Tumarkin, Hyperbolic Coxeter n-polytopes with n+ 2 facets, Math. Notes 75 (2004), 848-854.

OSAKA CITY UNIVERSITY ADVANCED MATHEMATICAL INSTITUTE AND DEPARTMENT OF MATHEMATICS, OSAKA CITY UNIVERSITY, 558-8585, OSAKA, JAPAN
E-mail address: komori@sci.osaka-cu.ac.jp

DEPARTMENT OF MATHEMATICS, OSAKA CITY UNIVERSITY, 558-8585, OSAKA, JAPAN
E-mail address: yuriko.ummt.77@gmail.com