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A PSEUDO-METRIC ON MODULI SPACE OF HYPERELLIPTIC CURVES

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ABSTRACT. We exhibit a pseudo-Hermitian metric on the moduli space/Teichmüller space of hyperelliptic curves of genus $g \geq 2$. The pseudo-metric is defined using the area form on the moduli space of Euclidean cone structures which are obtained by taking quotients of hyperelliptic curves by the involutions. We also express the signature of the pseudo-Hermitian metric in terms of the genus of curves.

1. INTRODUCTION

Let $\Sigma_{g,n}$ be a connected orientable surface of genus $g \geq 0$ with $n \geq 0$ punctures. Troyanov showed that when a set of n real positive numbers $\alpha_1, \ldots, \alpha_n$ satisfying the Gauss-Bonnet condition $2\pi\chi(S) = \Sigma_{i=0}^{n}(2\pi - \alpha_i)$ is given, for each conformal structure on $\Sigma_{g,n}$, there exists a Euclidean cone metric on the surface where at the i-th puncture the structure has cone angle α_i, which is unique up to similarities. Therefore, the moduli space of $\Sigma_{g,n}$ can be identified with the moduli space (up to similarities) of Euclidean cone structures.

For a hyperelliptic curve S of genus g with hyperelliptic involution ι, we can regard S/ι as a Riemann sphere with $2g+2$ points. By giving cone angles $(2g-1)\pi$ to one point and π to all the rest, this Riemann surface is developed to a (possibly singular) Euclidean $(2g+1)$-gon (immersed on the Euclidean plane). A similar construction can be done for a once-punctured hyperelliptic curve S of genus g, in which case we have a Euclidean $(2g+2)$-gon. This consideration, which we described in [6], has lead us to realise that these kinds of Euclidean cone structures are natural objects to study.

In this note, we shall show that the moduli space of Euclidean cone structures with cone angles $(n-2)\pi, \underbrace{\pi, \ldots, \pi}_{n}$ with odd n has a natural pseudo-Hermitian metric derived from the area form. We shall see that the area form is an Hermitian form of signature $((n-1)/2, (n-1)/2)$, and hence the pseudo-Hermitian metric has signature $((n-3)/2, (n-1)/2)$.

There is prior work by Thurston [7] on the moduli space of Euclidean cone structures on the 2-sphere when the cone angles are all less than 2π. He showed that the area function on the Euclidean cone structures gives rise to an Hermitian form with signature $(1, n-3)$, where n is the number of cone points, and that it induces a complex hyperbolic metric on the moduli space of such Euclidean cone structures on the 2-sphere. In our setting, Euclidean

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cone structures have one cone point where the cone angle is greater than 2π. This should be reflected in the signatures of our form and pseudo-Hermitian metrics which are quite different from Thurston's.

The space of Euclidean polygons with prescribed exterior angles up to similarities can be regarded as the real part of the space of Euclidean cone structures on the 2-sphere by doubling the polygon. From this point of view, Thurston's condition that all the cone angles are less than 2π corresponds to the condition that the polygon is convex. Then the spaces of convex polygons with prescribed exterior angles admit hyperbolic structures, and their real shapes were studied in [3], [1], [4], [5] [2]. Some of the non-convex cases were also dealt with in [1], [5], where the signature of the area form is given in terms of the exterior angles.

This note is organised as follows. In Section 2, we shall briefly review spaces of Euclidean polygons. In Section 3, we shall study the area form on the space of Euclidean cone structures on a sphere with cone angles $(n-2)\pi, \underbrace{\pi, \dots, \pi}_{n}$ to determine its signature. In Section 4 we shall introduce the notion of Teichmüller space of Euclidean cone structures and the moduli space of marked Euclidean cone structures, and then show they can be naturally identified.

2. Spaces of polygons

In this section, we shall summarise the work of [1], [4], [5] [2] to which our present work should be compared and contrasted.

2.1. Space of Euclidean polygons. Let $n \geq 3$. A Euclidean polygon P with n sides is a sequence of n vectors $x_1, \dots, x_n$ in the Euclidean plane $\mathbb{E}^2$. We call x_i the i-th vertex of P, and the difference $x_{i+1} - x_i$ the i-th edge of P, where the indices are understood as modulo n. We say that an n-gon is *centred*, if the sum of the x_i is zero. Any polygon can be translated to a unique centred polygon. Identifying the Euclidean plane $\mathbb{E}^2$ with the complex plane $\mathbb{C}$, we regard each vertex x_i as a complex number. Then the set $\mathcal{P}_n$ of centred n-gons is an $(n-1)$-dimensional complex vector space.

The signed area of a centred polygon $P = (x_1, \dots, x_n)$ is given by the formula

$$\mathcal{A}(P) = \frac{\sqrt{-1}}{4} \sum_{i=1}^{n} (x_i \bar{x}_{i+1} - \bar{x}_i x_{i+1}), \tag{1}$$

where $\sqrt{-1}(x_i \bar{x}_{i+1} - \bar{x}_i x_{i+1})/4$ is the singed area of the triangle spanned by the vector x_i and x_{i+1} counterclockwise. This defines an Hermitian form on the space $\mathcal{P}_n$.

2.2. Space of Euclidean polygons with fixed exterior angles. Let $\theta = (\theta_1, \dots, \theta_n)$ be an n-tuple of real numbers where $-\pi < \theta_i \leq \pi$ for $i = 1, \dots, n$ and $\sum_{i=1}^n \theta_i = 2\pi$. We say that a polygon $P = (x_1, \dots, x_n)$ has an exterior angle θ_i at the vertex x_i if the signed angle $\angle(x_i - x_{i-1}, x_{i+1} - x_i)$ is equal to θ_i. Let $\mathcal{P}_\theta$ be the subset of $\mathcal{P}_n$ consisting of n-gons with exterior angle θ_i at the i-th vertex x_i for $i = 1, \dots, n$. Let s_i be the length of the

i-th edge of P in $\mathcal{P}_\theta$. Then the s_i parametrise the space $\mathcal{P}_\theta$ and satisfy the equation

$$\sum_{i=1}^{n} s_i \exp(\sqrt{-1}\sum_{k=1}^{i} \theta_k) = 0, \tag{2}$$

so that $\mathcal{P}_\theta$ lies in a real $(n-2)$-dimensional vector space.

The area form $\mathcal{A}$ restricted to the space $\mathcal{P}_\theta$ induces a quadratic form $\mathcal{A}_\theta$. The signature of the quadratic form $\mathcal{A}_\theta$ is (k_+, k_-) where k_+ is the number of negative exterior angles θ_i plus 1, and k_+ is the number of positive exterior angles θ_i minus 3. Especially when all the exterior angles are neither zero nor π, it defines a non-degenerate quadratic form on $\mathcal{P}_\theta$ (see [1], [5]).

For convex n-gons, the signature is $(1, n-3)$ so that it gives rise to a hyperbolic structure on the projectivised space of positive vectors, that is, the space of positive area n-gons up to similarities. When there are negative exterior angles, the polygon is non-convex. The number of nonpositive exterior angles is at most $n-3$, and if there are exactly $n-3$ negative exterior angles, they are consecutive. In the latter case, the area form $\mathcal{A}_\theta$ has signature $(n-2, 0)$ (if there are neither zero nor π exterior angles) and it induces a spherical structure on the projectivised space of positive vectors.

2.3. **Interpretation by moduli.** The space of Euclidean polygons with fixed exterior angles up to similarities can be interpreted as the configuration space of n points on the real projective line $\mathbb{R}P^1$ ([4], [5]). Let $\theta = (\theta_1, \ldots, \theta_n)$ be the set of real numbers satisfying

$$0 < \theta_i < \pi \ (i = 1, \ldots, n), \ \sum_{i=1}^{n} \theta_i = 2\pi. \tag{3}$$

We regard the real projective line $\mathbb{R}P^1$ as the ideal boundary of the upper half plane $\mathbb{H}^2$. Then for an n-tuple of points $(x_1, \ldots, x_n) \in (\mathbb{R}P^1)^n$, if the x_i lie on $\mathbb{R}P^1$ in positive cyclic order $x_{i_1} < x_{i_2} < \ldots < x_{i_n}$, by the Schwarz-Christoffel formula, the upper half plane $\mathbb{H}^2$ is conformally mapped to a simply connected region whose boundary consists of n segments, which is a polygon with exterior angles $\theta_{i_1}, \ldots, \theta_{i_n}$ in cyclic order, uniquely up to similarities. This map extends to the boundary as a homeomorphism, in such a way that the point x_{i_k} is mapped to a vertex of the polygon where the exterior angle is θ_{i_k}. Conversely, if there given a simply connected bounded region with n sides, i.e.an n-gon, with exterior angles $\theta_{i_1}, \ldots, \theta_{i_n}$ in positive cyclic order, by the uniformisation theorem, it is conformally mapped to the upper half plane $\mathbb{H}^2$, and it induces a homeomorphism on the boundaries. The image of the vertices of the polygon determines the set of n points on $\mathbb{R}P^1$. These maps lead to a homeomorphism between the space $\mathcal{X}(n)$ of equivalence classes of n points on $\mathbb{R}P^1$, up to automorphisms of $\mathbb{R}P^1$ and the space P_θ of Euclidean polygons with exterior angles θ, up to similarities. (See [4], [5]). Note that the space $\mathcal{X}(n)$ is disconnected and has $(n-1)!/2$ connected components, each of which has labelling by the equivalence class of $\{1, \ldots, n\}$ up to dihedral action. The connected component labelled by $(i_1, \ldots, i_n)$ is homeomorphic to the interior of the polyhedron $P_{\sigma(\theta)}$ where $\sigma(\theta) = (\theta_{i_1}, \ldots, \theta_{i_n})$.

The hyperbolic metric on the interior of each polyhedron $P_{\sigma(\theta)}$ induces a metric on the configuration space $\mathcal{X}(n)$ and its completion is obtained by gluing the hyperbolic polyhedra $P_{\sigma(\theta)}$ along the faces of the same degenerate polygons. The resulting space is an $(n-3)$-dimensional hyperbolic cone manifold. The geometry of this space was studied in [4], [5], [10] for $n = 2, 3$ and in [2] in general.

Since the real projective line $\mathbb{R}P^1$ is regarded as the real part of the complex projective line $\mathbb{C}P^1$, the configuration space $\mathcal{X}(n)$ constitutes a subspace of the configuration space of points on $\mathbb{C}P^1$. As was shown in [7], the configuration space of points on $\mathbb{C}P^1$ can be identified with the space of Euclidean cone metrics on the sphere up to similarities, which admits a complex hyperbolic metric by the area form which is an Hermitian form of type $(1, n-3)$. On the other hand, the polygon space can be regarded as the subspace of Euclidean cone structures on the sphere by the double of the polygon obtained by gluing the polygon with its reflected image along their corresponding edges. Moreover the inclusion is an isometry (see [2]) with respect to the metrics defined above.

3. Space of Euclidean cone structures

3.1. Euclidean cone structures on a surface and Hermitian forms on parametrisations. A *Euclidean cone metric* on a surface is a singular metric which is Euclidean except at finite points whose neighbourhoods are modelled on Euclidean cones. If a Euclidean cone metric C on an oriented surface of genus g has cone points $p_1, \ldots, p_n$ with cone angles $\theta_1, \ldots, \theta_n$, it satisfies the Gauss-Bonnet formula

$$\sum_{i=1}^{n} \theta_i = 2(n - \chi(S))\pi. \tag{4}$$

The curvature at the cone of cone-angle θ_i is defined to be $2\pi - \theta_i$.

Let $C(g; \theta_1, \ldots, \theta_n)$ denote the moduli space of Euclidean cone metrics on a connected closed orientable surface of genus g with n labelled cone points of angles θ_i, up to equivalence by orientation and label-preserving similarities. We call each representative in $C(g, \theta_1, \ldots, \theta_n)$ a Euclidean cone structure. The following theorem is due to Troyanov.

Theorem 1. [8, 9] *There is a homeomorphism from the space of conformal structures on a connected orientable surface $\Sigma_{g,n}$ of genus g with $n > 0$ punctures to $C(g; \theta_1, \ldots, \theta_n)$,*

The space of Euclidean cone structures on the sphere is studied in [7] in the case where all cone singularities have positive curvature, that is the cone angles less than 2π. He showed that in this case, in the same way as the Euclidean polygon space, the area forms of Euclidean cone structures give rise to a complex hyperbolic metric on the space $C(\theta_1, \ldots, \theta_n)$. In contrast to Thurston's work, what we need to study is the case when Euclidean cone structures have one cone points where cone angle can be greater than 2π.

Let $C((n-2)\pi, \pi \times n)$ denote $C(0; (n-2)\pi, \underbrace{\pi, \ldots, \pi}_{n})$. For each Euclidean cone structure in $C((n-2)\pi, \pi \times n)$, we denote its cone points by $p_0, \ldots, p_n$

in accidence with the order of labels. An arc on a sphere C with marked Euclidean cone structure is called a *geodesic arc* when outside the endpoints the arc is a geodesic in the usual sense with respect to the Euclidean structure.

Theorem 2. *Let $n = 2m + 1$ $(m \geq 1)$ be an odd integer greater than or equal to 3. If C is a Euclidean cone metric on $\mathbb{C}P^1$ with labelled cone points of angles $(n-2)\pi$ and n π's, there is a complex $(n-1)$-dimensional local parametrisation of the Euclidean metrics near C up to isometries, with respect to which the signature of the area form $\mathcal{A}$ is an Hermitian form of type (m, m).*

Proof. Our proof is similar to that of Proposition 3.3 of Thurston [7].

We argue by induction on m. In the case when $m = 1$, the cone metric which we are considering has four cone points each of which has angle π. By cutting the sphere by geodesic arcs connecting adjacent (with respect to the labelling) cone points, we get two congruent parallelograms. If we let z_1 be the vector from the first cone point to the second, and z_2 that from the second to the third, then the area of this cone metric is expressed as $\sqrt{-1}(z_1\bar{z}_2 - \bar{z}_1 z_2)/2$. It is clear that the two variables z_1 and z_2 can move freely in a small neighbourhood of any fixed Euclidean cone metric. Therefore the area form has signature $(1, 1)$.

Now suppose that $m \geq 2$ and that our claim is valid up to $m-1$. Let C be a Euclidean cone metric on $\mathbb{C}P^1$ contained in $C((n-2)\pi, \pi \times n)$. We shall perform surgery on C as follows. Let p_0 be the cone point of C with angle $(n-2)\pi$ and p_1, p_2, p_3 be cone points of angle π following p_0 with respect to the labelling. Consider a geodesic loop a based at p_0 separating p_1, p_2, p_3 from the other cone points of angle π. Let D be the disc bounded by a on the side of p_1, p_2, p_3. By the Gauss-Bonnet formula, we see that a forms the angle 2π at p_0 on the side of D.

We cut open C along a, remove D, and paste a back to itself by the symmetry at its midpoint, which we denote by P' , making the midpoint a cone point at angle π. Thus we get a new cone metric C' on $\mathbb{C}P^1$ in which P_0 has cone angle $(n-4)\pi$, and P' has cone angle π whereas p_1, p_2 and p_3 disappear. Therefore, C' is contained in $C((n-4)\pi, \pi \times (n-2))$. By the assumption of induction $C((n-4)\pi, \pi \times (n-2))$ has area from of signature $(m-1, m-1)$. On the other hand, the area of C is equal to Area(C) + $\sqrt{-1}(z_1\bar{z}_2 - \bar{z}_1 z_2)/2$, where z_1 is the vector from p_1 to p_2 and z_2 that from p_2 to p_3. We can check easily that these two can move freely keeping the metric of C' unchanged, hence that z_1 and z_2 give two independent complex variables locally. Thus we have shown that $C((n-2)\pi, \pi \times (n-2))$ has area form of signature (m, m). □

Recall that the moduli space $C((n-2)\pi, \pi \times n)$ is defined to be the space of Euclidean cone metrics up to similarities. Therefore, in a neighbourhood of $[C]$ this space can be represented by the vectors x with $\mathcal{A}(x, x) = 1$, on which a pseudo-Hermitian metric is induced from the form $\mathcal{A}$. Recall that we defined a parametrisation above only within a neighbourhood of some Euclidean cone metric regarded as a base point. If we change base-points, the coordinates changes are induced from the changing of shortest geodesics. These correspond to linear transformations preserving the area form $\mathcal{A}$. Therefore, the pseudo-Hermitian metric is independent of the basepoints. The signature of $C((n-2)\pi, \pi \times n)$ is $(m-1, m)$.

3.2. Euclidean cone structures on hyperelliptic curves. Given a Euclidean cone metric C on $\mathbb{C}P^1$ with cone angle $(n-2)\pi$ at p_0 and angle π at $p_1, \ldots, p_n$, the metric restricted to the complement of the cone points is, by uniformization theorem, conformal to a metric on the Riemann sphere $\hat{\mathbb{C}}$ minus $n+1$ points. These points determine a configuration of points on $\hat{\mathbb{C}}$ up to automorphisms of $\hat{\mathbb{C}}$. By using the same argument in [7, Proposition 8.1], we can show that this correspondence induces a homeomorphism from $C((n-2)\pi, \pi \times n)$ to the configuration space $\mathcal{X}(n+1)$ of $n+1$ labelled distinct points on $\mathbb{C}P^1$ up to $Aut(\hat{\mathbb{C}})$. This means that the inverse map given by the Schwarz-Christoffel formula given there also works in the case of cone point of negative curvature. (This also follows from the Troyanov's result.)

We abuse the notation to denote the corresponding images of cone points in $\mathcal{X}(n+1)$ still by $p_0, \ldots, p_n$. Every configuration $(p_0, \ldots, p_n)$ of n points on $\mathbb{C}P^1$ uniquely determines a holomorphic structure on the double branched covering over $\mathbb{C}P^1$ branching at $p_0, \ldots, p_n$ when n is odd, and at $p_1, \ldots, p_n$ when n is even, up to isomorphisms. This correspondence leads to a homeomorphism between $\mathcal{X}(n+1)$ with the moduli space $\hat{\mathcal{H}}_g$ (see below) of hyperelliptic curves of genus g with labelled fixed points where $g = (n-1)/2$ when n is odd, and with the moduli space $\hat{\mathcal{H}}_{g,1}$ of hyperelliptic curves of genus $g = n/2 - 1$ with labelled fixed points and one distinguished point when n is even.

We now define the moduli spaces of hyperelliptic curves $\mathcal{H}_g, \mathcal{H}_{g,1}$, and those with labelled fixed points $\hat{\mathcal{H}}_g$ and $\hat{\mathcal{H}}_{g,1}$ as follows. Each element of $\mathcal{H}_g$ is a pair of a Riemann surface S of genus g and a holomorphic involution $\iota : S \to S$. Two pairs (S_1, ι_1) and (S_2, ι_2) are identified when there is a holomorphic bijection $f : S_1 \to S_2$ such that $f \circ \iota_1 = \iota_2$. Similarly, each element of $\mathcal{H}_{g,1}$ is a triple consisting of Riemann surface S of genus g, a distinguished one point x on S, and a holomorphic involution $\iota : S \to S$ which does not fix x. Two triples (S_1, x_1, ι_1) and (S_2, x_2, ι_2) are identified when there is a holomorphic bijection $f : S_1 \to S_2$ such that $f \circ \iota_1 = \iota_2$ and either $f(x_1) = x_2$ or $f(x_1) = \iota_2(x_2)$. Both $\mathcal{H}_g$ and $\mathcal{H}_{g,1}$ have topologies induced from the topologies of the moduli spaces.

We now turn to define $\hat{\mathcal{H}}_g$ and $\hat{\mathcal{H}}_{g,1}$. Each element of $\hat{\mathcal{H}}_g$ is a triple consisting of a Riemann surface S of genus g, a holomorphic involution ι and the fixed points of f labelled as $x_1, \ldots, x_{2g+2}$. Two triples $(S_1, \iota_1, (x_1^1, \ldots, x_{2g+2}^1))$ and $(S_2, \iota_2, (x_1^2, \ldots, x_{2g+2}^2))$ are identified when there is a holomorphic bijection $f : S_1 \to S_2$ with $f \circ \iota_1 = \iota_2$ which sends x_j^1 to x_j^2 for $j = 1, \ldots, 2g+2$. We define $\hat{\mathcal{H}}_{g,1}$ in the same way adding one distinguished point as we defined $\mathcal{H}_{g,1}$ above. These spaces $\hat{\mathcal{H}}_g$ and $\hat{\mathcal{H}}_{g,1}$ have natural topologies, and there are covering maps $\hat{\mathcal{H}}_g \to \mathcal{H}_g$ and $\hat{\mathcal{H}}_{g,1} \to \mathcal{H}_{g,1}$ obtained by forgetting labelling on fixed points, whose covering translation group is the symmetric group S_{2g+2}.

Now it is immediate to see conversely that every hyperelliptic curves in $\hat{\mathcal{H}}_g$ (resp. $\hat{\mathcal{H}}_{g,1}$) endows a Euclidean cone metric with one cone point of angle

$2(n-2)\pi$ when n is odd (resp. with two cone points of angle $(n-2)\pi$ when n is even) together with labelled cone points of angle π, which is unique up to similarities. Thus we have the following.

Proposition 1. *Suppose $n \geq 3$. Set $g = (n-1)/2$ when n is odd, and $g = (n-2)/2$ when n is even. Then the moduli space of Euclidean cone structures $C((n-2)\pi, \pi \times n)$ is homeomorphic to $\hat{\mathcal{H}}_g$ when n is odd, and to $\hat{\mathcal{H}}_{g,1}$ when n is even.*

For each cone metric C in $C((n-2)\pi, \pi \times n)$, there are shortest geodesics joining the cone points p_0 and p_i for $i = 1, \ldots, n$ which are mutually disjoint except at p_0. If we cut C along these geodesics and develope into the Euclidean plane $\mathbb{E}^2$, we obtain an n-gon $P(C)$ whose vertices correspond to the cone point of angle $(n-2)\pi$, and whose midpoints of the edges to the cone points p_i in order. When the n-gon $P = P(C)$ is a convex n-gon, there is a fairly good way to see the hyperelliptic curve corresponding to C. For a convex n-gon P in the Euclidean plane $\mathbb{E}^2$, take the image $-P$ of P by the π-rotation around the midpoint of an edge of P. Then $P \cup (-P)$ forms a $(2n-2)$-gon. Identifying the pairs of parallel sides of the same indices by translations, we obtain topologically a closed surface Σ_P of genus $[(n-1)/2]$ where all the vertices of P are identified into one point when n is odd, and into two points when n is even. Regarding the Euclidean plane $\mathbb{E}^2$ as the complex plane $\mathbb{C}$, the surface Σ_P is endowed with a holomorphic structure induced from that on $P \cup -P$ in $\mathbb{C}$ since the identification of the edges are by translations. We note that the holomorphic structure on Σ_P does not depend on the choice of P in its similarity class. Moreover, since the surface Σ_P admits an obvious holomorphic involution induced from the π rotation around the midpoint of an edge of P, it is a hyperelliptic curve.

We also note that the surface Σ_P is naturally endowed with a Euclidean cone structure with one cone point of angle $2(n-2)\pi$ at the point corresponding to the vertices of P when n is odd, and with two cone points of angle $(n-2)\pi$. Obviously the induced metric on the quotient of Σ_P by the involution is equivalent to the original metric C where the branching points are at the points corresponding to the vertices of P and the midpoints of the n edges, which are naturally indexed by the indices of the edges of P.

The pseudo-Hermitian metric which we introduced in §3.1 can be pushed forward to $\hat{\mathcal{H}}_g$ and then to $\mathcal{H}_g$.

Corollary 1. *Both $\hat{\mathcal{H}}_g$, and $\mathcal{H}_g$ admit a pseudo-Hermitian metric of the signature $(g-1, g)$ coming from the area form.*

4. The Teichmüller space of hyperelliptic curves

4.1. **Teichmüller space of a Euclidean cone structure.** For a surface with a Euclidean cone metric and labelled singular points, which we denote by C, we consider a pair (X, ϕ) where X is a Euclidean cone metric of the surface with cone points of the same indices and angles with those of C, and ϕ an orientation-preserving diffeomorphism from C to X which takes the cone points of C to those of X preserving their labels. Two such pairs (X, ϕ) and (X', ψ) are said to be equivalent if the diffeomorphism $\psi \circ \phi^{-1}: X \to X'$ is homotopic to a similarity which preserves the orientation and the labels of the cone points. Then the Teichmüller space $\mathcal{T}(C)$ of C to be the set of equivalence classes of pairs (X,ϕ). For a different choice of a cone metric C' of the same type, there is a noncanonical isomorphism between the Teichmüller spaces $\mathcal{T}(C)$ and $\mathcal{T}(C')$ induced by a diffeomorphism between C and C'.

4.2. **Marked euclidean cone structures.** For a Euclidean cone structure C on the sphere with cone singularities of angle $(n-2)\pi$ at the point p_0 and π at points $p_1,\dots,p_n$, let $t=(t_1,\dots,t_n)$ be a collection of disjoint smooth paths in C such that t_i connects the cone points p_0 and p_i. The image of t in C forms a tree with one internal vertex from which n edges issue. Two such collections t and t' are said to be *equivalent* if there is an ambient isotopy of C fixing the cone points from t to t'. We call its equivalence class a *marking* of C, and the pair $(C,[t])$ a *marked Euclidean cone structure.* Two marked Euclidean cone structures $(C,[t])$ and $(C',[t'])$ are said to be *equivalent* if there is an orientation and label-preserving similarity ϕ from C to C', such that the image $\phi(t)$ represents $[t']$. Let $\widetilde{C}((n-2)\pi,\pi\times n)$ be the space of equivalence classes $[(C,[t])]$ of the marked Euclidean cone structures $(C,[t])$, and $\widetilde{C}_0((n-2)\pi,\pi\times n)$ its subspace consisting of marked Euclidean cone structures whose marking has edges issuing from p_0 in the order of $t_1,\dots,t_n$ clockwise. There is a natural projection π from $\widetilde{C}_0((n-2)\pi,\pi\times n)$ to $C((n-2)\pi,\pi\times n)$ sending $(C,[t])$ to C by forgetting the marking.

Proposition 2. *Let C be a Euclidean cone structure in $C((n-2)\pi,\pi\times n)$. Then there is a bijection from the Teichmüller space $\mathcal{T}(C)$ of C to the moduli space of marked Euclidean cone structures $\widetilde{C}_0((n-2)\pi,\pi\times n)$.*

Proof. Fix a marking t_0 on C whose edges issue from p_0 in the order of their subscripts clockwise. For any pair (X,ϕ) with a diffeomorphism $\phi: C\to X$ representing a point in $\mathcal{T}(C)$, the pair $(X,[\phi(t_0)])$ defines an element in $\widetilde{C}_0((n-2)\pi,\pi\times n)$ whose equivalence class does not depend on the choice of the representative (X,ϕ). Thus we get a map $\Phi:\mathcal{T}(C)\to\widetilde{C}_0((n-2)\pi,\pi\times n)$ defined by

$$\Phi([(X,\phi)])=[(X,[\phi(t_0)])]$$

for $[(X,\phi)]\in\mathcal{T}(C)$. Then we shall show that the map Φ from the Teichmüller space $\mathcal{T}(C)$ to the space of marked Euclidean cone structures $\widetilde{C}_0((n-2)\pi,\pi\times n)$ is bijective.

First we shall show that the map Φ is injective. Suppose $[(X,[\phi(t_0)])]=[(X',[\psi(t_0)])]$ for $(X,\phi),(X',\psi)\in\mathcal{T}(C)$. By composing with ψ the isometry from X' to X realising their equivalence, we can assume that $X=X'$. By further composing with ψ an orientation preserving homeomorphism isotopic to the identity fixing the cone points, we can assume that $\phi|t_0=\psi|t_0$, that is, ϕ and ψ are identical on the marking t_0. Since the complement of t_0 in C is homeomorphic to an open disc, ϕ and ψ are isotopic relative to the cone points, and we see that $(X,\phi)=(X',\psi)$ in $\mathcal{T}(C)$.

Next we shall show the surjectivity. Let $(X,[t])$ be a pair representing a point in $\widetilde{C}_0((n-2)\pi,\pi\times n)$. Take a diffeomorphism $\phi: C\to X$ mapping the cone points of C to those of X preserving the labels. We consider the tree $\phi^{-1}(t)$ on C, which we denote by t_1. What we have to show is that there is an auto-diffeomorphism of C fixing the cone points which takes t_0 to t_1 (or t_1 to t_0). First, we take a diffeomorphism from t_0 to t_1 fixing the cone points. This map can be extended to a diffeomorphism from a regular neighbourhood of t_0 to that of t_1 since the order of labels of arcs issuing from p_0 are the same for t_0 and t_1. Such a diffeomorphism in its turn can be extended to the entire C since the complements of the regular neighbourhoods are discs. Thus we have proved the surjectivity. □

The fundamental group of $C((n-2)\pi, \pi \times n)$ is the pure braid group of the sphere that acts on $\widetilde{C}_0((n-2)\pi, \pi \times n)$. Evidently the area form $\mathcal{A}$ lifts to $\widetilde{C}_0((n-2)\pi, \pi \times n)$, it defines a pseudo-Hermitian metric on the Teichmüller space of $\mathcal{T}(C)$ of $C \in C((n-2)\pi, \pi \times n)$ of signature $(m-1, m)$ when $n = 2m+1$ for $m \geq 1$.

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