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On Triangle of numbers

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1 Introduction

The number series which is defined by the formula

$$a_{n,k} = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^n, \qquad (k = 1, ..., n)$$

is called triangle of numbers([O]).

As will be mentioned in Section 2, the triangule of numbers are deeply related with Stirling numbers of the second kind.

The goal of this article is to describe some properties of this number series and to study their relationship with Bernoulli numbers.

Some of formulas about $a_{n,k}$ in this paper has been already known in the context of Stirling numbers, but we will describe their proofs from the viewpoint of triangle of numbers.

We define $0^0 = 1$ below.

The author would like to express deep gratitude to Professor H. Tagawa for his useful comments.

2 The recurrence relation

Lemma 2.1. For any $k \in \mathbb{N}$, the following equality holds.

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} i^\ell = 0, \qquad (\ell = 0, 1, ..., k-1).$$

Proof. In the case of $k = 1$, the left hand side of the equality is $(-1)^0 \binom{1}{0} 0^0 + (-1)^1 \binom{1}{1} 1^0 = 0$ and the equality is correct.

In the case of $k \geq 2$, we will prove the equality by the induction on ℓ for the fixed k.

If $\ell = 0$, then the left hand side of the equality is the binomial expansion of $(1-1)^k$ and the equality is correct.

Assume that

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} i^\ell = 0 \tag{2.1}$$

is correct for $\ell = 0, 1, ..., n$ $(n \leq k-2)$.

Differentiating both sides of the equality of $(1-x)^k$,

$$(1-x)^k = \sum_{i=0}^{k} \binom{k}{i} (-x)^i,$$

$n+1$ times, we obtain the equality

$$(-1)^{n+1} k(k-1)\cdots(k-n)(1-x)^{k-n-1} = \sum_{i=n+1}^{k} (-1)^i i(i-1)\cdots(i-n)\binom{k}{i} x^{i-n-1}.$$

(Note that $k-n-1 \geq 1$ since $n \leq k-2$.) Substituting $x=1$, we have

$$0 = \sum_{i=n+1}^{k} (-1)^i i(i-1)\cdots(i-n)\binom{k}{i} = \sum_{i=1}^{k} (-1)^i i(i-1)\cdots(i-n)\binom{k}{i}. \tag{2.2}$$

We define integers a_j $(j = 0,1,..,n)$ by $x(x-1)\cdots(x-n) = x^{n+1} + a_n x^n + \cdots + a_0$. Then the equation (2.2) can be calculated as

$$\begin{aligned}
0 &= \sum_{i=1}^{k} (-1)^i (i^{n+1} + a_n i^n + \cdots + a_0)\binom{k}{i} \\
&= \sum_{i=1}^{k} (-1)^i i^{n+1}\binom{k}{i} + \sum_{i=1}^{k} (-1)^i (a_n i^n + \cdots + a_0)\binom{k}{i} \\
&= \sum_{i=1}^{k} (-1)^i \binom{k}{i} i^{n+1} + \sum_{\ell=0}^{n} a_\ell \sum_{i=1}^{k} (-1)^i \binom{k}{i} i^\ell \\
&= \sum_{i=1}^{k} (-1)^i \binom{k}{i} i^{n+1}. \qquad (\because\ (2.1))
\end{aligned}$$

Thus we have proved the $\ell = n+1$ case. □

Corollary 2.2. We can extend the domain of k of $a_{n,k}$ from $\{1,2,..,n\}$ (see Introduction) to the set of positive integers $\mathbb{N}$ and $a_{n,k} = 0$ if $n < k$.

Proof. For $n < k$, we have

$$\begin{aligned}
a_{n,k} &= \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^n \\
&= (-1)^k \sum_{i=0}^{k} (-1)^i \binom{k}{i} i^n \\
&= 0. \quad (\because \text{ Lemma 2.1})
\end{aligned}$$

□

We consider $a_{n,k}$ for any $n, k \in \mathbb{N}$ hereafter.

Proposition 2.3. The number series $\{a_{n,k}\}$ satisfies the following recurrence relation.

$$\begin{cases}
a_{n,1} = 1 \\
a_{n+1,k} = k a_{n,k-1} + k a_{n,k} \quad (k = 2,3,...,n+1) \\
a_{n,k} = 0 \quad (n < k)
\end{cases}$$

Proof. The first equality is easy to prove from the definition, and the last one is shown in Corollary 2.2. The right hand side of the second one is calculated as

$$\begin{aligned}
a_{n,k-1}+a_{n,k} &= \sum_{i=0}^{k-1}(-1)^{k-1-i}\binom{k-1}{i}i^n+\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}i^n \\
&= \sum_{i=0}^{k-1}(-1)^{k-1-i}\binom{k-1}{i}i^n+\sum_{i=0}^{k-1}(-1)^{k-i}\binom{k}{i}i^n+k^n \\
&= \sum_{i=0}^{k-1}(-1)^{k-i}\left\{-\binom{k-1}{i}+\binom{k}{i}\right\}i^n+k^n \\
&= \sum_{i=0}^{k-1}(-1)^{k-i}\binom{k-1}{i-1}i^n+k^n \\
&= \sum_{i=0}^{k}(-1)^{k-i}\binom{k-1}{i-1}i^n.
\end{aligned}$$

Using the relation $k\binom{k-1}{i-1}=i\binom{k}{i}$, we have

$$\begin{aligned}
ka_{n,k-1}+ka_{n,k} &= \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}i^{n+1} \\
&= a_{n+1,k}.
\end{aligned}$$

□

Corollary 2.4. $a_{n,n}=n!$.

Proof. From Lemma 2.2 and the third equality of the above proposition, we obtain $a_{n+1,n+1}=(n+1)a_{n,n}$. Since $a_{1,1}=1$ holds from the definition, $a_{n,n}=n!$ holds. □

From above discussions, we conclude that

$$\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}i^n=\begin{cases}0 & n=0,1,2,\dots,k-1\\ a_{n,k} & n=k,k+1,k+2,\dots\\ k! & n=k \text{ (special cases of the second equality)}\end{cases}$$

Remark 2.5. When the author first tried to solve this recurrence relation, he conjectured that the general term is

$$b_{n,k}=k\sum_{i=0}^{k-1}(-1)^{k-i-1}\binom{k-1}{i}(i+1)^{n-1}$$

and proved it is true. Indeed it is easy to see $a_{n,k}=b_{n,k}$ by using the relation $k\binom{k-1}{i}=(i+1)\binom{k}{i+1}$:

$$\begin{aligned}
b_{n,k} &= k\sum_{i=0}^{k-1}(-1)^{k-i-1}\binom{k-1}{i}(i+1)^{n-1} \\
&= \sum_{i=0}^{k-1}(-1)^{k-i-1}(i+1)\binom{k}{i+1}(i+1)^{n-1}
\end{aligned}$$

$$\begin{aligned} &= \sum_{i=0}^{k-1}(-1)^{k-i-1}\binom{k}{i+1}(i+1)^n \\ &= \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}i^n \\ &= a_{n,k}. \end{aligned}$$

Definition 2.6. (Stirling numbers) Let n, k be positive integers.

Integers $\left[\begin{matrix} n \\ k \end{matrix}\right]$ defined by following recurrence relations are called Stirling numbers of the first kind :

$$\left[\begin{matrix} n+1 \\ k \end{matrix}\right] = n\left[\begin{matrix} n \\ k \end{matrix}\right] + \left[\begin{matrix} n \\ k-1 \end{matrix}\right], \quad \left[\begin{matrix} n \\ 1 \end{matrix}\right] = (n-1)!, \quad \left[\begin{matrix} n \\ 0 \end{matrix}\right] = 0, \quad \left[\begin{matrix} n \\ k \end{matrix}\right] = 0 \ \text{ if } n < k.$$

This is the number of permutations on n letters having k cycles.

Integers $\left\{\begin{matrix} n \\ k \end{matrix}\right\}$ defined by following recurrence relations are called Stirling numbers of the second kind :

$$\left\{\begin{matrix} n+1 \\ k \end{matrix}\right\} = k\left\{\begin{matrix} n \\ k \end{matrix}\right\} + \left\{\begin{matrix} n \\ k-1 \end{matrix}\right\}, \quad \left\{\begin{matrix} n \\ 1 \end{matrix}\right\} = 1, \quad \left\{\begin{matrix} n \\ 0 \end{matrix}\right\} = 0, \quad \left\{\begin{matrix} n \\ k \end{matrix}\right\} = 0 \ \text{ if } n < k.$$

This is the number of way of partition a set of n elements into k nonempty disjoint subsets.

Proposition 2.7.

$$a_{n,k} = k!\left\{\begin{matrix} n \\ k \end{matrix}\right\}.$$

Proof. Multiplied by $k!$, the recurrence relation of the Stirling numbers of the second kind becomes

$$k!\left\{\begin{matrix} n+1 \\ k \end{matrix}\right\} = k\left(k!\left\{\begin{matrix} n \\ k \end{matrix}\right\} + (k-1)!\left\{\begin{matrix} n \\ k-1 \end{matrix}\right\}\right).$$

By this relation and $1!\left\{\begin{matrix} 1 \\ 1 \end{matrix}\right\} = 1$, we can conclude

$$a_{n,k} = k!\left\{\begin{matrix} n \\ k \end{matrix}\right\}.$$

□

The above equality is rewrited as

$$k!\left\{\begin{matrix} n \\ k \end{matrix}\right\} = \sum_{i=1}^{k}(-1)^{k-i}\binom{k}{i}i^n.$$

Of course, this equality can be thought of a definition of the Stirling number of the second kind ([Gr],[GKP]).

From this proposition and the combinatorial meaning of the Stirling numbers, we can see that the combinatorial meaning of $a_{n,k}$, that is, it is a number of all surjections from a set having n elements to a set having k elements.

We obtain the following triangle derived from $a_{n,k}$ by the recurrence relation : $a_{n,k}$ is the entry of the k-th number of the n-th row.

$$
\begin{array}{ccccccccccccccccc}
&&&&&&&&1&&&&&&&& \\
&&&&&&&\swarrow\times 1&&\searrow\times 2&&&&&&& \\
&&&&&&1&&&&2&&&&&& \\
&&&&&\swarrow\times 1&&\searrow\times 2&&\swarrow\times 2&&\searrow\times 3&&&&& \\
&&&&1&&&&6&&&&6&&&& \\
&&&\swarrow\times 1&&\searrow\times 2&&\swarrow\times 2&&\searrow\times 3&&\swarrow\times 3&&\searrow\times 4&&& \\
&&1&&&&14&&&&36&&&&24&& \\
&\swarrow\times 1&&\searrow\times 2&&\swarrow\times 2&&\searrow\times 3&&\swarrow\times 3&&\searrow\times 4&&\swarrow\times 4&&\searrow\times 5& \\
1&&&&30&&&&150&&&&240&&&&120
\end{array}
$$

Corollary 2.8. For $\forall n \geq 2$,

$$\sum_{k=1}^{n} \frac{(-1)^k}{k} a_{n,k} = 0.$$

Proof. From the recurrence relation of $a_{n,k}$, we can calculate as

$$
\begin{aligned}
& \sum_{k=1}^{n+1} \frac{(-1)^{k+1}}{k} a_{n+1,k} \\
= \quad & a_{n+1,1} + \sum_{k=2}^{n+1} \frac{(-1)^{k+1}}{k} k(a_{n,k-1} + a_{n,k}) \\
= \quad & a_{n+1,1} + \sum_{k=2}^{n+1} (-1)^{k+1} a_{n,k-1} + \sum_{k=2}^{n+1} (-1)^{k+1} a_{n,k} \\
= \quad & a_{n+1,1} + \sum_{k=1}^{n} (-1)^{k} a_{n,k} + \sum_{k=2}^{n+1} (-1)^{k+1} a_{n,k} \\
= \quad & a_{n+1,1} + (-1)^1 a_{n,1} + (-1)^n a_{n,n+1} \\
= \quad & 1 - 1 + 0 = 0.
\end{aligned}
$$

□

The relationship between the above triangle and Corollary 2.8 corresponds to the relationship between the Pascal's triangle and the equality in Lemma 2.1 ($\ell = 0$ case).

We don't know there exist equalities with respect to $a_{n,k}$ which correspond to $\ell \geq 1$ case of Lemma 2.1.

3 Relationship with Bernoulli numbers

In the beginning of this section, we study the relationship of $a_{n,k}$ and the generating function $F_n(x) = \sum_{m=0}^{\infty} m^n x^m$ of the number series $0^n, 1^n, 2^n, \ldots$.

Note that

$$F_0(x) \quad = \quad \sum_{n=0}^{\infty} n^0 x^n = \frac{1}{1-x},$$

$$\begin{aligned} xF_n'(x) &= x\sum_{m=1}^{\infty} m^{n+1}x^{m-1} \\ &= \sum_{m=0}^{\infty} m^{n+1}x^m \\ &= F_{n+1}(x) \qquad (n\in\mathbb{N}) \end{aligned} \tag{3.1}$$

as formal power series.

Lemma 3.1. For any integer $n \geq 1$, the function $F_n(x)$ is a sum of rational functions as

$$F_n(x) = \sum_{k=1}^{n} a_{n,k}x^k(1-x)^{-k-1}.$$

Proof. Put $n = 0$ in the equation (3.1), we have $F_1(x) = xF_0'(x) = x(1-x)^{-2} = a_{1,1}x(1-x)^{-2}$.

If we assume that $F_n(x) = \sum_{k=1}^{n} a_{n,k}x^k(1-x)^{-k-1}$, then, by using the equation (3.1), we can calculate as follows.

$$\begin{aligned} F_{n+1}(x) &= x\sum_{k=1}^{n} a_{n,k}\left\{kx^{k-1}(1-x)^{-k-1} + (k+1)x^k(1-x)^{-k-2}\right\} \\ &= \sum_{k=1}^{n} ka_{n,k}x^k(1-x)^{-k-1} + \sum_{k=1}^{n}(k+1)a_{n,k}x^{k+1}(1-x)^{-k-2} \\ &= \sum_{k=1}^{n} ka_{n,k}x^k(1-x)^{-k-1} + \sum_{k=2}^{n+1} ka_{n,k-1}x^k(1-x)^{-k-1} \\ &= a_{n,1}x(1-x)^{-2} + \sum_{k=2}^{n} k(a_{n,k}+a_{n,k-1})x^k(1-x)^{-k-1} + (n+1)a_{n,n}x^{n+1}(1-x)^{-n-2} \\ &= a_{n+1,1}x(1-x)^{-2} + \sum_{k=2}^{n} a_{n+1,k}x^k(1-x)^{-k-1} + a_{n+1,n+1}x^{n+1}(1-x)^{-n-2} \\ &\qquad\qquad (\because \text{ Proposition 2.3, Corollary 2.4}) \\ &= \sum_{k=1}^{n+1} a_{n+1,k}x^k(1-x)^{-k-1}. \end{aligned}$$

□

By the well known fact (see [GKP], [N] for example), $F_0(x)F_n(x)$ is the generating function of the number series $\sum_{m=0}^{k} m^n$.

In fact, for the generating function $G(x) = \sum_{m=0}^{\infty} a_m x^m$ of a number series $\{a_m\}$,

$$\begin{aligned} F_0(x)G(x) &= \sum_{n=0}^{\infty} x^n \sum_{m=0}^{\infty} a_m x^m \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_m x^{m+n} \\ &= \sum_{k=0}^{\infty}\left(\sum_{m=0}^{k} a_m\right)x^k \qquad (k = m+n = k) \end{aligned}$$

holds, and $F_0(x)G(x)$ is the generating function of $\sum_{m=0}^{k} a_m$.

From this fact, we can prove the following proposition.

Proposition 3.2. For any positive integers ℓ, n, the following equality holds.

$$\sum_{i=0}^{\ell} i^n = \sum_{k=1}^{n} \binom{\ell+1}{k+1} a_{n,k}. \tag{3.2}$$

Proof. Since easy calculation shows

$$\begin{aligned} F_0(x)F_n(x) &= \frac{1}{1-x}\sum_{k=1}^{n+1} c_{n,k}x^k(1-x)^{-k-1} \\ &= \sum_{k=1}^{n} a_{n,k}x^k(1-x)^{-k-2} \\ &= \sum_{k=1}^{n}\sum_{j=0}^{\infty} {}_{k+2}\mathrm{H}_j a_{n,k}x^{k+j} \\ &= \sum_{\ell=1}^{\infty}\sum_{k=1}^{\min\{\ell,n\}} {}_{k+2}\mathrm{H}_{\ell-k} a_{n,k}x^{\ell} \qquad \text{(we put } \ell = j+k\text{)}, \end{aligned}$$

we get

$$\sum_{i=0}^{\ell} i^n = \sum_{k=1}^{\min\{\ell,n\}} {}_{k+2}\mathrm{H}_{\ell-k} a_{n,k}.$$

Here, ${}_n\mathrm{H}_k$ denotes the number of combinations of n objects taken r at a time with repetition, that is,

$${}_n\mathrm{H}_k = \binom{n+r-1}{r} = \binom{n+k-1}{k}.$$

By the above formula and $\binom{p}{q} = a_{p,q} = 0$ if $p < q$, we finally obtain

$$\sum_{i=0}^{\ell} i^n = \sum_{k=1}^{\min\{\ell,n\}} \binom{\ell+1}{k+1} a_{n,k} = \sum_{k=1}^{\ell} \binom{\ell+1}{k+1} a_{n,k} = \sum_{k=1}^{n} \binom{\ell+1}{k+1} a_{n,k}.$$

□

The equation in the following corollary is found in [Gr].

Corollary 3.3.

$$\ell^n = \sum_{k=1}^{n} \binom{\ell}{k} a_{n,k}$$

Proof.

$$\begin{aligned} \ell^n &= \sum_{i=1}^{\ell} i^n - \sum_{i=1}^{\ell-1} i^n \\ &= \sum_{k=1}^{n} \binom{\ell+1}{k+1} a_{n,k} - \sum_{k=1}^{n} \binom{\ell}{k+1} a_{n,k} \\ &= \sum_{k=1}^{n} \left\{ \binom{\ell+1}{k+1} - \binom{\ell}{k+1} \right\} a_{n,k} \\ &= \sum_{k=1}^{n} \binom{\ell}{k} a_{n,k}. \end{aligned}$$

□

Example 3.4. In the case $n = 5$, the diagram in the previous section says $a_{5,1} = 1, a_{5,2} = 30, a_{5,3} = 150, a_{5,4} = 240$, $a_{5,5} = 120$, so we have

$$\begin{aligned}
\sum_{i=0}^{\ell} i^5 &= \binom{\ell+1}{2} + 30\binom{\ell+1}{3} + 150\binom{\ell+1}{4} + 240\binom{\ell+1}{5} + 120\binom{\ell+1}{6} \\
&= \frac{\ell(\ell+1)}{12}\{6 + 60(\ell-1) + 75(\ell-2)(\ell-1) + 24(\ell-3)(\ell-2)(\ell-1) \\
&\qquad +2(\ell-4)(\ell-3)(\ell-2)(\ell-1)\} \\
&= \frac{\ell(\ell+1)}{12}(2\ell^4 + 4\ell^3 + \ell^2 - \ell) \\
&= \frac{\ell^2(\ell+1)^2(2\ell^2+2\ell-1)}{12}.
\end{aligned}$$

Remark 3.5. We rewrite the formula in the above proposition with binomial coefficients as

$$\begin{aligned}
\sum_{i=0}^{\ell} i^n &= \sum_{k=1}^{n} \binom{\ell+1}{k+1} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^n \\
&= \sum_{k=1}^{n} \sum_{i=1}^{k} (-1)^{k-i} \binom{\ell+1}{k+1} \cdot \binom{k}{i} i^n.
\end{aligned}$$

Definition 3.6. (Bernoulli numbers)

For a positive integer n, we define a rational number B_n by

$$\sum_{i=0}^{n} \binom{n+1}{i} B_i = 0, \quad B_0 = 1,$$

which is called a Bernoulli number.

Some people use the following proposition as a definition of Bernoulli numbers.

Proposition 3.7. Bernoulli numbers are the coefficients in the following power series:

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k.$$

From this proposition and the fact that $\dfrac{x}{e^x-1} + \dfrac{x}{2} = \dfrac{x}{2}\coth\dfrac{x}{2}$ is an even function of x, we obtain the following proposition.

Proposition 3.8. $B_{2n+1} = 0$ for $n \geq 1$.

Bernoulli numbers were originally found when Jacobi Bernoulli studied a summation formula about $\sum_{i=0}^{\ell} i^n$. His conclusion was the following formula.

Proposition 3.9.

$$\sum_{i=0}^{\ell} i^n = \frac{1}{n+1} \sum_{k=0}^{n} \binom{n+1}{k} B_k (\ell+1)^{n+1-k}. \tag{3.3}$$

Since $\sum_{i=0}^{\ell} i^n$ is known to be expressed by Stirling numbers of the first and the second kinds as

$$\sum_{i=0}^{\ell} i^n = \sum_{k=0}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \frac{1}{k+1} \sum_{j=0}^{k+1} (-1)^{k+1-j} \left[\begin{matrix} k+1 \\ j \end{matrix} \right] (\ell+1)^j,$$

we can obtain a relation between Bernoulli numbers and Stirling numbers.

Proposition 3.10. B_n is described by the Stirling numbers of the first and the second kind as

$$\frac{1}{n+1}\binom{n+1}{j}B_{n+1-j} = \sum_{k=0}^{n}\left\{ \begin{matrix} n \\ k \end{matrix} \right\}\left[\begin{matrix} k+1 \\ j \end{matrix} \right]\frac{(-1)^{k+1-j}}{k+1}.$$

A simpler expression is presented in [AIK]. Note that their definition of Bernoulli numbers are different from ours, they call our $(-1)^n B_n$ as a Bernoulli number.

Proposition 3.11. B_n is described by the Stirling numbers of the second kind as

$$B_n = \sum_{k=1}^{n}\frac{(-1)^k}{k+1}k!\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$$

We can find proofs of above propositions concerned with Bernoulli numbers in [AIK], [GKP] and [K].

Now, we compare the equation (3.2) and (3.3).

We denote the coefficient of x^j in the polynomial $x(x-1)(x-2)\cdots(x-k)$ by $d_{k,j}$. Then, the right-hand side of the equation (3.2) is

$$\sum_{k=1}^{n}\binom{\ell+1}{k+1}a_{n,k} = \sum_{k=1}^{n}\frac{1}{(k+1)!}\sum_{m=1}^{k+1}d_{k,m}a_{n,k}(\ell+1)^m$$

since $\binom{\ell+1}{k+1} = (\ell+1)\ell(\ell-1)\cdots(\ell-k+1)/(k+1)!$. Therefore we obtain

$$\frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k}B_k(\ell+1)^{n+1-k} = \sum_{k=1}^{n}\frac{1}{(k+1)!}\sum_{m=1}^{k+1}d_{k,m}a_{n,k}(\ell+1)^m$$

and since this equation is satisfied by any integer ℓ, every coefficient of $(\ell+1)^p$ $(p=1,2,...,n)$ in both sides coincides each other. Especially, by comparing coefficients of $(\ell+1)^1$, we obtain

$$\frac{1}{n+1}\binom{n+1}{n}B_n = \sum_{k=1}^{n}\frac{1}{(k+1)!}d_{k,1}a_{n,k} \qquad (n=1,2,3,...).$$

We have defined that $d_{k,1}$ is the coefficient of x in $x(x-1)(x-2)\cdots(x-k)$, so $d_{k,1}=(-1)^k k!$. So the above equation becomes

$$B_n = \sum_{k=1}^{n}\frac{(-1)^k}{k+1}a_{n,k}.$$

Thus, together with the definition of $a_{n,k}$, we have proved the following theorem (c.f. Corollary 2.8).

Theorem 3.12. For a positive integer n, a Bernoulli number B_n can be described by triangle array of numbers, Stirling numbers of the second kind and binomial coefficients :

$$(1) \qquad B_n = \sum_{k=1}^{n}\frac{(-1)^k}{k+1}a_{n,k} \tag{3.4}$$

$$(2) \qquad B_n = \sum_{k=1}^{n}\frac{1}{k+1}\sum_{i=0}^{k}(-1)^i\binom{k}{i}i^n \tag{3.5}$$

Remark 3.13. (1) This theorem can be obtained as a corollary of Proposition 3.11.

(2) As a result, we obtain an algorithm to calculate Bernoulli numbers by Proposition 2.3 and the equation (3.4).

(3) Equation (3.5) has an old histry, as is asserted in [Go].

From Proposition 3.8 and Theorem 3.12, we have the following corollary.

Corollary 3.14.

$$\sum_{k=1}^{n} \frac{(-1)^k}{k+1} a_{n,k} = 0$$

if n is a odd integer which is greater than or equal to 3.

Example 3.15. The diagram in Section2 says $a_{4,1} = 1, a_{4,2} = 14, a_{4,3} = 36, a_{4,4} = 24$, so we have

$$\begin{aligned} B_4 &= \sum_{k=1}^{4} \frac{(-1)^k}{k+1} a_{4,k} \\ &= -\frac{1}{2}a_{4,1} + \frac{1}{3}a_{4,2} - \frac{1}{4}a_{4,3} + \frac{1}{5}a_{4,4} \\ &= -\frac{1}{2} + \frac{14}{3} - \frac{36}{4} + \frac{24}{5} \\ &= -\frac{1}{30}. \end{aligned}$$

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