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Prehomogeneous vector spaces and their regularity

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Abstract

In this paper, we gather the various known constructions of prehomogeneous vector spaces and give some new results. We consider everything over the complex number field $\mathbb{C}$.

Introduction

It is classically known that the Fourier transform of the complex power of the quadratic form $\sum_{i=1}^{n} x_i^2$ and the determinant $\det X$ of an $n \times n$ matrix X are again essentially the complex power of some polynomials. What is the real reason of these phenomena ? In 1961, Mikio Sato realized that these polynomials $\sum_{i=1}^{n} x_i^2$ and $\det X$ are relative invariants under some big action of algebraic groups. Thus he reached the notion of prehomogeneous vector spaces and showed that the Fourier transform of the complex power of a non-degenerate relative invariant of a reductive regular prehomogeneous vector space is again essentially the complex power of some polynomial. By using these results, one can construct the zeta function of a regular prehomogeneous vector space which satisfies the functional equations. For example, the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$ can be regarded as the zeta function of the simplest prehomogeneous vector space $(GL(1), \Lambda_1, V(1))$. To get a new zeta function, a classification of prehomogeneous vector spaces is important.

For the classification of all prehomogeneous vector spaces, to find the PV-equivalences is essential. For example, the discovery of castling transformation made the classification of irreducible prehomogeneous vector spaces possible ([SK]). In this paper, we gather all PV-equivalences so far known and investigate their relation of relative invariants and regularity.

Now we give the basic definitions of prehomogeneous vector spaces and their regularity. Let G be a linear algebraic group and $\rho : G \to GL(V)$ a rational representation on a finite-dimensional vector space V over the complex number field $\mathbb{C}$. If V has a dense G-orbit $\mathbb{O} = \rho(G)v_0$ with respect to the Zariski topology, a triplet (G, ρ, V) is called *a prehomogeneous vector space* (abbrev. PV). In general, an orbit $\rho(G)v_0$ is called an homogeneous space. So $V = \overline{\rho(G)v_0}$ implies that V is almost homogeneous, although it cannot be homogeneous since $\rho(G)0 = \{0\}$. In this sense, M. Sato named such a triplet (G, ρ, V) a prehomogeneous vector space.

Then $\mathbb{O}$ is Zariski-open and its complement S is called *the singular set* of (G, ρ, V). We call $v \in \mathbb{O}$ *a generic point*, and the isotropy subgroup at a generic point $G_v = \{g \in G \mid \rho(g)v = v\}$ $(v \in \mathbb{O})$ is called *a generic isotropy subgroup*. Note that $v \in V$ is a generic point if and only if $\dim G_v = \dim G - \dim V$. Note that (G, ρ, V) is a PV if and only if $(G^o, \rho|_{G^o}, V)$ is a PV where G^o is the connected component of G. Hence for the classification problem, we may assume that G is connected.

A non-zero rational function $P(v)$ on V is called *a relative invariant* if there exists a rational character $\chi : G \to GL(1)$ satisfying $P(\rho(g)v) = \chi(g)P(v)$ for $g \in G$. Let $S_i = \{v \in S \mid P_i(v) = 0\}$ $(i = 1, \ldots, N)$ be irreducible components of S with codimension one. When G is connected, these irreducible polynomials $P_1, \ldots, P_N$ are algebraically independent relative invariants and any relative invariant can be expressed uniquely as $P(v) = cP_1(v)^{m_1} \cdots P_N(v)^{m_N}$ with $c \in \mathbb{C}^\times$ and $(m_1, \ldots, m_N) \in \mathbb{Z}^N$. These $P_1, \cdots, P_N$ are called *the basic relative invariants* of (G, ρ, V). Note that a PV (G, ρ, V) has no relative invariant if and only if $\operatorname{codim} S \geq 2$.

By using the relations $\rho(\exp tA) = \exp td\rho(A)$ and $\chi(\exp tA) = \exp td\chi(A)$ for $A \in \mathrm{Lie}(G)$ and $t \in \mathbb{C}$, we differentiate $P(\rho(\exp tA)v) = \chi(\exp tA)P(v)$ by t. Then we have $\langle d\rho(A)v, \varphi_P(v)\rangle = d\chi(A)$ where $\varphi_P = \operatorname{grad}\log P : \mathbb{O} \to V^*$ satisfies $\varphi_P(\rho(g)v) = \rho^*(g)\varphi_P(v)$ for $g \in G$ and $v \in \mathbb{O}$. Here ρ^* is the dual representation of ρ on the dual vector space V^* of V. Hence $\varphi_P(\mathbb{O})$ is a G-orbit of the dual triplet (G, ρ^*, V^*). If $\varphi_P(\mathbb{O})$ is a Zariski-dense G-orbit, we call P *non-degenerate*. In this case, (G, ρ^*, V^*) is also a PV. If there exists a non-degenerate relative invariant, (G, ρ, V) is called *a regular PV*. When G is reductive, it is regular if and only if its generic isotropy subgroup is reductive.

In Section 1, we give the preliminaries for the later use.

In Section 2, we discuss the direct sum of PVs. So the classification of indecomposable PVs will be essential.

In Section 3, we discuss the adhesion of several PVs by trivial PVs. We also discuss the regularity.

In Section 4, we discuss the well known castling transformations and its generalization by Y. Teranishi. We also give some new generalization (cf. Proposition 4.5) from which the result of Y. Teranishi is induced.

In Section 5, we discuss Sato-Mori transformations. In this section, Proposition 5.3 is a new result.

In Section 6, we discuss symplectic PV-equivalence. The results about relative invariants and regularity are new.

In Section 7, we discuss orthogonal PV-equivalence. We give the complete proof.

Notation

In general, $V(n)$ denotes an n-dimensional vector space. If $V(n)$ and $V(n)^*$ appear at the same time, $V(n)^*$ denotes the dual vector space of $V(n)$.

If $\oplus$ and $\otimes$ appear at the same time, we write $+$ instead of $\oplus$ to avoid confusion.

1 Preliminaries

All equivalences of prehomogeneity are obtained so far from the following Key Lemma.

Lemma 1.1. (Key Lemma by M. Sato)

Assume that an algebraic group G acts on two irreducible algebraic varieties W and W'. Let $\varphi : W \to W'$ be a morphism satisfying

(I) $\varphi(gw) = g\varphi(w) \quad (g \in G, w \in W)$,

(II) $\overline{\varphi(W)} = W'$.

Then the following assertions (1) and (2) are equivalent:

(1) $W = \overline{G \cdot w}$ for some $w \in W$, i.e., W is G-prehomogeneous.

(2) (a) $W' = \overline{G \cdot w'}$ for some $w' \in W'$.

(b) For the above point $w' \in W'$ in (a), there exists a point $w \in \varphi^{-1}(w')$ such that $\varphi^{-1}(w') = \overline{G_{w'} \cdot w}$, where $G_{w'} = \{g \in G;\ gw' = w'\}$ is the isotropy subgroup of G at w'.

Note that a generic isotropy subgroup of (1) is isomorphic to that of (2)(b) since $(G_{w'})_w = G_w$. Also note that $W = \overline{G \cdot w}$ implies $W' = \overline{G \cdot \varphi(w)}$.

Proof. For the proof, see Proposition 7.6 in [K]. ∎

In particular, we have the following proposition.

Proposition 1.2. *Let V and V' be finite-dimensional vector spaces on which an algebraic group G acts. Assume that $\varphi : V \to V'$ is a polynomial map satisfying (I) $\overline{\varphi(V)} = V'$, (II) $\varphi(gv) = g\varphi(v)$ for $g \in G$, $v \in V$. Then we have the following assertions.*

(1) If (G, V) is a PV, then (G, V') must be a PV. Moreover if $v_0 \in V$ is a generic point of (G, V), then $\varphi(v_0) \in V'$ is always a generic point of (G, V').

(2) If $P'(v')$ is a relative invariant of (G, V') corresponding to the character χ, then $P(v) = P'(\varphi(v))$ is a relative invariant of (G, V) corresponding to the same character χ.

Proof. (1) This is a special case of the last statement of Lemma 1.1. (2) Since $P(gv) = P'(\varphi(gv)) = P'(g\varphi(v)) = \chi(g)P'(\varphi(v)) = \chi(g)P(v)$, we have our assertion. ∎

Proposition 1.3. *Let H be a connected algebraic group which has no nontrivial rational character, and G a connected algebraic group. Let $\rho : H \times G \to GL(V)$ and $\sigma : G \to GL(V')$ be rational representations. Assume that there exists a surjective polynomial map $\varphi : V \to V'$ satisfying $\varphi(\rho(g,h)v) = \sigma(g)\varphi(v)$ $(h \in H, g \in G, v \in V)$. Let $\pi : H \times G \to G$ be the projection. Assume that $(H \times G, \rho, V)$ is a PV with a generic point v, and $\pi((H \times G)_v) = G_{\varphi(v)}$ holds. Then the basic relative invariants of $(H \times G, \rho, V)$ are given by $\{P_1, \ldots, P_N\}$ with $P_i(v) = P_i'(\varphi(v))$ $(i = 1, \ldots, N)$ where $\{P_1', \ldots, P_N'\}$ are basic relative invariants of (G, σ, V'). If $(H \times G, \rho, V)$ is a reductive regular PV, then (G, σ, V') is also a regular PV.*

Proof. See Theorem 1.11 in [KKS]. ∎

Remark 1.4. *In (2) of Proposition 1.2, $P(v) = P'(\varphi(v))$ is a polynomial whenever P' is a polynomial. However even if $P(v) = P'(\varphi(v))$ is a polynomial, P' might not be a polynomial. For example, let $G = GL(1)^2$ act on $V = V' = \mathbb{C}^2$ as $V \ni x = (x_1, x_2) \mapsto (\alpha x_1, \beta x_2)$ and $V' \ni z = (z_1, z_2) \mapsto (\alpha\beta z_1, \beta z_2)$ $((\alpha, \beta) \in G)$. Define $\varphi : V \to V'$ by $(x_1, x_2) \mapsto (x_1 x_2, x_2)$. Then for $P(x) = x_1, P'(z) = z_1/z_2$, we have $P(x) = P'(\varphi(x))$.*

To prove the prehomogeneity of non-irreducible triplets, the following Proposition is fundamental, which can be obtained immediately from the Key Lemma.

Proposition 1.5. *The following assertions are equivalent.*

(1) $(G, \rho_1 \oplus \rho_2, V_1 \oplus V_2)$ is a PV.

(2) (a) (G, ρ_1, V_1) is a PV with a generic isotropy subgroup H.

(b) $(H, \rho_2|_H, V_2)$ is a PV.

2 Direct Sums

We define *the direct sum* $(G_1, \rho_1, V_1) \oplus (G_2, \rho_2, V_2)$ of (G_1, ρ_1, V_1) and (G_2, ρ_2, V_2) by $(G_1 \times G_2, \rho_1 \otimes 1 + 1 \otimes \rho_2, V_1 \oplus V_2)$. Similarly we can define the direct sum $\oplus_{i=1}^n (G_i, \rho_i, V_i)$ of (G_i, ρ_i, V_i) $(i = 1, \ldots, n)$.

Theorem 2.1. *(1) The direct sum* $\oplus_{i=1}^{n}(G_i, \rho_i, V_i)$ *is a PV if and only if each component* (G_i, ρ_i, V_i) $(i = 1, \ldots, n)$ *is a PV.*

(2) If $\{P_1^i, \ldots, P_{k_i}^i\}$ *are the basic relative invariants of* (G_i, ρ_i, V_i)*, then* $\{P_1^1, \ldots, P_{k_1}^1, \ldots, P_1^n, \ldots, P_{k_n}^n\}$ *are the basic relative invariants of* $\oplus_{i=1}^{n}(G_i, \rho_i, V_i)$.

(3) The direct sum $\oplus_{i=1}^{n}(G_i, \rho_i, V_i)$ *is a regular PV if and only if each component* (G_i, ρ_i, V_i) $(i = 1, \ldots, n)$ *is a regular PV.*

Proof. By Proposition 1.5, we have (1). The statement of (2) is obvious. For (3), see Proposition 1.5 in [KKTI].

∎

Hence if some PVs (G_i, ρ_i, V_i) $(i = 1, \ldots, n)$ are given, we can construct a new PV as their direct sum. We say that a triplet is *indecomposable* if it is not the direct sum of m triplets with $m \geq 2$.

3 Adhesion by Trivial PVs

Lemma 3.1. *Let* G' *be a subgroup of an algebraic group* G.

(1) (G, ρ, V) *is a PV whenever* $(G', \rho|_{G'}, V)$ *is a PV.*

(2) (G, ρ, V) *has no relative invariant if* $(G', \rho|_{G'}, V)$ *has no relative invariant.*

(3) If (G, ρ, V) *is a regular PV and* $(G', \rho|_{G'}, V)$ *is a PV, then* $(G', \rho|_{G'}, V)$ *is a regular PV.*

Proof. (1) is obvious. Since a relative invariant of (G, ρ, V) is a relative invariant of $(G', \rho|_{G'}, V)$, we have (2). For (3), if (G, ρ, V) has a non-degenerate relative invariant, it is also a non-degenerate relative invariant of $(G', \rho|_{G'}, V)$.

∎

Theorem 3.2. *Let* $\rho : H \to GL(m)$ *be any rational representation of any algebraic group* H.

(1) For any $n \geq m$*, a triplet* $(H \times GL(n), \rho \otimes \Lambda_1, M(m, n))$ *is always a PV, which is called a trivial PV.*

(2) For $n > m$*, a trivial PV* $(H \times GL(n), \rho \otimes \Lambda_1, M(m, n))$ *has no relative invariant. In particular, it is a non-regular PV.*

(3) For $n = m$*, a trivial PV* $(H \times GL(n), \rho \otimes \Lambda_1, M(m, n))$ *is a regular PV.*

Proof. (1) By (1) of Lemma 3.1, we may assume that $H = \{e\}$. Define the map $\varphi : V = M(n) \to V' = M(m,n)$ by $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \mapsto X_1$. Then $GL(n)$ acts on $V = M(n)$ by $X \mapsto X^tA$ ($X \in M(n), A \in GL(n)$) which is clearly a PV with a generic point I_n. Hence by (1) of Proposition 1.2, a triplet $(\{e\} \times GL(n), \rho|_{\{e\}} \otimes \Lambda_1, M(m,n))$ is a PV with a generic point $(I_m|O)$.

(2) By (2) of Lemma 3.1, we may assume that $H = \{e\}$. The singular set $S = \{X \in M(m,n) \mid \operatorname{rank} X < m\}$ is the common zeros of minors so that $\operatorname{codim} S \geq 2$ for $n > m$ and hence there is no relative invariant (see Preliminaries).

(3) If $m = n$, then $P(X) = \det X \in M(n)$ is a non-degenerate relative invariant since $\operatorname{grad}\log\det X = {}^tX^{-1}$ for $X \in GL(n)$ (see Lemma 1.21 in [KKTI]).

∎

Theorem 3.3. (Adhesion by Trivial PVs) *Let ρ and σ be rational representations of an algebraic group G satisfying $\deg \sigma \leq n$.*

(1) $(G \times GL(n), \rho \otimes 1 + \sigma \otimes \Lambda_1, V \oplus W)$ is a PV if and only if (G, ρ, V) is a PV.

(2) If $\deg \sigma < n$, then $(G \times GL(n), \rho \otimes 1 + \sigma \otimes \Lambda_1, V \oplus W)$ is a non-regular PV. Note that (G, ρ, V) can be a regular or a non-regular PV.

(3) If $\deg \sigma = n$, then $(G \times GL(n), \rho \otimes 1 + \sigma \otimes \Lambda_1, V \oplus W)$ is a regular PV if and only if (G, ρ, V) is a regular PV.

Proof. (1) Assume that $(G \times GL(n), \rho \otimes 1 + \sigma \otimes \Lambda_1, V \oplus W)$ is a PV . Then by applying (1) of Proposition 1.2 to $\varphi : V \oplus W \to V$, we know that (G, ρ, V) is a PV. On the other hand, if (G, ρ, V) is a PV with a generic isotropy subgroup H, then by Proposition 1.5, $(G \times GL(n), \rho \otimes 1 + \sigma \otimes \Lambda_1, V \oplus W)$ is a PV if and only if $(H \times GL(n), \sigma|_H \otimes \Lambda_1, W)$ is a PV. However it is a trivial PV.

(2) If $m = \deg \sigma < n$, we may assume that $W = M(m,n)$. Then $(v, X) \in V \oplus M(m,n)$ is a generic point if and only if v is a generic point of (G, ρ, V) and $\operatorname{rank} X = m$. Hence the singular set S of $(G \times GL(n), \rho \otimes 1 + \sigma \otimes \Lambda_1, V \oplus M(m,n))$ is the union $\{(v,X)|v \in S_V, X \in M(m,n)\} \cup \{(v,X)|v \in V, X \in S_W\}$ where S_V is the singular set of (G, ρ, V) and $S_W = \{X \in M(m,n)|\operatorname{rank} X < m\}$. Since $\operatorname{codim}\{(v,X)|v \in V, X \in S_W\} \geq 2$ (see the proof of Theorem 3.2), any relative invariant on $V \oplus M(m,n)$ is a relative invariant on V. This implies that any relative invariant is degenerate.

(3) Since $\deg \sigma = n$, we may assume that $W = M(n)$. Then $(v, X) \in V \oplus M(n)$ is a generic point if and only if v is a generic point of (G, ρ, V) and $\operatorname{rank} X = n$. Hence the singular set S of $(G \times GL(n), \rho \otimes 1 + \sigma \otimes \Lambda_1, V \oplus M(n))$ is the union $\{(v,X)|v \in$

$S_V, X \in M(n)\} \cup \{(v,X)|v \in V, \det X = 0\}$. This implies that any relative invariant $P(v,X)$ on $V \oplus M(n)$ is of the form $Q(v)(\det X)^r$ $((v,X) \in V \oplus M(n))$ for some integer r where $Q(v)$ is a relative invariant of (G,ρ,V). Hence we have $\operatorname{grad}\log P(v,X) = (\operatorname{grad}\log Q(v),0) + (0, r\,{}^tX^{-1}) \in V^* \oplus M(n)$ $((v,X) \in V \oplus M(n) \setminus S)$. This implies that $P(v,X)$ is non-degenerate if and only if $Q(v)$ is non-degenerate and $r \neq 0$. Thus we obtain our assertion.

∎

4 Castling Transformation and its Generalization

As we see in the previous section, $(H \times GL(n), \sigma \otimes \Lambda_1, V \otimes V(n))$ is always a PV if $m = \dim V \leq n$. In this section, we shall consider the case $m = \dim V > n$. Let $\mathrm{Grass}_n(V) = \{U|\ U$ is an n-dimensional subspace of $V\}$ be the Grassmann variety. If $\sigma : H \to GL(V)$ is a rational representation of an algebraic group H, then H acts on $\mathrm{Grass}_n(V)$ by $U \mapsto \sigma(h)U$ $(h \in H, U \in \mathrm{Grass}_n(V))$. We identify $V \otimes V(n)$ with $M(m,n)$ and put $W = \{X \in M(m,n)|\operatorname{rank} X = n\}$. Then $(H \times GL(n), \sigma \otimes \Lambda_1, V \otimes V(n))$ is a PV if and only if W is $H \times GL(n)$-prehomogeneous. For $X = (x_1|\cdots|x_n) \in W$, let $\langle X\rangle = \mathbb{C}x_1 + \cdots + \mathbb{C}x_n$ be an n-dimensional subspace of $\mathbb{C}^m$, i.e., $\langle X\rangle \in \mathrm{Grass}_n(V)$. Then the map $\varphi : W \to W' = \mathrm{Grass}_n(V)$ defined by $X \mapsto \langle X\rangle$ is a surjective and it satisfies $\varphi(\sigma(h)X{}^tA) = \sigma(h)\varphi(X)$ $(h \in H, A \in GL(n))$. If $\langle X\rangle = \langle Y\rangle \in \mathrm{Grass}_n(V)$, then as a base change, there exists a unique $A \in GL(n)$ satisfying $Y = X{}^tA$, i.e., a fiber $\varphi^{-1}(\langle X\rangle)$ is $GL(n)$-homogeneous. Hence by Key Lemma, we obtain the former part of the following lemma.

Lemma 4.1. *Assume that* $m = \dim V > n \geq 1$. *Then the following assertions are equivalent.*
(1) $(H \times GL(n), \sigma \otimes \Lambda_1, V \otimes V(n))$ *is a PV.*
(2) The Grassmann variety $\mathrm{Grass}_n(V)$ *is* H*-prehomogeneous by* σ.
Their generic isotropy subgroups are isomorphic.

Proof. We show the last assertion. If $X \in M(m,n)$ is a generic point, then $\langle X\rangle$ is a generic point of $\mathrm{Grass}_n(V)$ by Lemma 1.1. Let $p : H \times GL(n) \to H$ be a projection. We show that this induces an isomorphism $p : (H \times GL(n))_X \to H_{\langle X\rangle}$. For $(h,A) \in (H \times GL(n))_X$, we have $\langle\sigma(h)X\rangle = \langle\sigma(h)X{}^tA\rangle = \langle X\rangle$, and hence $h \in H_{\langle X\rangle}$. Conversely for any $h \in H_{\langle X\rangle}$, there exists a unique $A \in GL(n)$ satisfying $\sigma(h)X{}^tA = X$. This implies that the restriction of p to $(H \times GL(n))_X$ is a bijection. ∎

Proposition 4.2. (Castling transformation)

(1) $(H \times GL(n), \sigma \otimes \Lambda_1, V \otimes V(n))$ *with* $m = \dim V > n \geq 1$ *is a PV if and only if* $(H \times GL(m-n), \sigma^* \otimes \Lambda_1, V^* \otimes V(m-n))$ *is a PV where* $\sigma^* : H \to GL(V^*)$ *is*

the dual representation of σ. These triplets are called the castling transforms of each other. Their generic isotropy subgroups are isomorphic. There exists a one-to-one correspondence of their relative invariants (cf. p. 68 in [SK]).

(2) A castling transform of a regular PV is a regular PV.

Proof. (1) The bijection $f : \mathrm{Grass}_n(V) \to \mathrm{Grass}_{m-n}(V^*)$ defined by $U \mapsto U^{\perp}$ is H-equivariant, i.e., $f(\sigma(h)U) = \sigma^*(h)f(U)$ $(h \in H, U \in \mathrm{Grass}_n(V))$. Hence $\mathrm{Grass}_n(V)$ is H-prehomogeneous by σ if and only if $\mathrm{Grass}_{m-n}(V^*)$ is H-prehomogeneous by σ^*. Hence by Lemma 4.1, we obtain our assertion.

(2) When G is reductive, it is obvious since a reductive PV is regular if and only if a generic isotropy subgroup is reductive. However, even if it is not reductive, this assertion holds (see Theorem 1.30 in [KKTI], [SO]).

∎

Remark 4.3. (Grassmann construction) *To construct a new PV $(H \times GL(m-n), \sigma^* \otimes \Lambda_1, V^* \otimes V(m-n))$ from a given PV $(H \times GL(n), \sigma \otimes \Lambda_1, V \otimes V(n))$ with $m = \dim V > n \geq 1$ is sometimes called the Grassmann construction. In particular, for any given PV (G, ρ, V), we can construct a new PV $(G \times GL(m-1), \rho^* \otimes \Lambda_1, V^* \otimes V(m-1))$. Note that the discovery of castling transform, i.e., Grassmann construction, made the classification of irreducible PVs possible ([SK]).*

Theorem 4.4. *Assume that $m > n \geq 1$. Then the following assertions are equivalent and their generic isotropy subgroups are isomorphic.*
(1) $(G \times GL(n), \rho \otimes 1 + \sigma \otimes \Lambda_1, V \oplus M(m,n))$ is a PV.
(2)$(G \times GL(m-n), \rho \otimes 1 + \sigma^* \otimes \Lambda_1, V \oplus M(m, m-n))$ is a PV.
There exists a bijective correspondence between relative invariants of (1) and (2).

Proof. By Propositions 1.5 and 4.2, we obtain our results. For the relative invariants, see Proposition 1.16 in [KKS]. ∎

Proposition 4.5. *Assume that $m > n + l$. Let $\rho : H \to GL(m)$ and $\sigma : K \to GL(n)$ be rational representations of algebraic groups H and K. Then the following assertions are equivalent.*

(1) $\left(H \times \begin{pmatrix} GL(l) & * \\ O & \sigma(K) \end{pmatrix}, \ \rho \otimes \Lambda_1^*, \ M(m, n+l)\right)$ *is a PV.*

(2) $\left(H \times \begin{pmatrix} GL(m-n-l) & * \\ O & \sigma^*(K) \end{pmatrix}, \ \rho^* \otimes \Lambda_1^*, \ M(m, m-l)\right)$ *is a PV.*

Moreover their generic isotropy subgroups are isomorphic.

Proof. The $GL(l+n)$-part of the generic isotropy subgroup of $(GL(l+n)\times K, \Lambda_1\otimes\sigma, M(l+n,n))$ at $\begin{pmatrix} O \\ I_n \end{pmatrix}$ is $\begin{pmatrix} GL(l) & O \\ * & \sigma^*(K) \end{pmatrix}$ $(=\Lambda_1^*(\begin{pmatrix} GL(l) & * \\ O & \sigma(K) \end{pmatrix}))$. Hence, by Proposition 1.5, (1) is equivalent to (3) $(H\times GL(l+n)\times K, \rho\otimes\Lambda_1\otimes 1+1\otimes\Lambda_1\otimes\sigma, M(m,l+n)\oplus M(l+n,n))\cong((H\times K)\times GL(l+n), (\rho\otimes 1+1\otimes\sigma)\otimes\Lambda_1, M(m+n,l+n))$ is a PV. Its castling transform with respect to $GL(l+n)$ is $((H\times K)\times GL(m-l), (\rho^*\otimes 1+1\otimes\sigma^*)\otimes\Lambda_1, M(m+n,m-l))$. Since this is a PV if and only if (2) holds, we obtain our assertion. ∎

Example 4.6. *For $2m > l+3$, $(Sp(m)\times \begin{pmatrix} GL(l) & * \\ O & GO(3) \end{pmatrix}, \Lambda_1\otimes\Lambda_1^*, M(2m,l+3))$ is a PV if and only if $(Sp(m)\times \begin{pmatrix} GL(2m-l-3) & * \\ O & GO(3) \end{pmatrix}, \Lambda_1\otimes\Lambda_1^*, M(2m,2m-l))$ is a PV. Actually they are PVs. Note that $\Lambda_1^*=\Lambda_1$ for $Sp(m)$ and $SO(3)$.*

Let $P(e_1,\ldots,e_r)$ be the standard parabolic subgroup of $GL(n)$. Note that $\Lambda_1^*(P(e_1,\ldots,e_r))$ $\{{}^tA^{-1}|A\in P(e_1,\ldots,e_r)\}$ is conjugate to $P(e_r,\ldots,e_1)$.

As a corollary of Proposition 4.5, we obtain the new proof of the following proposition by Y. Teranishi.

Proposition 4.7. (Parabolic transformation [T])

Assume that $m > n = e_1+\cdots+e_r \geq 1$. Let $\sigma : H \to GL(m)$ be a rational representation of an algebraic group H. Then the following assertions are equivalent.

(1) $(H\times P(e_1,\ldots,e_r), \sigma\otimes\Lambda_1^*, M(m,n))$ is a PV.

(2) $(H\times P(m-n,e_r,\ldots,e_2), \sigma^*\otimes\Lambda_1^*, M(m,m-e_1))$ is a PV.

Proof. By Proposition 4.5, $(H\times P(e_1,\ldots,e_r), \sigma\otimes\Lambda_1^*, M(m,n))$

$=(H\times \begin{pmatrix} GL(e_1) & * \\ O & P(e_2,\ldots,e_r) \end{pmatrix},\ \sigma\otimes\Lambda_1^*,\ M(m,n))$ is a PV if and only if

$(H\times \begin{pmatrix} GL(m-n) & * \\ O & P(e_r,\ldots,e_2) \end{pmatrix},\ \sigma^*\otimes\Lambda_1^*,\ M(m,m-e_1))$

$=(H\times P(m-n,e_r,\ldots,e_2), \sigma^*\otimes\Lambda_1^*, M(m,m-e_1))$ is a PV. Also see p. 141 in [T] or p. 238 in [KKO]. ∎

Note that if $r=1$, then it is a usual castling transform.

Remark 4.8. *(1) In Proposition 4.7, there exists a one-to-one correspondence between the relative invariants of (1) and (2) (see Lemma 1.4 in [T]).*

(2) Although the castling transformation keeps the regularity, the parabolic transformation does not keep the regularity in general as we see in the following example. In Example 4.6, if we use $GL(3)$ *instead of* $GO(3)$*, then we obtain a PV* $(Sp(m) \times P(l,3), \Lambda_1 \otimes \Lambda_1^*, M(2m, l+3))$*. If* l *is odd, then it is a regular PV (cf. (2) of Proposition 6.3). However its parabolic transform is* $(Sp(m) \times P(2m-l-3,3), \Lambda_1 \otimes \Lambda_1^*, M(2m, 2m-l))$ *which is not regular since* $2m-l$ *is odd.*

5 Sato-Mori Transformation

Theorem 5.1. (M. Sato) *Assume that* $n \geq \max\{m_1, m_2\}$*. Let* $\rho_i : G \to GL(m_i)$ $(i = 1, 2)$ *be rational representations of* G*. Then* $(G \times GL(n), \rho_1 \otimes \Lambda_1 + \rho_2 \otimes \Lambda_1^*, M(m_1, n) \oplus M(m_2, n))$ *is a PV if and only if* $(G, \rho_1 \otimes \rho_2, M(m_1, m_2))$ *is a PV.*

Proof. See Theorem 7.8 in [K]. ∎

Corollary 5.2. *A triplet* (G, ρ, V) *is a PV if and only if* $(G \times GL(n), \rho \otimes \Lambda_1 + 1 \otimes \Lambda_1^*, V \otimes V(n) + V(n)^*)$ *for any natural number* n *satisfying* $\dim V \leq n$*.*

Proposition 5.3. *Assume that* $n > \max\{m_1, m_2\}$*. Let* $P_1', \ldots, P_N'$ *be basic relative invariants of* $(G, \rho_1 \otimes \rho_2, M(m_1, m_2))$*. Then* $P_1, \ldots, P_N$ *are basic relative invariants of* $(G \times GL(n), \rho_1 \otimes \Lambda_1 + \rho_2 \otimes \Lambda_1^*, M(m_1, n) \oplus M(m_2, n))$ *where* $P_i(X, Y) = P_i'(X^tY)$ $(i = 1, \ldots, N)$ *for* $(X, Y) \in M(m_1, n) \oplus M(m_2, n)$*.*

Proof. The number of basic relative invariants are the same (see Proposition 1.18 in [KKTI]). Hence it is enough to prove that P_i is irreducible. If not, we have $P_i = QR$ for some polynomials Q and R. Since the group is connected, Q and R are also relative invariants so that there exist Q' and R' such that $P_i' = Q'R'$. In general, if a polynomial F on $M(m_1, n) \oplus M(m_2, n)$ is defined by $F(X, Y) = F'(X^tY)$ for some rational function F' on $M(m_1, m_2)$, then F' is also a polynomial since $F([I_{m_1}|O], [{}^tZ|O]) = F'(Z)$ holds. Hence Q' and R' are polynomials, which contradicts an irreducibility of P_i' (cf. Remark 1.4). ∎

Proposition 5.4. *If* $n > \max\{m_1, m_2\}$*, then a PV* $(G \times GL(n), \rho_1 \otimes \Lambda_1 + \rho_2 \otimes \Lambda_1^*, M(m_1, n) \oplus M(m_2, n))$ *is a regular PV if and only if* $m_1 = m_2$*.*

Proof. Assume that $m_1 = m_2 (= m)$. Then the image of $G \times GL(n)$ is a subgroup of the image of the group of $(GL(m) \times GL(m) \times GL(n), \Lambda_1 \otimes 1 \otimes \Lambda_1 + 1 \otimes \Lambda_1 \otimes \Lambda_1^*, M(m, n) \oplus M(m, n))$. It is castling equivalent to $((GL(m) \times GL(n-m)) \times GL(n), (\Lambda_1 \otimes 1 + 1 \otimes \Lambda_1) \otimes \Lambda_1, M(n))$ which is a regular trivial PV. Hence by (3) of Lemma 3.1 and Theorem 4.4, our PV is a regular PV. If $m_1 \neq m_2$, it is a non-regular PV by Proposition 1.22 in [KKTI]. ∎

Proposition 5.5. *If $(G, \rho_1 \otimes \rho_2, M(m_1, m_2))$ is a regular PV, then $(G \times GL(n), \rho_1 \otimes \Lambda_1 + \rho_2 \otimes \Lambda_1^*, M(m_1, n) \oplus M(m_2, n))$ with $n = \max\{m_1, m_2\}$ is a regular PV. So far, the converse holds under some conditions, but not proved in general yet.*

Proof. See (2) of Proposition 1.20 in [KKTI].

∎

Proposition 5.6. *Assume that $n = m_1 \geq m_2$. Then a relative invariant $P(X, Y)$ of a PV $(G \times GL(m_1), \rho_1 \otimes \Lambda_1 + \rho_2 \otimes \Lambda_1^*, M(m_1) \oplus M(m_2, m_1))$ is of the form $P(X, Y) = P'(X^tY)(\det X)^r$ where $P'(Z)$ is a relative invariant of $(G, \rho_1 \otimes \rho_2, M(m_1, m_2))$ and $r \in \mathbb{Z}$. In this case, even if $P'(Z)$ is an irreducible polynomial, $P(X, Y) = P'(X^tY)$ is not necessarily irreducible.*

Proof. By (2) of Proposition 1.18 in [KKTI], we have the former result. For the latter assertion, for example, if $n = m_1 = m_2$, then $P'(Z) = \det Z$ is an irreducible polynomial, but $P(X, Y) = P'(X^tY) = \det X \det Y$ is not irreducible.

∎

Theorem 5.7. *The following assertions are equivalent.*

(1) $(G, \rho + \rho_1 \otimes \rho_2, V \oplus M(m_1, m_2))$ is a PV.

(2) $(G \times GL(n), \rho \otimes 1 + \rho_1 \otimes \Lambda_1 + \rho_2 \otimes \Lambda_1^*, V \oplus M(m_1, n) \oplus M(m_2, n))$ is a PV for all n satisfying $n \geq \max\{m_1, m_2\}$

If $n > \max\{m_1, m_2\}$, there exists a bijective correspondence between the relative invariants of (1) and (2).

Proof. By Proposition 1.5 and Theorem 5.1, we obtain the equivalence of (1) and (2). For the latter assertion, one can prove similarly as Proposition 5.3.

∎

Now consider the case $m_1 > n \geq m_2$.

Theorem 5.8. ([KKS2])
For $m_1 > n > m_2$, the following assertions are equivalent.

(1) $(G \times GL(n),\ \rho_1 \otimes \Lambda_1 + \rho_2 \otimes \Lambda_1^*,\ M(m_1, n) \oplus M(m_2, n))$ is a PV.

(2) (a) $(G,\ \rho_1 \otimes \rho_2,\ M(m_1, m_2))$ is a PV with a generic point $Z_0 \in M(m_1, m_2)$.

*(b) Let H be the isotropy subgroup of G at Z_0, ρ_1^* the dual action of ρ_1 and $\langle Z_0\rangle^\perp = \{f \in V(m_1)^* | f(v) = 0$ for all $v \in \langle Z_0\rangle\}$.
Then $(H \times GL(m_1 - n),\ \rho_1^* \otimes \Lambda_1,\ \langle Z_0\rangle^\perp \otimes V(m_1 - n))$ is a PV.*

Moreover generic isotropy subgroups of (1) and (2)(b) are isomorphic. In particular, when G is reductive, (1) is a regular PV if and only if a generic isotropy subgroup of (2)(b) is reductive.

Note that if $m_1 > n = m_2$, then (1) and (2)(a) in Theorem 5.8 are equivalent since the generic isotropy subgroup at I_n of $(G \times GL(m_2),\ \rho_2 \otimes \Lambda_1^*,\ M(n))$ is $\{(A, \rho_2(A));\ A \in G\}$. The special case $n = m_1 - 1$ and $m_2 = 1$ of Theorem 5.8 was already known (See Theorem 1.1 in [KUY], Theorem 7.10 in [K]) and used to prove that $(GL(1)^4 \times G,\ \rho,\ V((2n+1)(2m+2n-1)))$ is a PV and $(GL(1)^5 \times G,\ \rho + 1 \otimes \Lambda_1^{(*)} \otimes 1,\ V((2n+1)(2m+2n-1)) \oplus V(2m))$ is a non-PV where $G = Sp(n) \times SL(2m) \times SL(2n-1)$ and $\rho = \Lambda_1 \otimes \Lambda_1 \otimes 1 + \Lambda_1 \otimes 1 \otimes \Lambda_1 + 1 \otimes \Lambda_1^{(*)} \otimes 1 + 1 \otimes 1 \otimes \Lambda_1^*$ (See Lemmas 3.31 and 3.33 in [KUY]).

For the case $\min\{m_1, m_2\} \geq n$, we have the following result.

Let G be a linear algebraic group. Let $\rho_1 : G \to GL(m_1)$ and $\rho_2 : G \to GL(m_2)$ be its rational representations. Then G acts on $M(m_1, m_2)$ by $\rho_1 \otimes \rho_2$, i.e., $X \mapsto \rho_1(g) X {}^t\rho_2(g)$ $(X \in M(m_1, m_2),\ g \in G)$. By this action, G also acts on a rank variety $M^{(n)}(m_1, m_2) = \{X \in M(m_1, m_2) |\ \operatorname{rank} X = n\}$.

Theorem 5.9. ([OHK]) *The following assertions are equivalent.*

(1) $M^{(n)}(m_1, m_2)$ has a Zariski-dense G-orbit by the action $\rho_1 \otimes \rho_2$.

(2) $(G \times GL(n),\ \rho_1 \otimes \Lambda_1 + \rho_2 \otimes \Lambda_1^*,\ M(m_1, n) \oplus M(m_2, n))$ is a PV.

Here the action of *(2)* is given by $(X, Y) \mapsto (\rho_1(g) X {}^t A, \rho_2(g) Y A^{-1})$ for $(X, Y) \in M(m_1, n) \oplus M(m_2, n)$ and $(g, A) \in G \times GL(n)$.

6 Symplectic PV-equivalence

Let $\sigma : H \to GL(n)$ be a rational representation of a linear algebraic group H. Then the action $\Lambda_2(\sigma)$ of H on $\operatorname{Alt}(n) = \{X \in M(n) |\ {}^tX = -X\}$ is given by $X \mapsto \sigma(h) X {}^t\sigma(h)$ $(h \in H, X \in \operatorname{Alt}(n))$. Let $Sp(m) = \{A \in GL(2m) |\ {}^tAJA = J\}$ be the symplectic group where $J = \begin{pmatrix} O & I_m \\ -I_m & O \end{pmatrix}$. For $x, y \in \mathbb{C}^{2m}$, if we define $\langle x, y \rangle = {}^tx J y = \sum_{i=1}^{m} (x_i y_{m+i} - x_{m+i} y_i)$, we have $\langle Ax, Ay \rangle = \langle x, y \rangle$ for $A \in Sp(m)$. Now we investigate the relation of 2 triplets $(Sp(m) \times H, \Lambda_1 \otimes \sigma, M(2m, n))$ and $(H, \Lambda_2(\sigma), \operatorname{Alt}(n))$.

Lemma 6.1. *For $2m \geq n$, define the map $\varphi : M(2m, n) \to \operatorname{Alt}(n)$ by $X \mapsto {}^tXJX = (\langle x_i, x_j \rangle)$ $(X = (x_1 | \cdots | x_n) \in M(2m, n))$. Then φ is surjective and $\varphi(AX {}^t\sigma(h)) = \sigma(h) \varphi(X) {}^t\sigma(h)$ $(X \in M(2m, n), (A, h) \in Sp(m) \times H)$.*

Proof. For $(A, B) \in Sp(m) \times GL(n)$, we have $\varphi(AX {}^tB) = {}^t(AX {}^tB) J (AX {}^tB) = B \varphi(X) {}^tB$. Since $(GL(n), \Lambda_2, \operatorname{Alt}(n))$ is a PV with finitely many orbits $\{X \in \operatorname{Alt}(n) |\ \operatorname{rank} X =$

$2l\}$ $(2l \le n)$, and $\operatorname{rank}\varphi(X_{2l}) = 2l$ for $X_{2l} = (e_1|e_{m+1}|\cdots|e_l|e_{m+l}|O) \in M(2m, n)$, we obtain the surjectivity of φ. ∎

Lemma 6.2. *For $X, Y \in M(2m, n)$ with $2m \ge n$ satisfying $\varphi(X) = \varphi(Y)$ and $\operatorname{rank} X = \operatorname{rank} Y = n$, there exists $A \in Sp(m)$ satisfying $Y = AX$.*

Proof. For $X = (x_1|\cdots|x_n)$ and $Y = (y_1|\cdots|y_n)$, $\varphi(X) = \varphi(Y)$ implies $\langle x_i, x_j\rangle = \langle y_i, y_j\rangle$ for $i, j = 1, \ldots, n$, we have our result from Lemma 7.49 in [K]. ∎

Proposition 6.3. *Assume that $2m \ge n$.*

(1) $(Sp(m) \times H, \Lambda_1 \otimes \sigma, M(2m, n))$ is a PV if and only if $(H, \Lambda_2(\sigma), \mathrm{Alt}(n))$ is a PV. If $P'_1, \ldots, P'_N$ are basic relative invariants of $(H, \Lambda_2(\sigma), \mathrm{Alt}(n))$, then $P_1, \ldots, P_N$ are basic relative invariants of $(Sp(m) \times H, \Lambda_1 \otimes \sigma, M(2m, n))$ where $P_i(X) = P'_i({}^tXJX)$ $(i = 1, \ldots, N)$.

(2) A PV $(Sp(m) \times H, \Lambda_1 \otimes \sigma, M(2m, n))$ is a regular PV if and only if n is even.

(3) A PV $(H, \Lambda_2(\sigma), \mathrm{Alt}(n))$ is a regular PV if n is even. If n is odd, A PV $(H, \Lambda_2(\sigma), \mathrm{Alt}(n))$ can be a regular PV or a non-regular PV.

Proof. (1) The former part is obtained by Key Lemma and Lemmas 6.1, 6.2. To prove the latter part, we use Proposition 1.3. Let $X \in M(2m, n)$ be a generic point of $(Sp(m) \times H, \Lambda_1 \otimes \sigma, M(2m, n))$. Since $AX{}^t\sigma(h) = X$ implies $\sigma(h)({}^tXJX){}^t\sigma(h) = {}^tXJX$, we have $\pi((Sp(m) \times H)_X) \subset H_{\varphi(X)}$. Now assume that $h \in H_{\varphi(X)}$ and put $Y = X{}^t\sigma(h)$. Then we have $\varphi(Y) = \sigma(h)\varphi(X){}^t\sigma(h) = \varphi(X)$ and hence there exists $A \in Sp(m)$ satisfying $X = AY (= AX{}^t\sigma(h))$,i.e.,$(A, h) \in (Sp(m) \times H)_X$ and $h \in \pi((Sp(m) \times H)_X)$.

(2) If n is even, then $(Sp(m) \times GL(n), \Lambda_1 \otimes \Lambda_1, M(2m, n))$ is a regular irreducible PV (see [SK]). Hence a PV $(Sp(m) \times H, \Lambda_1 \otimes \sigma, M(2m, n))$ is a regular PV by (3) of Lemma 3.1. Now assume that $(Sp(m) \times H, \Lambda_1 \otimes \sigma, M(2m, n))$ is a regular PV with a non-degenerate relative invariant $P(X)$. By (1), there exists a relative invariant P' of $(H, \Lambda_2(\sigma), \mathrm{Alt}(n))$ satisfying $P(X) = P'({}^tXJX)$. For a generic point $X \in M(2m, n)$, we have $\operatorname{grad}\log P(X) = -JX \operatorname{grad}\log P'({}^tXJX) \in M(2m, n)$ by direct calculation. Since P is non-degenerate, we have $n = \operatorname{rank}\operatorname{grad}\log P(X) \le \operatorname{rank}\operatorname{grad}\log P'({}^tXJX) \le n$. Since $\operatorname{grad}\log P'({}^tXJX) \in \mathrm{Alt}(n)$ with rank $= n$, this implies that n is even.

(3) If n is even, then $(GL(n), \Lambda_2, \mathrm{Alt}(n))$ is a regular irreducible PV (see [SK]), and hence a PV $(H, \Lambda_2(\sigma), \mathrm{Alt}(n))$ is a regular PV by (3) of Lemma 3.1. If n is odd, $(GL(n), \Lambda_2, \mathrm{Alt}(n))$ is a non-regular irreducible PV (see [SK]) while if $\sigma : H =$

$\{\mathrm{diag}(\alpha,\beta,\gamma)|\alpha,\beta,\gamma \in GL(1)\} \hookrightarrow GL(3)$, then $(H, \Lambda_2(\sigma), \mathrm{Alt}(3)) = (GL(1)\times GL(1)\times GL(1), \Lambda_1\otimes\Lambda_1\otimes 1 + \Lambda_1\otimes 1\otimes\Lambda_1 + 1\otimes\Lambda_1\otimes\Lambda_1, \mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C})$ is a regular PV.

∎

Theorem 6.4. *(1) For $2m \geq n$, the triplet $T = (Sp(m)\times G, 1\otimes\rho + \Lambda_1\otimes\sigma, V\oplus M(2m,n))$ is a PV if and only if the triplet $T' = (G, \rho\oplus\Lambda_2(\sigma), V\oplus\mathrm{Alt}(n))$ is a PV.*

(2) If $P'(v,Z)$ is a relative invariant of T', then $P(v,X) = P'(v, {}^tXJX)$ is a relative invariant of T and this gives the bijection of relative invariants.

(3) When G is reductive and T is a regular PV, then T' is also a regular PV. However the converse does not hold.

Proof. (1) If (G,ρ,V) is a non-PV, then both are non-PVs by (1) of Proposition 1.2. If (G,ρ,V) is a PV with a generic isotropy subgroup H, then it is enough to show that $(H\times Sp(m), \sigma|_H\otimes\Lambda_1, M(n,2m))$ is a PV if and only if $(H, \Lambda_2(\sigma|_H), \mathrm{Alt}(n))$ is a PV. This is obtained from Proposition 6.3.

(2) By Theorem 1.13 in [KKS] and Proposition 6.3, we obtain our result.

(3) If a generic isotropy subgroup of T is reductive, then a generic isotropy subgroup of T' is reductive. Hence it is regular. The converse does not hold as we see the following example: $(GL(1)^2\times SL(2l+1), \Lambda_1\oplus\Lambda_2, V(2l+1)\oplus V(l(2l+1)))$ is a regular PV where $GL(1)^2$ acts on each irreducible component as scalar multiplications (see p. 95 in [K2]) while $(GL(1)^2\times Sp(m)\times SL(2l+1), \Lambda_1\otimes\Lambda_1 + 1\otimes\Lambda_1, V(2m)\otimes V(2l+1) + V(2l+1))$ $(2m > 2l+1)$ is a non-regular PV. (see p. 398 in [KKIY]).

∎

7 Orthogonal PV-equivalence

Let $\sigma : H \to GL(n)$ be a rational representation of a linear algebraic group H. Then the action $2\Lambda_1(\sigma)$ of H on $\mathrm{Sym}(n) = \{X \in M(n)|\ {}^tX = X\}$ is given by $X \mapsto \sigma(h)X{}^t\sigma(h)$ $(h \in H, X \in \mathrm{Sym}(n))$. For $K = {}^tK \in GL(m)$, let $O(K) = \{A \in GL(m)|{}^tAKA = K\}$ (resp., $SO(K) = O(K)\cap SL(m)$) be the orthogonal group (resp., the special orthogonal group) with respect to K. For $x,y \in \mathbb{C}^m$, if we define $(x,y) = {}^txKy$, we have $(Ax,Ay) = (x,y)$ for $A \in O(K)$.

The following lemma is well-known (cf. [W]).

Lemma 7.1. *For $m \geq n$, define the map $\varphi : M(m,n) \to \mathrm{Sym}(n)$ by $X \mapsto {}^tXKX = ((x_i,x_j))$ $(X = (x_1|\cdots|x_n) \in M(m,n))$. Then φ is surjective and we have $\varphi(AX{}^t\sigma(h)) = \sigma(h)\varphi(X){}^t\sigma(h)$ $(X \in M(m,n),\ (A,h) \in O(K)\times H)$.*

Proof. For any $(A,g) \in O(K) \times GL(n)$, we have $\varphi(AX^tg) = {}^t(AX^tg)K(AX^tg) = g^tX({}^tAKA)X^tg = g\varphi(X)^tg$. Since $(GL(n), 2\Lambda_1, \mathrm{Sym}(n))$ is a PV with finitely many orbits with representatives $T_d = \begin{pmatrix} I_d & O \\ O & O \end{pmatrix}$ $(d = 0, 1, \ldots, n)$, and for $X_d = (z_1|\cdots|z_d|O) \in M(m,n)$, we have $\varphi(X_d) = T_d$, we obtain the surjectivity of φ. Here $z_1, \ldots, z_m$ is an orthonormal basis of $\mathbb{C}^m$ with respect to $(\ ,\ )$. Since $\sigma(H) \subset GL(n)$, we obtain the latter result.

∎

Lemma 7.2. *For $m > n \geq 1$, assume that $X, Y \in M(m,n)$ satisfy the condition ${}^tXKX = {}^tYKY$ with* rank ${}^tXKX = n$. *Then there exist non-zero elements x and y in $\mathbb{C}^m$ satisfying ${}^t\tilde{X}K\tilde{X} = \begin{pmatrix} {}^tXKX & 0 \\ 0 & {}^txKx \end{pmatrix} = \begin{pmatrix} {}^tYKY & 0 \\ 0 & {}^tyKy \end{pmatrix} = {}^t\tilde{Y}K\tilde{Y}$ with* rank ${}^t\tilde{X}K\tilde{X} = n+1$ *where $\tilde{X} = (X|x)$ and $\tilde{Y} = (Y|y) \in M(m, n+1)$.*

Proof. For $X = (x_1|\cdots|x_n)$, put $\langle X\rangle = \mathbb{C}x_1 + \cdots + \mathbb{C}x_n$. We show that rank ${}^tXKX = n$ implies that $\langle X\rangle \oplus \langle X\rangle^\perp = \mathbb{C}^m$ where $\langle X\rangle^\perp = \{y \in \mathbb{C}^m | (x,y) = 0$ for all $x \in \langle X\rangle\}$. It is enough to show that $\langle X\rangle \cap \langle X\rangle^\perp = \{0\}$. For any $x = a_1x_1 + \cdots + a_nx_n \in \langle X\rangle \cap \langle X\rangle^\perp$, we have ${}^tXKX{}^t(a_1,\ldots,a_n) = {}^tXKx = {}^t((x_1,x),\ldots,(x_n,x)) = {}^t(0,\ldots,0)$. Since ${}^tXKX \in GL(n)$, we have ${}^t(a_1,\ldots,a_n) = {}^t(0,\ldots,0)$, i.e., $x = 0$. Hence there exists a non-zero element x of $\langle X\rangle^\perp$. We show that we may assume $(x,x) = {}^txKx \neq 0$. If $(x,x) = 0$ for all $x \in \langle X\rangle^\perp$, then we have $(x,x') = \frac{1}{2}(x+x', x+x') = 0$ for all $x, x' \in \langle X\rangle^\perp$. Hence if x is a non-zero element of $\langle X\rangle^\perp$, then we have $(x,y) = (x,z) + (x,w) = 0$ for all $y = z+w \in \mathbb{C}^m = \langle X\rangle \oplus \langle X\rangle^\perp$. This implies $x \in (\mathbb{C}^m)^\perp = \{0\}$, a contradiction. Thus there exists $x \in \langle X\rangle^\perp$ satisfying $(x,x) = {}^txKx \neq 0$. Similarly we see that there exists $y \in \langle Y\rangle^\perp$ satisfying $(y,y) = {}^tyKy \neq 0$. By multiplying a scalar if necessary, we may assume that $(x,x) = (y,y) \neq 0$. Then we have ${}^t\tilde{X}K\tilde{X} = \begin{pmatrix} {}^tXKX & 0 \\ 0 & {}^txKx \end{pmatrix} = \begin{pmatrix} {}^tYKY & 0 \\ 0 & {}^tyKy \end{pmatrix} = {}^t\tilde{Y}K\tilde{Y}$ with rank ${}^t\tilde{X}K\tilde{X} = n+1$. ∎

Proposition 7.3. (cf. Witt's Theorem)

(1) *For $X, Y \in M(m)$ satisfying ${}^tXKX = {}^tYKY$ with* rank ${}^tXKX = m$, *there exists $A \in O(K)$ satisfying $Y = AX$.*

(2) *Assume that $m > n \geq 1$. For $X, Y \in M(m,n)$ satisfying ${}^tXKX = {}^tYKY$ with* rank ${}^tXKX = n$, *there exists $A \in SO(K)$ satisfying $Y = AX$.*

Proof. (1) We have $X, Y \in GL(m)$ and ${}^tXKX = {}^tYKY$. Hence if we put $A = YX^{-1}$, then we have ${}^tAKA = K$, i.e., $A \in O(K)$ satisfying $Y = AX$.

(2) First assume that $n = m-1$. By Lemma 7.2, there exist non-zero elements x and y in $\mathbb{C}^m$ satisfying ${}^t\tilde{X}K\tilde{X} = {}^t\tilde{Y}K\tilde{Y}$ with $\tilde{X}, \tilde{Y} \in GL(m)$ where $\tilde{X} = (X|x)$ and $\tilde{Y} = (Y|y)$. If we put $A = \tilde{Y}\tilde{X}^{-1}$, then we have $A \in O(K)$ with $\tilde{Y} = A\tilde{X}$ and hence $Y = AX$. In the case $A \in O(K) \setminus SO(K)$, take $Y' = (Y| -y)$ instead of $\tilde{Y}$. If we put $A_- = Y'\tilde{X}^{-1}$, then we have $A_- \in SO(K)$ satisfying $Y = A_-X$ since $\det A_- = -\det A$ and ${}^t\tilde{Y}K\tilde{Y} = {}^tY'KY'$. Now assume that the assertion holds for n and we show that the assertion also holds for $n-1$. Assume that $X, Y \in M(m, n-1)$ satisfy ${}^tXKX = {}^tYKY$ with rank ${}^tXKX = n-1$. By Lemma 7.2, there exist non-zero elements $x, y \in \mathbb{C}^m$ satisfying ${}^t\tilde{X}K\tilde{X} = {}^t\tilde{Y}K\tilde{Y}$ with rank ${}^t\tilde{X}K\tilde{X} = n$ where $\tilde{X} = (X|x)$ and $\tilde{Y} = (Y|y)$. By the assumption of induction, there exists $A \in SO(K)$ satisfying $\tilde{Y} = A\tilde{X}$ and hence we have $Y = AX$.

∎

Proposition 7.4. *(1) For $m \geq n$, $(SO(K) \times H, \Lambda_1 \otimes \sigma, M(m,n))$ is a PV if and only if $(H, 2\Lambda_1(\sigma), \mathrm{Sym}(n))$ is a PV.*

(2) Assume that $m > n$. If $P'_1, \ldots, P'_N$ are basic relative invariants of $(H, 2\Lambda_1(\sigma), \mathrm{Sym}(n))$, then $P_1, \ldots, P_N$ are basic relative invariants of $(SO(K) \times H, \Lambda_1 \otimes \sigma, M(m,n))$ where $P_i(X) = P'_i({}^tXKX)$ $(i = 1, \ldots, N)$.

(3) $(SO(K) \times H, \Lambda_1 \otimes \sigma, M(m,n))$ $(m \geq n)$ and $(H, 2\Lambda_1(\sigma), \mathrm{Sym}(n))$ are regular PVs.

Proof. (1) If $m > n \geq 1$, then by (2) of Proposition 7.3, a generic fiber of $\varphi : M(m,n) \to \mathrm{Sym}(n)$ is $SO(K)$-homogeneous. Hence, applying Key Lemma to $\varphi : M(m,n) \to \mathrm{Sym}(n)$ in Lemma 7.1, we have (1) for $m > n$. If $m = n$, then by (1) of Proposition 7.3, a generic fiber of $\varphi : M(m) \to \mathrm{Sym}(m)$ is $O(K)$-homogeneous. Hence, by Key Lemma, $(O(K) \times H, \Lambda_1 \otimes \sigma, M(m))$ is a PV if and only if $(H, 2\Lambda_1(\sigma), \mathrm{Sym}(n))$ is a PV. Since $(O(K) \times H, \Lambda_1 \otimes \sigma, M(m))$ is a PV if and only if $(SO(K) \times H, \Lambda_1 \otimes \sigma, M(m))$ is a PV, we have (1) for $m = n$.

(2) Let $X \in M(m,n)$ be a generic point. By Proposition 1.3, it is enough to show that $\pi((SO(K) \times H)_X) = H_{\varphi(X)}$. By using (2) of Proposition 7.3, we can prove this assertion similarly as (1) of Proposition 6.3.

(3) Since $(SO(K) \times GL(n), \Lambda_1 \otimes \sigma, M(m,n))$ and $(GL(n), 2\Lambda_1(\sigma), \mathrm{Sym}(n))$ are regular PVs, we have our assertion by (3) of Lemma 3.1.

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Remark 7.5. *Note that (2) of Proposition 7.4 does not hold for $n = m$. For example, $(GL(m), 2\Lambda_1, \mathrm{Sym}(m))$ has the basic relative invariant $P'(Z) = \det Z$. However $P(X) = P'({}^tXKX) = (\det K)(\det X)^2$ is not the basic relative invariant of $(SO(K) \times GL(m), \Lambda_1 \otimes \Lambda_1, M(m))$.*

Theorem 7.6. *For $m > n$, a triplet $T = (SO(K) \times G, 1 \otimes \rho + \Lambda_1 \otimes \sigma, V \oplus M(m,n))$ is a PV if and only if a triplet $T' = (G, \rho \oplus 2\Lambda_1(\sigma), V \oplus \mathrm{Sym}(n))$ is a PV. If $P'(v, Z)$ is a relative invariant of T', then $P(v, X) = P'(v, {}^tXKX)$ is a relative invariant of T. Moreover this gives the one to one correspondence of relative invariants.*

Proof. By Propositions 1.5 and 7.4, we obtain the former part. By Theorem 1.13 in [KKS] and Proposition 7.4, we obtain the latter part.

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