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A classification of some prehomogeneous vector spaces related with hypergeometric functions

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Abstract

In this paper, we give the detailed proof of a classification of finite reductive prehomogeneous vector spaces of type $((Sp_{m_1} \times GL_{m_2} \times GL_1) \times GL_n, (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)(m_1 \geq 2, n \geq 4)$ under various restricted scalar multiplications, which are omitted in [KKMOT]. They are related with hypergeometric functions [O].

Introduction

Let G be a connected linear algebraic group, V a finite dimensional vector space (dim $V \geq 1$), and ρ a rational representation of G on V, all defined over the complex number field $\mathbb{C}$. If V has a Zariski-dense G-orbit, we call a triplet (G, ρ, V) a *prehomogeneous vector space* (abbrev. PV). When there is no confusion, we sometimes write (G, ρ) instead of (G, ρ, V). When G is reductive, we call it a reductive PV. For any rational representation $\rho : G \to GL(V)$ with finitely many orbits, (G, ρ, V) must be a PV. Such a PV is called a finite PV (abbrev. FP). We would like to classify all reductive FPs of type $(G \times GL_n, \rho \otimes \Lambda_1)(n \geq 2)$ which are related with hypergeometric functions. All reductive FPs with full scalar multiplications are completely classified in [KKY]. However if we restrict the scalar multiplications, then the difficulty of different type arises, and only the special cases of the restriction of acalar multiplications are studied. In [KKMOT], all reductive FPs of $((G \times GL_1) \times SL_n, (\rho \otimes \Lambda_1) \otimes \Lambda_1, (V(m) \otimes V(1)) \otimes V(n))$ with $n \geq 2$ under various restricted scalar multiplications are completely classified, but the main part of the proof of the most complicated type $((Sp_{m_1} \times GL_{m_2} \times GL_1) \times GL_n, (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ with $m_1 \geq 2$ and $n \geq 4$ are not written in details. In this paper, we give the complete proof for this omitted case. Note that such FPs with $m_1 = 1$ (i.e., $Sp_1 = SL_2$)(resp. $n = 2, 3$) are classified in [Ka] (resp. Theorem 3.11 in [KKMOT]). We denote the representation $(\Lambda_1 \otimes 1 \otimes 1) \oplus (1 \otimes \Lambda_1 \otimes 1) \oplus (1 \otimes 1 \otimes \Lambda_1)$ of $Sp_{m_1} \times GL_{m_2} \times GL_1$ by $\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1$.

In Section 1, we give the preliminaries. In particular, we review some basic facts related with Grassmann variety and the orbits. We also give the orbital decomposition of

$(Sp_m \times GL_n, \Lambda_1 \otimes \Lambda_1)$ and the isotropy subalgebra of each orbit in the convenient form for later use.

In Section 2, we quote Theorems in [KKMOT], by which we classify FPs of type $((Sp_{m_1} \times GL_{m_2} \times GL_1) \times GL_n, (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ with $m_1 \geq 2$ and $n \geq 4$ under various restricted scalar multiplications.

In Section 3, we give the list of finite prehomogeneous vector spaces of type $((Sp_{m_1} \times GL_{m_2} \times GL_1) \times GL_n, (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ with $m_1 \geq 2$ and $n \geq 4$ under various restricted scalar multiplications.

Notation We denote $\mathbb{C}^n$ by $V(n)$. As usual, $\mathbb{C}$ stands for the field of complex numbers. We denote by $e_i^{(n)}$ the i-th fundamental vector in $\mathbb{C}^n$. We often write e_i for simplicity. For positive integers m, n, we denote by $M(m,n)$ the totality of $m \times n$ matrices over $\mathbb{C}$. If $m = n$, we simply write $M(n)$ instead of $M(n,n)$. We also use the notations $M(m,n)' = \{X \in M(m,n) \mid \operatorname{rank} X = \min\{m,n\}\}$ and $M(m,n)'' = \{X \in M(m,n) \mid \operatorname{rank} X < \min\{m,n\}\}$. For $r < n$, we put $M^r_{m,n} = \{(X|O) \in M(m,n) \mid X \in M(m,r)\}$. We denote by I_n (or $I(n)$) the identity matrix of size n. We denote by ${}^tA$ the transposed matrix of a matrix A. Two triplets are called isomorphic and denoted by $(G,\rho,V) \cong (G',\rho',V')$ if there exits a group isomorphism $\sigma : \rho(G) \to \rho'(G)$ and an isomorphism $\tau : V \to V'$ of vector spaces satisfying $\tau(\rho(g)(v)) = (\sigma\rho(g))\tau(v)$ for all $g \in G$ and $v \in V$.

We denote by GL_n (resp. $SL_n, SO_n, Spin_n, Sp_n, (G_2), E_6, E_7$) the general linear group $\{X \in M(n) | \det X \neq 0\}$ (resp. the special linear group $\{X \in GL_n | \det X = 1\}$, the special orthogonal group $\{X \in SL_n | \ {}^tXX = I_n\}$, the spin group, the symplectic group $\{X \in GL_{2n} | \ {}^tXJ_nX = J_n\}$ where $J_n = \left(\begin{smallmatrix} O & I_n \\ -I_n & O \end{smallmatrix}\right)$, exceptional algebraic groups $(G_2), E_6, E_7$). When the expression of n is complicated, we also write $GL(n)$ instead of GL_n etc. Further we denote by GSp_n the general symplectic group $\{X \in GL_{2n} \mid {}^tXJ_nX = xJ_n \text{with } x \in GL_1\} = \{\alpha A \mid \alpha \in GL_1, A \in Sp_n\} \cong (GL_1 \times Sp_n)/\{(1, I_{2n}), (-1, -I_{2n})\}$. We denote by $T_u(n)$ the group of all nonsingular upper matrices and put $ST_u(n) = T_u(n) \cap SL_n$. Then we write $H_{n,q} = \{\left(\begin{smallmatrix} A & C \\ O & B \end{smallmatrix}\right) \in GL_n \mid A \in Sp_q, B \in T_u(n-2q), C \in M(2q, n-2q)\}$ and $SH_{n,q} = SL_n \cap H_{n,q}$ with $2q \leq n$.

We denote by Λ_1 the standard representation of GL_n on $V(n)$. For a subgroup H of GL_n, the restriction $\Lambda_1|_H$ is also simply denoted by Λ_1. More generally, Λ_k $(k = 1, \ldots, r)$ denotes the fundamental irreducible representation of a simple algebraic group of rank r. We have $(GSp_n, \Lambda_1) \cong (GL_1 \times Sp_m, \Lambda_1 \otimes \Lambda_1)$. In general, we denote by ρ^* the dual representation of a rational representation ρ. It is known that (H,σ,V) is a FP if and only if (H,σ^*,V^*) is a FP for any algebraic group H, not necessarily reductive (see [P]). Hence $(G, \rho_1^{(*)} \oplus \cdots \oplus \rho_l^{(*)})$ is a FP if and only if $(G, \rho_1 \oplus \cdots \oplus \rho_l)$ is a FP where $\rho^{(*)}$ implies ρ or its dual ρ^*. Also if G_1 and G_2 are reductive, then we have $(G_1 \times G_2, \rho_1^{(*)} \otimes \rho_2^{(*)}) \cong (G_1 \times G_2, \rho_1 \otimes \rho_2)$. Using these facts and by the form of FPs (see [KKY]), it is not necessary to consider the dual representation as far as we deal with FPs. For a representation $\rho : G \to GL(V)$ and a point v of V, we denote by G_v the isotropy subgroup $\{g \in G \mid \rho(g)v = v\}$ at v.

1 Preliminaries

Proposition 1.1. ([KKMOT, Proposition 1.1]) *Assume that* $(H \times GL_n, \rho \otimes \Lambda_1)$ *is a FP. Then* $(H \times SL_n, \rho \otimes \Lambda_1)$ *is also a FP if and only if the* GL_n*-part of the connected component of the isotropy subgroup of each orbit is not contained in* SL_n. *In this case, they have the same orbits.*

Proposition 1.2. ([KKMOT, Proposition 1.2]) *Let* $\sigma : H \to GL_m$ *be a representation of an algebraic group* H.

1. *If* $m < n$, *then* $(H \times SL_n, \sigma \otimes \Lambda_1, M(m,n))$ *is a FP if and only if* $(H \times GL_n, \sigma \otimes \Lambda_1, M(m,n))$ *is a FP. In this case, they have the same orbits.*
2. *If* $m \geq n$ *and the number of orbits of* $H \times SL_n$ *in* $M(m,n)'$ *is finite, then* $(H \times SL_n, \sigma \otimes \Lambda_1, M(m,n))$ *is a FP if and only if* $(H \times GL_n, \sigma \otimes \Lambda_1, M(m,n))$ *is a FP. In this case, they have the same orbits.*

Next we shall review the relation between the Grassmann variety and finite prehomogeneity ([SK, Section 8]).

Definition 1.3. Let V be an m-dimensional vector space. For any n satisfying $m \geq n \geq 0$, $Grass_n(V) = \{W | W \text{ is an n-dimensional subspace of } V\}$ is an $n(m-n)$-dimensional variety which is called the Grassmann variety.

Then the following assertion holds.

Proposition 1.4. ([SK, Proposition 1 in Section 8]) (Correspondence of orbits). *Let* G *be any algebraic group. For* $m \geq n \geq 1$, *and for any representation* $\rho : G \to GL_m$, *consider a triplet* $(G \times GL_n, \rho \otimes \Lambda_1, M(m,n))$ *and a triplet* $(G, \rho, \cup_{k=0}^{n} Grass_k(V(m)))$ *without assuming the prehomogeneity. Then* $G \times GL_n$*-orbits in* $M(m,n)$ *correspond bijectively to* G*-orbits in* $\cup_{k=0}^{n} Grass_k(V(m))$.

In particular, when we assume a number of $G \times GL_n$-orbits on $M(m,n)$ is finite, also a number of G-orbits on $\cup_{k=0}^{n} Grass_k(V(m))$ is finite. Moreover for any t satisfying $n > t \geq 1$, a number of G-orbits on $\cup_{k=0}^{t} Grass_k(V(m))$ is finite. Therefore a number of $G \times GL_t$-orbits on $M(m,t)$ is finite. In general, if an irreducible algebraic variety W is decomposed into finitely many orbits by the action of a algebraic group H, W has a Zarisaki dense H-orbit. Hence the following Lemma is obtained, which is fundamental for a classification of FPs.

Lemma 1.5. ([KKMOT, Lemma 1.3]) *Let G be any algebraic group, not necessarily reductive, and ρ its representation, not necessarily irreducible.*

1. *For $m > n \geq 2$, if $(G \times GL_n, \rho \otimes \Lambda_1, V(m) \otimes V(n))$ is a FP, then a triplet $(G \times GL_k, \rho \otimes \Lambda_1, V(m) \otimes V(k))$ is also a FP for any k satisfying $n \geq k \geq 1$.*
2. *For $n \geq m \geq 2$, if $(G \times GL_n, \rho \otimes \Lambda_1, V(m) \otimes V(n))$ is a FP, then a triplet $(G \times GL_k, \rho \otimes \Lambda_1, V(m) \otimes V(k))$ is also a FP for any k.*

Remark 1.6. (Castling transform) ([SK, Proposition 7 in section 2]) Let ρ be a representation of an algebraic group H on an m-dimensional vector space V. For any n satisfying $m > n \geq 1$, the following conditions are equivalent.

1. $(H \times GL_n, \rho \otimes \Lambda_1, V \otimes V(n))$ is a PV.
2. $(H \times GL_{m-n}, \rho^* \otimes \Lambda_1, V \otimes V(m-n))$ is a PV.
3. $(H \times GL_{m-n}, \rho \otimes \Lambda_1, V \otimes V(m-n))$ is a PV if H is reductive.

We say the triplets 1, 2 (resp. 1, 3 if H is reductive) in Remark 1.6 are castling transforms of each other. This castling transformation is essential for the classification of irreducible PVs. However, in general, a castling transform of a FP is not necessarily a FP although it is a PV. For example, a castling transform $(SL_2 \times GL_3, 3\Lambda_1 \otimes \Lambda_1, V(4) \otimes V(3))$ of a FP $(GL_2, 3\Lambda_1, V(4))$ is a PV, but it is not a FP. If it is a FP, then by 1 of Lemma 1.5, $(SL_2 \times GL_2, 3\Lambda_1 \otimes \Lambda_1, V(4) \otimes V(2))$ must be a PV, which is a contradiction by dimension reason.

Proposition 1.7. ([KKMOT, Proposition 1.4]) *If $(G \times GL_n, \rho \otimes \Lambda_1)$ with $n \geq 2$ is a FP, then we have $\rho = \rho_1 + \cdots + \rho_k$ with $k = 1, 2, 3$ where $\rho_1, \ldots, \rho_k$ are irreducible representations.*

Here we review the symplectic group Sp_m. The action Λ_1 of Sp_m on $V(2m)$ is given by $x \mapsto Ax$ ($A \in Sp_m$, $x \in V(2m)$) which satisfies $\langle Ax, Ay \rangle = \langle x, y \rangle$ where $\langle x, y \rangle = {}^t x J y$. Note that this condition is equivalent to $A \in Sp_m$.

Lemma 1.8. ([K, Lemma 7.49]) *Let $v_1, \ldots, v_r$ and $u_1, \ldots, u_r$ be linearly independent elements of $V(2m)$ satisfying $\langle v_i, v_j \rangle = \langle u_i, u_j \rangle$ for $i, j = 1, \ldots, r$. Then there exists $A \in Sp_m$ satisfying $u_i = Av_i$ $(i = 1, \ldots, r)$.*

Now consider the action $\Lambda_1 \otimes \Lambda_1$ of $Sp_m \times GL_n$ on $M(2m, n)$ given by $X \mapsto AX^tB$ for $(A, B) \in Sp_m \times GL_n$ and $X \in M(2m, n)$. Note that this is essentially the same as the action $\Lambda_1 \otimes \Lambda_1$ of $GSp_m \times SL_n$ on $M(2m, n)$ given by $X \mapsto AX^tB$ for $(A, B) \in GSp_m \times SL_n$ and $X \in M(2m, n)$. It is clear that $\operatorname{rank} X$ is invariant under the action of the group. Since ${}^tXJX \mapsto {}^t(AX^tB)J(AX^tB) = B({}^tXJX)^tB$, $\operatorname{rank}({}^tXJX)$ is also invariant. Since ${}^tXJX$ is an alternating matrix, its rank is always even. The condition $(\operatorname{rank} X, \operatorname{rank} {}^tXJX) \neq (\operatorname{rank} Y, \operatorname{rank} {}^tYJY)$ implies that X and Y do not belong to the same orbit. We shall show the converse.

Proposition 1.9. ([KKMOT, Proposition 1.5]) (The orbital decomposition of $(Sp_m \times GL_n, \Lambda_1 \otimes \Lambda_1)$) *If $X, Y \in M(2m,n)$ satisfy $\operatorname{rank} X = \operatorname{rank} Y$ and $\operatorname{rank} {}^tXJX = \operatorname{rank} {}^tYJY$, then we have $Y = AX{}^tB$ for some $(A,B) \in Sp_m \times GL_n$. Hence the orbits of $(Sp_m \times GL_n, \Lambda_1 \otimes \Lambda_1, M(2m,n))$ are given by*

$$O_{p,q} = \{X \in M(2m,n) \mid \operatorname{rank} X = p+q,\ \operatorname{rank} {}^tXJX = 2q\}$$

with $m \geq p \geq q \geq 0$ and $n \geq p+q$. The orbit $O_{p,q}$ is represented by $X_{p,q} = \begin{pmatrix} I'_p & O & O \\ O & I'_q & O \end{pmatrix} \in M(2m,n)$ where $I'_p = \begin{pmatrix} I_p \\ O \end{pmatrix} \in M(m,p)$ and $I'_q = \begin{pmatrix} I_q \\ O \end{pmatrix} \in M(m,q)$.

Now we shall calculate the isotropy subalgebra at $X_{p,q}$. The Lie algebra of Sp_m is given by $Lie(Sp_m) = \{\left(\begin{smallmatrix} A & B \\ C & -{}^tA \end{smallmatrix}\right) \mid A \in M(m), B, C \in Sym(m)\}$. We divide this matrix to the block size $(q, p-q, m-p, q, p-q, m-p)$ as follows:

$$\tilde{A} = \begin{pmatrix} A_1 & A_{12} & A_{13} & B_1 & B_{12} & B_{13} \\ A_{21} & A_2 & A_{23} & {}^tB_{12} & B_2 & B_{23} \\ A_{31} & A_{32} & A_3 & {}^tB_{13} & {}^tB_{23} & B_3 \\ C_1 & C_{12} & C_{13} & -{}^tA_1 & -{}^tA_{21} & -{}^tA_{31} \\ {}^tC_{12} & C_2 & C_{23} & -{}^tA_{12} & -{}^tA_2 & -{}^tA_{32} \\ {}^tC_{13} & {}^tC_{23} & C_3 & -{}^tA_{13} & -{}^tA_{23} & -{}^tA_3 \end{pmatrix} \in Lie(Sp_m).$$

Similarly we divide $X_{p,q}$ to the block size $(q, p-q, m-p, q, p-q, m-p) \times (q, p-q, q, n-p-q)$ and also divide $D \in Lie(GL_n)(= M(n))$ to the block size $(q, p-q, q, n-p-q)$ as follows:

$$X_{p,q} = \begin{pmatrix} I_q & O & O & O \\ O & I_{p-q} & O & O \\ O & O & O & O \\ O & O & I_q & O \\ O & O & O & O \\ O & O & O & O \end{pmatrix} \in M(2m,n), D = \begin{pmatrix} D_1 & D_{12} & D_{13} & D_{14} \\ D_{21} & D_2 & D_{23} & D_{24} \\ D_{31} & D_{32} & D_3 & D_{34} \\ D_{41} & D_{42} & D_{43} & D_4 \end{pmatrix} \in Lie(GL_n).$$

Then $\tilde{A}X_{p,q} + X_{p,q}({}^tD) = O$ if and only if $(\tilde{A}, D) =$

$$\left(\begin{pmatrix} A_1 & O & O & B_1 & B_{12} & O \\ A_{21} & A_2 & A_{23} & {}^tB_{12} & B_2 & B_{23} \\ O & O & A_3 & O & {}^tB_{23} & B_3 \\ C_1 & O & O & -{}^tA_1 & -{}^tA_{21} & O \\ O & O & O & O & -{}^tA_2 & O \\ O & O & C_3 & O & -{}^tA_{23} & -{}^tA_3 \end{pmatrix}, \begin{pmatrix} -{}^tA_1 & -{}^tA_{21} & -{}^tC_1 & D_{14} \\ O & -{}^tA_2 & O & D_{24} \\ -{}^tB_1 & -B_{12} & A_1 & D_{34} \\ O & O & O & D_4 \end{pmatrix}\right).$$

By changing the rows and columns from $(1, \dots, 6)$ to $(2, 1, 4, 3, 6, 5)$ and from $(1, 2, 3, 4)$ to $(1, 3, 2, 4)$, we obtain the following result.

Proposition 1.10. (cf.[KKMOT, Proposition 1.6]) *The isotropy subalgebra of $(Sp_m \times GL_n, \Lambda_1 \otimes \Lambda_1)$ at $X_{p,q} \in M(2m, n)(m \geq p \geq q \geq 0, n \geq p + q)$ is isomorphic to $\mathfrak{g}_{p,q} = \{(A, D)\}$ where*

$$A = \begin{pmatrix} A_2 & A_{21} & {}^tB_{12} & A_{23} & B_{23} & B_2 \\ O & A_1 & B_1 & O & O & B_{12} \\ O & C_1 & -{}^tA_1 & O & O & -{}^tA_{21} \\ O & O & O & A_3 & B_3 & {}^tB_{23} \\ O & O & O & C_3 & -{}^tA_3 & -{}^tA_{23} \\ O & O & O & O & O & -{}^tA_2 \end{pmatrix}, D = \begin{pmatrix} -{}^tA_1 & -{}^tC_1 & -{}^tA_{21} & D_{14} \\ -{}^tB_1 & A_1 & -B_{12} & D_{34} \\ O & O & -{}^tA_2 & D_{24} \\ O & O & O & D_4 \end{pmatrix}.$$

with the block size $(p - q, q, q, m - p, m - p, p - q) \times (q, q, p - q, n - p - q)$. Hence the isotropy subgroup $G_{p,q}$ at $X_{p,q}$ is locally isomorphic to

$$(GL(p - q) \times GL(n - p - q) \times Sp_q \times Sp(m - p)) \cdot U(k)$$

where $k = (p - q)(2m - 2p + 2q) + \frac{1}{2}(p - q)(p - q + 1) + (p + q)(n - p - q)$.

Similarly the isotropy subalgebra of $(GSp_m \times SL_n, \Lambda_1 \otimes \Lambda_1, M(2m, n))$ at $X_{p,q}$ $(m \geq p \geq q \geq 0, n \geq p + q)$ is isomorphic to $\mathfrak{g}'_{p,q} = \{(A', D')\}$ where $A' = \alpha I_{2m} + A, D' =$
$$\begin{pmatrix} -\alpha -{}^t A_1 & -{}^tC_1 & -{}^tA_{21} & D_{14} \\ -{}^tB_1 & -\alpha + A_1 & -B_{12} & D_{34} \\ O & O & -\alpha -{}^t A_2 & D_{24} \\ O & O & O & D_4 \end{pmatrix}$$
with $\alpha = \frac{1}{p+q}(\operatorname{tr} D_4 - \operatorname{tr} A_2)$.

Here we put $H_{n,q} = \begin{pmatrix} Sp_q & M(2q, n - 2q) \\ O & T_u(n - 2q) \end{pmatrix}, H^*_{n,m} = \begin{pmatrix} Sp_m & M(2m, n - 2m) \\ O & GL(n - 2m) \end{pmatrix}$,
$SH_{n,q} = \begin{pmatrix} Sp_q & M(2q, n - 2q) \\ O & ST_u(n - 2q) \end{pmatrix}, SH^*_{n,m} = \begin{pmatrix} Sp_m & M(2m, n - 2m) \\ O & SL(n - 2m) \end{pmatrix}$,
$H'_{n,q} = SL_n \cap \begin{pmatrix} GSp_q & M(2q, n - 2q) \\ O & T_u(n - 2q) \end{pmatrix}$, and $(H'_{n,m})^* = SL_n \cap \begin{pmatrix} GSp_m & M(2m, n - 2m) \\ O & GL(n - 2m) \end{pmatrix}$.

Proposition 1.11. ([KKMOT, Proposition 1.7])

1. *The GL_n (resp. SL_n)-part of an isotropy subgroup of $(Sp_m \times GL_n, \Lambda_1 \otimes \Lambda_1)$(resp. $(Sp_m \times SL_n, \Lambda_1 \otimes \Lambda_1)$) of any orbit contains a subgroup isomorphic to $H_{n,q}$(resp. $SH_{n,q}$) for some q satisfying $m \geq q$ and $n \geq 2q \geq 0$. If $n > 2m = 2q$, we can replace $H_{n,m}$(resp. $SH_{n,m}$) by $H^*_{n,m}$(resp. $SH^*_{n,m}$).*

2. *The $(GL_1 \times SL_n)$-part of an isotropy subgroup of $(GL_1 \times Sp_m \times SL_n, \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1)$ contains a subgroup isomorphic to $\{(\alpha, (\begin{smallmatrix} \alpha^{-1}A & C \\ O & B \end{smallmatrix})) \mid \alpha \in GL_1, A \in Sp_q, B \in T_u(n-2q), \det B = \alpha^{2q}, C \in M(2q, n-2q)\}$ for some q satisfying $m \geq q$ and $n > 2q > 0$.*

3. *The SL_n-part of an isotropy subgroup of $(GSp_m \times SL_n, \Lambda_1 \otimes \Lambda_1)$ contains $H'_{n,q}$. If $n > 2m = 2q$, we can replace $H'_{n,m}$ by $(H'_{n,m})^*$.*

2 A classification

In this section, we classify FPs of type $((Sp_{m_1} \times GL_{m_2} \times GL_1) \times GL_n, (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ with $m_1 \geq 2$ and $n \geq 4$ under various restricted scalar multiplications. In the following Theorem 2.1 to Theorem 2.3, we gather the known results which we will use for our classification.

Theorem 2.1. ([Kac, Theorem 2; SK, Section 8])

1. *$(SL_m \times GL_n, \Lambda_1 \otimes \Lambda_1, V(m) \otimes V(n))$ with $m \geq 1$ and $n \geq 2$,*

2. *$(SL_m \times SL_n, \Lambda_1 \otimes \Lambda_1, V(m) \otimes V(n))$ with $m \neq n$ and $n \geq 2$,*

3. *$(Sp_m \times GL_n, \Lambda_1 \otimes \Lambda_1)$ is a FP if and only if $m \geq 1$ and $n \geq 1$.*

4. *$(Sp_m \times SL_n, \Lambda_1 \otimes \Lambda_1)$ is a FP if and only if $2m < n$ or $n =$ odd (≥ 1).*

Theorem 2.2. ([KKY])

1. *$((GL_{m_1} \times GL_{m_2}) \times GL_n, (\Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ is a FP if and only if $m_1 \geq 1$ and $n \geq 1$.*

2. *$((Sp_{m_1} \times GL_{m_2}) \times GL_n, (\Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ is a FP if and only if $m_1 \geq 1$ and $n \geq 1$.*

Theorem 2.3. ([KKMOT, Theorem 2.3])

1. *$((SL_{m_1} \times GL_{m_2}) \times SL_n, (\Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ $(n \geq 2)$ is a FP if and only if $m_1 \neq n$.*

2. *$((SL_{m_1} \times SL_{m_2}) \times GL_n, (\Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ $(n \geq 2)$ is a FP if and only if $m_1 \neq m_2$ or $m_1 = m_2 > n$.*

3. $((SL_{m_1} \times SL_{m_2}) \times SL_n, (\Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ $(n \geq 2)$ *is a FP if and only if* $(n \neq m_1, n \neq m_2, n \neq m_1 + m_2\ , m_1 \neq m_2)$ *or with* $m_1 = m_2 > n$.

4. $((GSp_{m_1} \times SL_{m_2}) \times SL_n, (\Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ $(m_1 \geq 2,\ n \geq 2)$ *is a FP if and only if* $m_2 > n$ *or* $n = odd > m_2$ *or* $n > m_2 = odd$ *or* $n > \max\{2m_1, m_2\}$.

5. $((Sp_{m_1} \times GL_{m_2}) \times SL_n, (\Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ $(m_1 \geq 2,\ n \geq 2)$ *is a FP if and only if* $n > 2m_1$ *or* $n = odd$.

6. $((Sp_{m_1} \times SL_{m_2}) \times GL_n, (\Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ $(m_1 \geq 2,\ n \geq 2)$ *is a FP if and only if* $m_2 > n$ *or* $m_2 > 2m_1$ *or* $m_2 = odd$.

7. $((Sp_{m_1} \times SL_{m_2}) \times SL_n, (\Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ $(m_1 \geq 2,\ n \geq 2)$ *is a FP if and only if one of the following conditions holds.*
   (a) $m_2 > n > 2m_1$ *or* $m_2 > n = odd$,
   (b) $n > 2m_1 + m_2$ *and* $(m_2 > 2m_1$ *or* $m_2 = odd)$,
   (c) $2m_1 + m_2 > n > m_2$, $(m_2 > 2m_1$ *or* $m_2 = odd)$, $n > 2m_1 + 1$ *and* $n \not\equiv m_2 \mod 2$.

Here we put $S(i_1, \ldots, i_t) = \sum_{k=1}^{t} E_{i_k,k} \in M(n,m)$ $(n \geq i_1 > \cdots > i_t \geq 1)$ where $E_{i,j}$ denotes the matrix unit in $M(n,m)$. We also write $S(i_1, \ldots, i_t)' = \sum_{k=1}^{t} E'_{i_k,k} \in M(n,t)$ $(n \geq i_1 > \cdots > i_t \geq 1)$ where $E'_{i,j}$ denotes the matrix unit in $M(n,t)$. Hence we have $S(i_1, \ldots, i_t) = (S(i_1, \ldots, i_t)' \mid O) \in M(n,m)$.

Lemma 2.4. ([KKMOT, Lemma 2.4])

1. *For any* q *and* m, $(Sp_q \times GL_m, \Lambda_1 \otimes \Lambda_1) \cong (GSp_q \times SL_m, \Lambda_1 \otimes \Lambda_1)$ *is a FP while* $(Sp_q \times SL_m, \Lambda_1 \otimes \Lambda_1)$ *is a FP if and only if* $2q < m$ *or* $m = odd$.

2. *For any* m *and* n, $(ST_u(n) \times GL_m, \Lambda_1 \otimes \Lambda_1, M(n,m)) \cong (T_u(n) \times SL_m, \Lambda_1 \otimes \Lambda_1, M(n,m)) \cong (T_u(n) \times GL_m, \Lambda_1 \otimes \Lambda_1, , M(n,m))$ *is a FP with the orbits represented by* $S(i_1, \ldots, i_t) \in M(n,m)$ $(n \geq i_1 > \cdots > i_t \geq 1)$.

3. *If* $m \neq n$, *then a triplet* $(ST_u(n) \times SL_m, \Lambda_1 \otimes \Lambda_1, M(n,m))$ *is also a FP with the orbits represented by* $S(i_1, \ldots, i_t) \in M(n,m)$ $(n \geq i_1 > \cdots > i_t \geq 1)$.

4. *For any* m, n *and* q *with* $n > 2q > 0$, *a triplet* $(SH_{n,q} \times GL_m, \Lambda_1 \otimes \Lambda_1, M(n,m))$ *is a FP where* $SH_{n,q} = \begin{pmatrix} Sp_q & M(2q, n-2q) \\ O & ST_u(n-2q) \end{pmatrix}$.

5. *For any* m, n *and* q *with* $n > 2q > 0$ *where* $2q < m$ *or* $m = odd$, *a triplet* $(H_{n,q} \times SL_m, \Lambda_1 \otimes \Lambda_1, M(n,m))$ *is a FP where* $H_{n,q} = \begin{pmatrix} Sp_q\, M(2q, n-2q) \\ O \quad T_u(n-2q) \end{pmatrix}$.

6. *For any m, n and q with $n > 2q > 0$ and $n \neq m$, a triplet* $(H'_{n,q} \times SL_m, \Lambda_1 \otimes \Lambda_1, M(n,m))$ *is a FP where* $H'_{n,q} = SL_n \cap \begin{pmatrix} GSp_q & M(2q, n-2q) \\ O & T_u(n-2q) \end{pmatrix}$.
7. *For m, n and q with $n > 2q > 0$ and $n \not\equiv m \mod 2$ where $2q < m$ or $m = odd$, a triplet* $(SH_{n,q} \times SL_m, \Lambda_1 \otimes \Lambda_1, M(n,m))$ *is a FP.*

Theorem 2.5. ([KKY])
$((Sp_{m_1} \times GL_{m_2} \times GL_1) \times GL_n, (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ *is a FP if and only if* $m_1 \geq 1$ *and* $n \geq 1$.

When we classify FPs of type $((Sp_{m_1} \times GL_{m_2} \times GL_1) \times GL_n, (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ with $m_1 \geq 2$ and $n \geq 4$ under various restricted scalar multiplications, the following lemmas are essential.

Lemma 2.6. ([KKMOT, Lemma 3.3])

1. $(Sp_m \times (GL_n \times SL_1), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1))$
$(\cong (Sp_m \times GL_n, \Lambda_1 \otimes \Lambda_1 + \Lambda_1 \otimes 1))$ *is a FP.*
2. $(Sp_m \times (SL_n \times GL_1), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1))$ *is a FP if and only if $n > 2m$ or $n = odd$. More generally, let S_k be a subgroup of $GSp_m \times (SL_n \times GL_1)$ defined by* $S_k = \{(A, B, \alpha) \mid \alpha \in GL_1, A \in GSp_m, \det A = \alpha^k, B \in SL_n\}$. *Then* $(S_k, \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1))$, *i.e.,* $M(2m, n) \oplus V(2m) \ni (X, y) \mapsto (AX^tB, \alpha Ay) = (\alpha^{k/2m} A'X^tB, \alpha^{(2m+k)/2m} A'y)$ *with $(A, B, \alpha) \in S_k$ and $A' \in SP_m$, is a FP if and only if* ($n = 1$; $k \neq -m$) *or* ($2m \geq n = even$; $k \neq 0$) *or* ($2m > n = odd \geq 3$; $k \neq 2m/(n-1), -2m/(n+1)$) *or* $n > 2m$.
3. $(GSp_m \times (SL_n \times SL_1), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1))$ *is a FP if and only if* $n \geq 2$.
4. $(Sp_m \times (SL_n \times SL_1), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1))$ *is a FP if and only if* $n > 2m$.

Lemma 2.7. ([KKMOT, Lemma 3.4])

1. $(T_u(m) \times (GL_n \times GL_1), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1))$
$\cong (ST_u(m) \times (GL_n \times GL_1), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1))$
$\cong (T_u(m) \times (GL_n \times SL_1), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1))$
$\cong (T_u(m) \times (SL_n \times GL_1), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1))$ *is a FP.*
2. $(T_u(m) \times (SL_n \times SL_1), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1))$ *is a FP if and only if* $n \geq 2$.
3. $(ST_u(m) \times (GL_n \times SL_1), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1))$ *is a FP if and only if* $m \geq 3$.
4. $(ST_u(m) \times (SL_n \times SL_1), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1))$ *is a FP if and only if* $m \geq 3, n \geq 2, m \neq n$ *and* $m \neq n + 1$.

5. $(ST_u(m) \times (SL_n \times GL_1), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1))$ *is a FP if and only if* $(n = 1, m \geq 3)$ *or* $(n \geq 2, m \neq n)$. *More generally, let* G_r *be a subgroup of* $T_u(m) \times (SL_n \times GL_1)$ *defined by* $G_r = \{(A, B, \alpha) \mid \alpha \in GL_1, A \in T_u(m), \det A = \alpha^r, B \in SL_n\}$. *Then* $(G_r, \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1))$, *i.e.,* $M(m,n) \oplus V(m) \ni (X, y) \mapsto (AX^tB, \alpha Ay)$ *with* $\det \alpha A = \alpha^{m+r}$ *and* $(A, B, \alpha) \in G_r$, *is a FP if and only if* $(n = 1, m \geq 3)$ *or* $(n \geq 2,\ r \neq 0, -1, -m)$ *or* $(n \geq 2, r = 0;\ m \neq n)$ *or* $(n \geq 2, r = -1;\ m \neq n + 1)$ *or* $(n \geq 2, r = -m;\ m \geq 3)$.

Lemma 2.8. ([KKMOT, Lemma 3.5])

1. $(H'_{n,q} \times (GL_m \times SL_1), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1))$ $(n \geq 2q \geq 0)$ *is a FP.*
2. $(H'_{n,q} \times (SL_m \times GL_1), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1))$ $(n > 2q > 0)$ *is a FP if* $n > m$.
3. $(H'_{n,q} \times (SL_m \times SL_1), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1))$ *is a FP if* $n > m + 2 \geq 5$ *and* $n > n - 2q \geq 3$.

Proposition 2.9. $((GSp_{m_1} \times GL_{m_2} \times SL_1) \times SL_n, (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ $(m_1 \geq 2,\ n \geq 4)$ *is a FP.*

Proof. By Proposition 1.11, the SL_n-part of an isotropy subgroup of $(GSp_{m_1} \times SL_n, \Lambda_1 \otimes \Lambda_1)$ contains $H'_{n,q}$. Hence by 1 of Lemma 2.8, we have our result. ∎

Proposition 2.10. $((GSp_{m_1} \times SL_{m_2} \times GL_1) \times SL_n, (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ $(m_1 \geq 2,\ n \geq 4)$ *is a FP if and only if* $m_2 > n$ *or* $n = odd > m_2$ *or* $n > m_2 = odd$ *or* $n > \max\{2m_1, m_2\}$.

Proof. By 4 of Theorem 2.3, these conditions are necessary. If $m_2 > n$, then it is a FP by Proposition 1.2 and Theorem 2.5. So we may assume that $n > m_2$. By Proposition 1.10, the SL_n-part H of an isotropy subgroup of $(GSp_{m_1} \times SL_n, \Lambda_1 \otimes \Lambda_1)$ of any orbit contains $Sp_{n'}$ $(2m_1 \geq n = 2n')$ or $ST_u(n)$ or $H'_{n,q}$ $(n > 2q > 0)$. By 2 of Lemma 2.6, $(Sp_{n'} \times (SL_{m_2} \times GL_1), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1))$ is a FP if and only if $(n >)\ m_2 =$ odd. By 5 of Lemma 2.7, $(ST_u(n) \times (SL_{m_2} \times GL_1), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1))$ is a FP in our case. By 2 of Lemma 2.8, $(H'_{n,q} \times (SL_{m_2} \times GL_1), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1), M(n, m_2) \oplus V(n))$ is a FP for $n > m_2$. Hence we obtain our result. ∎

Proposition 2.11. $((GSp_{m_1} \times SL_{m_2} \times SL_1) \times SL_n, (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ $(m_1 \geq 2,\ n \geq 4)$ *is a FP if and only if* $m_2 > n$ *or* $n > \max\{2m_1 + 1, m_2 + 1 (\geq 3)\}$.

Proof. By 3 of Theorem 2.3, $((SL_{m_2} \times SL_1) \times SL_n, (\Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ is a FP if and only if $n \neq 1, n \neq m_2, n \neq m_2 + 1$ and $m_2 \neq 1$. Note that we deal with the case $n \geq 4$. If $m_2 > n$, then it is a FP by Propositions 1.2 and 2.9. So we assume that $n > m_2 + 1 \geq 3$. If $2m_1 \geq n =$ even $(= 2n')$, it is a non FP since $(Sp_{n'} \times (SL_{m_2} \times SL_1), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1))$ with $n = 2n' > m_2 + 1$ is a non FP by Lemma 2.6. Now we show that it is a non FP when $2m_1 + 1 \geq n =$ odd. If we put $n = 2q + 1$, the SL_n-part of a generic isotropy subgroup of $(GSp_{m_1} \times SL_n, \Lambda_1 \otimes \Lambda_1)$ is $H = \{\begin{pmatrix} \alpha A & * \\ O & \alpha^{-2q} \end{pmatrix} \mid \alpha \in GL_1, A \in Sp_q\}$. We show that $(H\times(SL_{m_2}\times SL_1),\Lambda_1\otimes(\Lambda_1\boxplus\Lambda_1))$ is a non FP. First assume that $m_2=$ even. If $((\begin{smallmatrix}X\\O\end{smallmatrix}),(\begin{smallmatrix}0\\1\end{smallmatrix}))\in M(n,m_2)\oplus V(n)$ is transfered to $((\begin{smallmatrix}X'\\O\end{smallmatrix}),(\begin{smallmatrix}0\\1\end{smallmatrix}))$ by $H\times(SL_{m_2}\times SL_1)$, the action $X\mapsto X'$ is $(Sp_q\times SL(m_2),\Lambda_1\otimes\Lambda_1)$ which is a non FP. When $m_2=$ odd, we consider similarly an element of type $((\begin{smallmatrix}X&0\\0&1\end{smallmatrix}),(\begin{smallmatrix}0\\1\end{smallmatrix}))$ with $X\in M(n-1,m_2-1)$, then the action $X\mapsto X'$ is $(Sp_q\times SL(m_2-1),\Lambda_1\otimes\Lambda_1)$ which is a non FP. If $n=m_2+2$ (resp. $m_2=2$), see Lemma 2.12 (resp. Lemma 2.13). Hence we may assume $n>\max\{2m_1+1,m_2+2\}$ with $m_2\geq 3$. In this case, the SL_n-part of an isotropy subgroup of $(GSp_{m_1}\times SL_n,\Lambda_1\otimes\Lambda_1)$ of any orbit contains $ST_u(n)$ or $H'_{n,q}$ $(n>n-2q\geq 2)$ by Proposition 1.11. By 4 of Lemma 2.7, $(ST_u(n)\times(SL_{m_2}\times SL_1),\Lambda_1\otimes(\Lambda_1\boxplus\Lambda_1))$ is a FP in our case. If $n-2q\geq 3$, $(H'_{n,q}\times SL_{m_2},\Lambda_1\otimes\Lambda_1+\Lambda_1\otimes 1,M(n,m_2)\oplus V(n))$ is a FP by Lemma 2.8. If $n-2q=2$, we have $q=m_1$ and we can replace $H'_{n,q}$ by H''_{n,m_1} by Proposition 1.11 and hence it is a FP. ∎

Lemma 2.12. *$((GSp_{m_1}\times SL_{m_2}\times SL_1)\times SL(m_2+2),(\Lambda_1\boxplus\Lambda_1\boxplus\Lambda_1)\otimes\Lambda_1)$ with $m_2\geq 2m_1$, is a FP.*

Proof. The process of the proof is similar as that of Proposition 2.11. It is enough to show that $(H'_{n,q}\times(SL_{m_2}\times SL_1),\Lambda_1\otimes(\Lambda_1\boxplus\Lambda_1))$ is a FP when $m_2>t>0$ and $2q=m_2-t+1$ since other cases are proved in Lemma 2.8. The number of orbits related with $M(m_2+2,m_2)''$ is finite by Proposition 1.2. Any point in $M(m_2+2,m_2)'$ is $H'_{n,q}\times SL_{m_2}$-equivalent to $\begin{pmatrix}O & I'_{2q-1}\\ S(i_1,\ldots,i_t) & O\end{pmatrix}$ with $S(i_1,\ldots,i_t)\in M(t+1,t)$ and $I'_{2q-1}=\begin{pmatrix}I_{2q-1}\\O\end{pmatrix}$. Since the $H'_{n,q}$-part of the isotropy subalgebra at this point contains $\{(-aI_{2q-2}+A)\oplus(\begin{smallmatrix}-a-d&*\\0&a+d\end{smallmatrix})\oplus(\begin{smallmatrix}-a+d&*\\O&B\end{smallmatrix})\mid A\in Lie(Sp_{q-1}),B\in Lie(T_u(n-2q-1)),\operatorname{tr}B=(2q-1)a-d\}$, it is a FP. ∎

Lemma 2.13. *$((GSp_{m_1}\times SL_2\times SL_1)\times SL_n,(\Lambda_1\boxplus\Lambda_1\boxplus\Lambda_1)\otimes\Lambda_1)$ with $n>2m_1+1$, is a FP.*

Proof. Similarly as Lemma 2.12, it is enough to show that $(H'_{n,q}\times(SL_2\times SL_1),\Lambda_1\otimes(\Lambda_1\boxplus\Lambda_1))$ is a FP when $m_2=2>t=1>0$. Any point in $M(n,2)'$ is transformed to (e_i,e_1) $(n\geq i\geq 2q+1)$ by $H'_{n,q}\times SL_2$ and the $H'_{n,q}$-part of the isotropy subalgebra contains $\{(\begin{smallmatrix}d&*\\0&-d\end{smallmatrix})\oplus(-aI_{2q-2}+A)\oplus(\begin{smallmatrix}-2a-d&*\\0&B\end{smallmatrix})\mid A\in Lie(Sp_{q-1}),B\in Lie(T_u(n-2q-1)),\operatorname{tr}B=2qa+d\}$, and hence it is a FP. ∎

Lemma 2.14. *Let $SH_{n,q},SH^*_{n,q}$ and $H_{n,q}$ $(n>2q>0)$ be as in Proposition 1.11.*

1. *$(SH_{n,q}\times(GL_m\times GL_1),\Lambda_1\otimes(\Lambda_1\boxplus\Lambda_1))$ is a FP.*
2. *(a) $(SH_{n,q}\times(GL_m\times SL_1),\Lambda_1\otimes(\Lambda_1\boxplus\Lambda_1))$ is a FP if $n-2q\geq 3$.*
   (b) $(SH^*_{n,q}\times(GL_m\times SL_1),\Lambda_1\otimes(\Lambda_1\boxplus\Lambda_1))$ is a FP if $n-2q\geq 2$.

3. (a) $(SH_{n,q} \times (SL_m \times GL_1), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1))$ *is a FP if* $n - 2q \geq 3$, $(n - 2q > m$ *or* $n \not\equiv m \mod 2)$ *and* $(m > 2q$ *or* $m =$ *odd*).
(b) $(SH^*_{n,q} \times (SL_m \times GL_1), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1))$ *is a FP if* $n - 2q \geq 2$, $(n - 2q > m$ *or* $n \not\equiv m \mod 2)$ *and* $(m > 2q$ *or* $m =$ *odd*).

4. $(H_{n,q} \times (GL_m \times SL_1), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1))$ *is a FP.*

5. $(H_{n,q} \times (SL_m \times GL_1), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1))$ *is a FP if* $m > 2q$ *or* $m =$ *odd.*

6. $(H_{n,q} \times (SL_m \times SL_1), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1))$ *is a FP if* $m > 2q + 1$.

Proof. Just similarly as in the beginning part of the proof of Lemma 2.8, it is enough to show that, for $m \geq t \geq 0$,
(1) $M(2q, m-t) \oplus V(2q) \ni (W, x) \mapsto (AW^tD, \alpha Ax)$,
(2) $M(n-2q, t) \oplus V(n-2q) \ni (S, y) \mapsto (BS^tC, \alpha By)$
are FPs at the same time where $A \in Sp_q, B \in ST_u(n-2q)$ (resp. $B \in SL(n-2q)$ for $SH^*_{n,q}$, $B \in T_u(n-2q)$ for $H_{n,q}$) and the full subgroup of type $\{((\begin{smallmatrix} C & O \\ O & D \end{smallmatrix}), \alpha) \mid C \in GL_t, D \in GL(m-t)\}$ of $GL_m \times GL_1$ (resp. $GL_m \times SL_1$, $SL_m \times GL_1$, $SL_m \times SL_1$) acts. Hence we have 1 by 1 of Lemmas 2.6 and 2.7. We have 2 by 1 of Theorem 2.3, Lemma 2.6 and 3 of Lemma 2.7.

For 3, (1) and (2) are related with $(\det C)(\det D) = 1$ and $\alpha \in GL_1$. First assume that $t = 0$. Then $D \in SL_m$ and (1) is a FP by 2 of Lemma 2.6. (2) becomes $y \mapsto \alpha By$ which is a FP even when $\alpha = 1$ since $n - 2q \geq 2$. Next assume that $t = m$. Then $C \in SL_m$ and (1) becomes just $x \mapsto \alpha Ax$ which is a FP even when $\alpha = 1$. If $m = 1$, (2) for $SH_{n,q}$ (resp. $SH^*_{n,q}$) is a FP by 5 of Lemma 2.7 (resp. 1 of Theorem 2.3) since $n - 2q \geq 3$ (resp. $n - 2q \geq 2$). If $m \geq 2$, (2) is a FP since $m \neq n - 2q$. Finally assume that $m > t > 0$. Then (1) is always a FP (cf. 1 of Lemma 2.6) and the restriction of scalars occurs in the following 3 cases (a)-(c). (a) When $2q \geq m - t =$ even, we have $\det D = 1$ (and hence $\det C = 1$) in a generic isotropy subgroup of (1). Then (2) for $SH_{n,q}$ with $t = 1$ is a FP by 5 of Lemma 2.7 since $n - 2q \geq 3$. Since $(n - 2q > m(> t)$ or $n \not\equiv m \mod 2)$ and $m \equiv t \mod 2$ implies that $n - 2q \neq t$, (2) for $SH_{n,q}$ with $t \geq 2$ (resp. $SH^*_{n,q}$ with $t \geq 1$) is a FP by 5 of Lemma 2.7 (resp. 1 of Theorem 2.3). (b) When $2q \geq m - t + 1 =$ even, we have $\alpha \det D = 1$ in a generic isotropy subgroup of (1). In this case, we have $t \geq 2$ since $m > 2q$ or $m =$ odd. If we put $(BS^tC, \alpha By) = (B'S^tC', \alpha' B'y)$ with $B' \in T_u(n-2q)$, $C' \in SL_t$, $\alpha' \in GL_1$, we see easily that $\det B' = (\alpha')^r$ with $r = (n-2q)/(t-1)$. Hence this reduces to 5 of Lemma 2.7. We have $r \neq -(n-2q)$ since otherwise $t = 0$, a contradiction. When $n - 2q = t$, we have $r \neq 0$. When $n - 2q = t + 1$, we have $r \neq -1$ since otherwise $t = 0$, a contradiction. Hence (2) is a FP. (c) When $2q \geq m - t + 1 =$ even$(= 2(u+1))$, we have $\det D = \alpha$ in the isotropy subgroup of (1) at $(e_1, \ldots, e_{u+1}, e_{q+1}, \ldots, e_{q+u}, e_{u+1})$, and hence $\det C = \alpha^{-1}$. If we put $(BS^tC, \alpha By) = (B'S^tC', \alpha' B'y)$ with $B' \in T_u(n-2q)$, $C' \in SL_t$, $\alpha' \in GL_1$, we see easily that $\det B' = (\alpha')^r$ with $r = -(n-2q)/(t+1)$. Hence this reduces to 5 of Lemma 2.7.

When $t = 1$, (2) is a FP since $n - 2q \geq 3$ for $SH_{n,q}$ (resp. $n - 2q \geq 2$ for $SH^*_{n,q}$). When $t \geq 2$, we have $r \neq -(n - 2q)$ since otherwise $t = 0$, a contradiction. When $n - 2q = t$, then clearly $r \neq 0$. Since $n - 2q > m$ or $n \not\equiv m \mod 2$, we have $n - 2q \neq t + 1 = m$. Hence (2) is a FP.

For 4, (1) and (2) are FPs at the same time by 1 of Lemmas 2.6 and 2.7.

For 5, (1) and (2) are related with $(\det C)(\det D) = 1$ and $\alpha \in GL_1$. If $t = 0$, then $D \in SL_m$ and (1) is a FP by 2 of Lemma 2.6 since $m > 2q$ or $m =$ odd. (2) becomes $y \mapsto \alpha By$ which is a FP even when $\alpha = 1$. If $t = m$, then $C \in SL_m$ and (2) is a FP by 1 of Lemma 2.7. (1) becomes just $x \mapsto \alpha Ax$ which is a FP even when $\alpha = 1$. Finally assume that $m > t > 0$. Then (1) is always a FP (cf. 1 of Lemma 2.6) and the restriction of scalars occurs in the following 3 cases (a)-(c). (a) When $2q \geq m - t =$ even, we have $\det D = 1$ (and hence $\det C = 1$) in a generic isotropy subgroup of (1). However α remains and (2) is a FP by 1 of Lemma 2.7. (b) When $2q \geq m - t + 1 =$ even, we have $\alpha \det D = 1$ in a generic isotropy subgroup of (1). In this case, we have $t \geq 2$ since $m > 2q$ or $m =$ odd. If we put $(BS^tC, \alpha By) = (B'S^tC', \alpha'B'y)$ with $B' \in T_u(n - 2q)$, $C' \in SL_t$, $\alpha' \in GL_1$, we see easily that $\det B' = (\det B)(\alpha')^r$ with $r = (n - 2q)/(t - 1)$. Hence $\det B'$ and α' have no relation and (2) is a FP by Lemma 2.7. (c) When $2q \geq m - t + 1 = \text{even}(= 2(u + 1))$, we have $\det D = \alpha \in GL_1$ in the isotropy subgroup of (1) at $(e_1, \ldots, e_{u+1}, e_{q+1}, \ldots, e_{q+u}, e_{u+1})$, and hence $\det C = \alpha^{-1}$. If we put $(BS^tC, \alpha By) = (B'S^tC', \alpha'B'y)$ with $B' \in T_u(n - 2q)$, $C' \in SL_t$, $\alpha' \in GL_1$, we see easily that $\det B' = (\det B)(\alpha')^r$ with $r = -(n - 2q)/(t + 1)$. Hence $\det B'$ and α' have no relation, and (2) is a FP by 1 of Lemma 2.7.

For 6, (1) and (2) are related with $(\det C)(\det D) = 1$ and $\alpha = 1$. If $t = 0$, (1) is a FP by 4 of Lemma 2.6 since $m > 2q + 1$. (2) becomes just $y \mapsto By$ with $B \in T_u(n - 2q)$ which is a FP. If $t = m$, (2) is a FP by 2 of Lemma 2.7 since $m \geq 2$. (1) becomes just $x \mapsto Ax$ with $A \in Sp_q$ which is a FP. Finally assume that $m > t > 0$. (1) is always a FP by 1 of Lemma 2.6, and the restriction of scalars occurs in the following 3 cases (a)-(c). When (a) $2q \geq m - t =$ even (resp. (b) $2q \geq m - t + 1 =$ even), then $\det D = 1$ (and hence $C \in SL_t$) in a generic isotropy subgroup. However since $t \geq m - 2q > 1$ in our case, (2) is a FP by 2 of Lemma 2.7. (c) When $2q \geq m - t + 1 = \text{even}(= 2(u + 1))$, we have $\det D = 1$ (and hence $C \in SL_t$) in the isotropy subgroup of (1) at $(e_1, \ldots, e_{u+1}, e_{q+1}, \ldots, e_{q+u}, e_{u+1})$. Since $m - 1 > 2q \geq m - t + 1$, we have $t \geq 3$, and hence (2) is a FP by 2 of Lemma 2.7. ∎

Proposition 2.15. $((Sp_{m_1} \times GL_{m_2} \times GL_1) \times SL_n, (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ $(m_1 \geq 2,\ n \geq 4)$ *is a FP if and only if* $2m_1 < n$ *or* $n =$ *odd.*

Proof. By 1 of Lemma 2.4, the condition is necessary. If $2m_1 < n$ or $n =$ odd, the SL_n-part of an isotropy subgroup of $(Sp_{m_1} \times SL_n, \Lambda_1 \otimes \Lambda_1)$ contains $SH_{n,q}$ $(n > 2q \geq 0)$ by Proposition 1.11. Hence we obtain our result by 1 of Lemmas 2.7 and 2.14. ∎

Proposition 2.16. $((Sp_{m_1} \times GL_{m_2} \times SL_1) \times SL_n, (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ $(m_1 \geq 2,\ n \geq 4)$ *is a FP if and only if* $n > 2m_1 + 1$.

Proof. $((Sp_{m_1} \times SL_1) \times SL_n, (\Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ is a FP if and only if $n > 2m_1 + 1$ by 7 of Theorem 2.3. Under this condition, the SL_n-part of an isotropy subgroup of $(Sp_{m_1} \times SL_n, \Lambda_1 \otimes \Lambda_1)$ contains $SH_{n,q}$ $(n - 2q \geq 3, m_1 \geq q)$ or $SH^*_{n,m_1}(n - 2m_1 = 2, q = m_1)$ by Proposition 1.11. Hence we obtain our result by 2 of Lemma 2.14 ∎

Proposition 2.17. $((Sp_{m_1} \times SL_{m_2} \times GL_1) \times SL_n, (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ $(m_1 \geq 2,\ n \geq 4)$ *is a FP if and only if one of the following conditions holds.*

1. $m_2 > n > 2m_1$ *or* $m_2 > n =$ *odd,*
2. $n > 2m_1 + m_2$ *and* ($m_2 > 2m_1$ *or* $m_2 =$ *odd*),
3. $2m_1 + m_2 > n > m_2$, ($m_2 > 2m_1$ *or* $m_2 =$ *odd*), $n > 2m_1 + 1$ *and* $n \not\equiv m_2 \mod 2$.

Proof. By 7 of Theorem 2.3, if it is a FP, these conditions are necessary. Assume that $m_2 > n$. Then by Propositions 1.2 and 2.15, it is a FP if and only if $n > 2m_1$ or $n =$ odd. Now assume that the condition 2 or 3 is satisfied. By Proposition 1.11, the SL_n-part of an isotropy subgroup of $(Sp_{m_1} \times SL_n, \Lambda_1 \otimes \Lambda_1)$ contains $ST_u(n)$ or $SH_{n,q}$ $(n > 2q > 0)$ or $SH^*_{n,m}$ $(n - 2q = 2$ and $q = m_1)$. By 5 of Lemma 2.7, $(ST_u(n) \times (SL_{m_2} \times GL_1), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1))$ is a FP in our case. The condition 2 or 3 implies the condition in 3 of Lemma 2.14, and hence we have our result. ∎

Proposition 2.18. $((Sp_{m_1} \times GL_{m_2} \times SL_1) \times GL_n, (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ $(m_1 \geq 2,\ n \geq 4)$ *is a FP. Note that in this case, it is always FP without the condition on n by Lemma 1.5. This is isomorphic to* $((GL_1 \times (Sp_{m_1} \times SL_1) \times GL_{m_2}) \times SL_n, (\Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1) \boxplus \Lambda_1) \otimes \Lambda_1)$.

Proof. If the GL_n-part of a generic isotropy subgroup contains $Sp_{n'}$ $(n = 2n')$ or $T_u(n)$, it is a FP by 1 of Lemma 2.6 (resp. by the 3rd form of 1 of Lemma 2.7). Otherwise it contains $H_{n,q}$ $(n > 2q > 0)$ by Proposition 1.11. Then by 4 of Lemma 2.14, we have our result. ∎

Proposition 2.19. $((Sp_{m_1} \times SL_{m_2} \times GL_1) \times GL_n, (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ $(m_1 \geq 2,\ n \geq 4)$ *is a FP if and only if* $m_2 > n$ *or* $m_2 > 2m_1$ *or* $m_2 =$ *odd. Note that this is isomorphic to* $((GL_1 \times (Sp_{m_1} \times SL_{m_2}) \times GL_1) \times SL_n, (\Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1) \boxplus \Lambda_1) \otimes \Lambda_1)$.

Proof. These conditions are necessary by 6 of Theorem 2.3. If $m_2 > n$, it is a FP by Proposition 1.2. So we may assume that $n \geq m_2 > 2m_1$ or $n \geq m_2 =$ odd. By Proposition 1.11, the GL_n-part of an isotropy subgroup of $(Sp_{m_1} \times GL_n, \Lambda_1 \otimes \Lambda_1)$ contains $Sp_{n'}$ $(2m_1 \geq n = 2n'), T_u(n)$ or $H_{n,q}$ $(n > 2q > 0)$. Hence by 2 of Lemma 2.6, 1 of Lemma 2.7 and 5 of Lemma 2.14, we have our result. ∎

Proposition 2.20. $((Sp_{m_1} \times SL_{m_2} \times SL_1) \times GL_n, (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ $(m_1 \geq 2,\ n \geq 4)$ *is a FP if and only if* $m_2 > n$ *or* $m_2 > 2m_1 + 1$.

Proof. First assume that $n \geq m_2$ and $2m_1 + 1 \geq m_2$. Then the GL_n-part of the isotropy subgroup of $(SL_{m_2} \times GL_n, \Lambda_1 \otimes \Lambda_1, M(m_2, n))$ at $(I_{m_2} | \ O)$ is $H = \begin{pmatrix} SL(m_2) & * \\ O & * \end{pmatrix}$. Then $(Sp_{m_1} \times SL_1) \times H$ acts on $\{(X | \ O) \in M(2m_1 + 1, n) \mid X \in M(2m_1 + 1, m_2)\}$ as $((Sp_{m_1} \times SL_1) \times SL_{m_2}, (\Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ which is a non FP in our case by 7 of Theorem 2.3. If $m_2 > n$, then by Propositions 1.2 and 2.18, it is a FP. So we may assume that $n \geq m_2 > 2m_1 + 1$. Then, by Proposition 1.11, the GL_n-part of an isotropy subgroup of $(Sp_{m_1} \times GL_n, \Lambda_1 \otimes \Lambda_1)$ contains $T_u(n)$ or $H_{n,q}$ $(n > 2q > 0)$. Since $m_2 \geq 2$, $(T_u(n) \times (SL_{m_2} \times SL_1), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1))$ is a FP by 2 of Lemma 2.7. Since $m_2 > 2m_1 + 1 \geq 2q + 1$, we have our result by 6 of Lemma 2.14. ∎

Proposition 2.21. $((Sp_{m_1} \times SL_{m_2} \times SL_1) \times SL_n, (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ $(m_1 \geq 2,\ n \geq 4)$ *is a FP if and only if* $m_2 > n > 2m_1 + 1$ *or* $(n > 2m_1 + m_2 + 1$ *and* $m_2 > 2m_1 + 1)$.

Proof. Assume that $m_2 > n$. Then by Propositions 1.2 and 2.16, it is a FP if and only if $n > 2m_1 + 1$. If $n = m_2$, it is a non FP since $(SL_{m_2} \times SL_n, \Lambda_1 \otimes \Lambda_1)$ is a non FP. Hence we assume that $n > m_2$. We shall show that if $2m_1 + m_2 + 1 \geq n (> m_2)$, it is a non FP. If $n = 2m_1 + m_2 + 1$ or $n = 2m_1 + m_2$, it is clearly a non FP since $(Sp_{m_1} \times SL_{m_2}) \subset SL_{2m_1 + m_2}$ etc. Hence we may assume that $n > m_2 > n - 2m_1$. Then there exists q satisfying $n - 2q = m_2$ or $n - 2q = m_2 + 1$ $(m_1 \geq q \geq 0)$. The SL_n-part of some isotropy subgroup of $(Sp_{m_1} \times SL_n, \Lambda_1 \otimes \Lambda_1)$ is contained in $SH^*_{n,q} = \begin{pmatrix} Sp(q) & * \\ O & SL(n-2q) \end{pmatrix}$ by Proposition 1.10. Then $(SH^*_{n,q} \times (SL_{m_2} \times SL_1), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1)$ $(n - 2q = m_2$ or $m_2 + 1)$ is a non FP. Hence we may assume that $n > 2m_1 + m_2 + 1$. Then by Propositions 1.2 and 2.20, we obtain our result. ∎

Proposition 2.22. $((GSp_{m_1} \times SL_{m_2} \times SL_1) \times GL_n, (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ $(m_1 \geq 2,\ n \geq 4)$ *is a FP if and only if* $m_2 \geq 2$.

Proof. If $m_2 = 1$, it is a non FP since $((SL_1 \times SL_1) \times GL_n, (\Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ is a non FP by 2 of Theorem 2.3. Assume that $m_2 \geq 2$. The GL_n-part of an isotropy subgroup of $(GSp_{m_1} \times GL_n, \Lambda_1 \otimes \Lambda_1)$ contains GSp_q $(n = 2q), T_u(n)$ or $H = \begin{pmatrix} GSp_q & M(2q, n-2q) \\ O & T_u(n-2q) \end{pmatrix}$ with $n > 2q > 0$. By 3 of Lemma 2.6, $(GSp_q \times (SL_{m_2} \times SL_1), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1))$ is a FP. By 2 of Lemma 2.7, $(T_u(n) \times (SL_{m_2} \times SL_1), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1))$ is a FP. Hence it is enough to show that $(H \times (SL_{m_2} \times SL_1), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1))$ $(m_2 \geq 2)$ is a FP. For this, just by the same argument of the beginning part of the proof of Lemma 2.8, it is enough to show that, for any t satisfying $m_2 \geq t \geq 0$,

(1) $M(2q, m_2 - t) \oplus V(2q) \ni (W, x) \mapsto (AW^tD, Ax)$

(2) $M(n - 2q, t) \ni (S, y) \mapsto (BS^tC, By)$

are FPs at the same time, where $A \in GSp_q, D \in GL(m_2 - t), B \in T_u(n - 2q), C \in GL_t$ and $(\det C)(\det D) = 1$. If $t = 0$, then $D \in SL_{m_2}$ and (1) is a FP by 3 of Lemma 2.6. (2) becomes just $y \mapsto By$ which is a FP. If $t = m_2$, then $C \in SL_{m_2}$ and (2) is a FP by 2 of

Lemma 2.7. (2) becomes just $x \mapsto Ax$ which is a FP. Finally assume that $m_2 > t > 0$. Then (1) is a FP by 1 of Lemma 2.6. The restriction of scalars occurs in the following 3 cases (a)-(c). (a) When $2q \geq m_2 - t =$ even (resp. (b) $2q \geq m_2 - t + 1 =$ even), we have $(\det A)^{m_2-t}(\det D)^{2q} = 1$ and $\det C = (\det A)^{(m-t)/2q}$ (resp. $(\det A)^{m_2-t+1}(\det D)^{2q} = 1$ and $\det C = (\det A)^{(m-t+1)/2q}$) in a generic isotropy subgroup. Hence no restriction of scalars occurs in (2). So by 1 of Lemma 2.7, (2) is a FP. (c) When $2q \geq m_2 - t + 1 =$ even, we have $\det D = (\det A)^{(1-(m_2-t))/2q}$ (and hence $\det C = (\det A)^{(m_2-t-1)/2q}$) in the isotropy subgroup at $(e_1, \ldots, e_{u+1}, e_{q+1}, \ldots, e_{q+u}, e_{u+1}) \in M(2q, m_2 - t + 1)$ with $m_2 - t = 2u + 1$. Note that if we write $(AW^tD, Ax) = (A'W^tD', \alpha' A' x)$ with $A' \in Sp_q$, the condition $\det D' = \alpha'$ implies that $\det D = (\det A)^{(1-(m_2-t))/2q}$. If $m_2 - t > 1$, we have $\det C \neq 1$ and hence (2) is a FP by 1 of Lemma 2.7. If $m_2 - t = 1$ and $t \geq 2$, then (2) is a FP by 2 of Lemma 2.7. Now assume that $m_2 = 2$ and $t = 1$. By the simple calculation of the isotropy subalgebra, we see that the H-part of the isotropy subgroup of $(H \times SL_2, \Lambda_1 \otimes \Lambda_1)$ at $\begin{pmatrix} 0 & e_1^{(2q)} \\ e_i^{(n-2q)} & 0 \end{pmatrix}$ contains $\{(\begin{pmatrix} ab & * & * \\ 0 & ab^{-1} & 0 \\ 0 & 0 & (ab)^{-1} \end{pmatrix}, aA) \mid A \in Sp_{q-1},\ a, b \in GL_1\} \times T_u(n-2q-1) \subset GL_n$. Hence $(H \times (SL_2 \times SL_1), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1))$ is a FP, and we obtain our result. ∎

Proposition 2.23. $((GL_1 \times (Sp_{m_1} \times SL_1) \times SL_{m_2}) \times SL_n, (\Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1) \boxplus \Lambda_1) \otimes \Lambda_1)$ $(m_1 \geq 2,\ n \geq 4)$ *is a FP if and only if* $m_2 > n$ *or* $(n > m_2$ *and* $n > 2m_1 + 1)$.

Proof. First we show that it is a non FP for $2m_1 + 1 \geq n \geq m_2$. If $n = m_2$, it is clearly a non FP. If $n = 2m_1 + 1$, it is a non FP since the SL_n-part of a generic isotropy subgroup of $(GL_1 \times Sp_{m_1} \times SL_1 \times SL_n, \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ is $(Sp_{m_1} \times SL_1, \Lambda_1 \boxplus \Lambda_1) \subset (SL_n, \Lambda_1)$ and $((Sp_{m_1} \times SL_1) \times SL_{m_2}, (\Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ with $2m_1 + 1 > m_2$ is a non FP by 7 of Theorem 2.3. So we may assume that $2m_1 \geq n > m_2$. If $n = 2n'$, it is a non FP since $(Sp_{n'} \times (SL_{m_2} \times SL_1), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1)$ with $n = 2n' > m_2$ is a non FP by 4 of Lemma 2.6. If $n = 2n' + 1$, it is a non FP since the SL_n-part of a generic isotropy subgroup of $(GL_1 \times (Sp_{m_1} \times SL_1) \times SL_n, \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ is $(Sp_{n'} \times SL_1, \Lambda_1 \boxplus \Lambda_1) \subset (SL_n, \Lambda_1)$ and $((Sp_{n'} \times SL_1) \times SL_{m_2}, (\Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1))$ $(2n' + 1 > m_2)$ is a non FP by 7 of Theorem 2.3. If $m_2 > n$, then by Propositions 1.2, it reduces to Proposition 2.18, and it is a FP. Finally assume that $n > m_2$ and $n > 2m_1 + 1$. The $(GL_1 \times SL_n)$-part H of an isotropy subgroup of $(GL_1 \times Sp_{m_1} \times SL_n, \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1)$ contains $(GL_1 \times ST_u(n))$ or $\{(\alpha, (\begin{smallmatrix} \alpha^{-1}A & C \\ O & B \end{smallmatrix})) \mid \alpha \in GL_1, A \in Sp_q, B \in T_u(n-2q), \det B = \alpha^{2q}, C \in M(2q, n-2q)\}$ with $n > 2q > 0$. By 5 of Lemma 2.7, $(ST_u \times (SL_{m_2} \times GL_1), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1))$ is a FP in our case. Hence, to prove that $(H \times SL(m_2), \Lambda_1 \otimes \Lambda_1)$ is a FP, just similarly as the beginning part of the proof of Lemma 2.8, it is enough to show that for $n > 2q > 0$,

(1) $M(2q, m_2 - t) \oplus V(2q) \ni (W, x) \mapsto (\alpha^{-1} AW^tD, Ax)$,

(2) $M(n - 2q, t) \oplus V(n - 2q) \ni (S, y) \mapsto (BS^tC, \alpha By)$

are FPs at the same time where $\alpha \in GL_1, A \in Sp_q, D \in GL(m_2 - t), B \in T_u(n - 2q), C \in$

$GL_t, (\det C)(\det D) = 1, \det B = \alpha^{2q}$. If $t = 0$, then $D \in SL(m_2)$ and (1) is a FP by 1 of Lemma 2.6. (2) becomes just $y \mapsto \alpha By$ which is a FP even when $\det(\alpha B) = \alpha^n = 1$ since $n > 2m_1 + 1$ implies $n - 2q \geq 2$. If $t = m_2$, then $C \in SL(m_2)$ and (1) becomes just $x \mapsto Ax$, which is always a FP. Now (2) reduces to 5 in Lemma 2.7 with $r = 2q$. So if $m_2 = 1$ and $n - 2q \geq 3$, it is a FP. If $m_2 = 1$ and $n - 2q = 2$, the condition $n > 2m_1 + 1 (m_1 \geq q)$ implies $q = m_1$. In this case, the SL_n-part of a generic isotropy subgroup of $(GL_1 \times Sp_{m_1} \times SL_n, \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1)$ is $\begin{pmatrix} GSp(m_1) & * \\ O & GL(2) \end{pmatrix} \cap SL_n$. Since $V(2) \oplus V(2) \ni (x, y) \mapsto (Bx, \alpha By)$ with $B \in GL_2, \det B = \alpha^{2m_1}$, is a FP, (2) is a FP. If $m_2 \geq 2$, it is a FP by 5 of Lemma 2.7 since $r = 2q \neq 0, -1, -(n - 2q)$. Finally assume that $m_2 > t > 0$. Then (1) is a FP by (1) of Lemma 2.6. The restriction of scalars occurs in the following 3 cases (a)-(c). (a) When $2q \geq m_2 - t =$ even (resp. (b),(c) When $2q \geq m_2 - t + 1 =$ even), then we have $\alpha^{-2q(m_2 - t)}(\det D)^{2q} = 1$ in a generic isotropy subgroup for (a),(b) (resp. in the isotropy subgroup at $(e_1, \ldots, e_{u+1}, e_{q+1}, \ldots, e_{q+u}, e_{u+1}) \in M(2q, m_2 - t + 1)$ with $m_2 - t = 2u + 1$ for (c)). Hence we have $\det C = \alpha^{-(m_2 - t)}$ for (a)-(c). If we write $(BS^tC, \alpha By) = (B'S^tC', \alpha'B'y)$ with $C' \in SL_t$, we have $\det B' = (\alpha')^r$ with $r = (tn - m_2 n + 2qm_2)/m_2$. Hence (2) reduces to 5 of Lemma 2.7. If $t = 1$, then it is a FP for $n - 2q \geq 3$. If $n - 2q = 2$, by the same argument as above, it is also a FP. Assume that $t \geq 2$. Then $r \neq -(n - 2q)$ since otherwise we have $tn = 0$. If $n - 2q = t$, then we have $r \neq 0$. If $n - 2q = t + 1$, then we see that $r \neq -1$. In both cases, otherwise we have $t(n - m_2) = 0$. Hence (2) is also a FP by 5 of Lemma 2.7. ∎

Proposition 2.24. $((GL_1 \times (Sp_{m_1} \times SL_{m_2}) \times SL_1) \times SL_n, (\Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1) \boxplus \Lambda_1) \otimes \Lambda_1)$ $(m_1 \geq 2, n \geq 4)$ *is a FP if and only if one of the following conditions holds.*

1. $m_2 > n$,
2. $m_2 = n > 2m_1 + 1$,
3. $n > m_2$ *and* $n > 2m_1 + 1$ *and* $(m_2 > 2m_1$ *or* $m_2 =$ *odd*$)$.

Proof. First assume that $n > m_2$ and $2m_1 \geq m_2 =$ even. Then it is a non FP since $(GL_1 \times (Sp_{m_1} \times SL_{m_2}) \times SL_n, \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ is a non FP in this case by 6 of Theorem 2.3. Next assume that $2m_1 + 1 \geq n \geq m_2$. If $m_2 = n$, then the SL_n-part of a generic isotropy subgroup of $(GL_1 \times SL_{m_2} \times SL_n, \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1)$ is SL_n and $((Sp_{m_1} \times SL_1) \times SL_n, (\Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ is a non FP in this case by 7 of Theorem 2.3. So it is a non FP. If $n = 2m_1 + 1 > m_2$, then the SL_n-part of a generic isotropy subgroup of $(GL_1 \times (Sp_{m_1} \times SL_1) \times SL_n, \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ is $Sp_{m_1} \times SL_1$ and $((Sp_{m_1} \times SL_1) \times SL_{m_2}, (\Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ is a non FP in this case by 7 of Theorem 2.3. Hence it is a non FP. If $2m_1 \geq n = 2n' > m_2$, it is a non FP since $(Sp_{n'} \times (SL_{m_2} \times SL_1), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1))$ is a non FP by 4 of Lemma 2.6. If $2m_1 \geq n = 2n' + 1 > m_2$, it is a non FP since the SL_n-part of a generic isotropy subgroup of $((GL_1 \times Sp_{m_1} \times SL_1) \times SL_n, (\Lambda_1 \otimes \Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ is

$\begin{pmatrix} Sp(n') & O \\ O & 1 \end{pmatrix}$ and $((Sp_{n'} \times SL_1) \times SL_{m_2}, (\Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ is a non FP in this case by 7 of Theorem 2.3. If $m_2 > n$, then by Propositions 1.2 and 2.9, it is a FP. If $m_2 = n > 2m_1+1$, for the orbits related with $M(n)'$, it reduces to $((Sp_{m_1} \times SL_1) \times SL_n, (\Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ which is a FP in this case by 7 of Theorem 2.3. For the orbits related with $M(n)''$, by Proposition 1.2, it reduces to Proposition 2.9. Finally assume that $n > m_2$ and $n > 2m_1+1$ and ($m_2 > 2m_1$ or $m_2 =$ odd). Then the $(GL_1 \times SL_n)$-part of an isotropy subgroup of $(GL_1 \times Sp_{m_1} \times SL_n, \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1)$ contains $(GL_1 \times ST_u(n))$ or $H = \{(\alpha, \begin{pmatrix} \alpha^{-1}A & C \\ O & B \end{pmatrix}) \mid \alpha \in GL_1, A \in Sp_q, B \in T_u(n-2q), \det B = \alpha^{2q}, C \in M(2q, n-2q)\}$ with $n - 2 \geq 2q > 0$. Note that $(ST_u(n) \times (GL_{m_2} \times SL_1), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1))$ is a FP by 3 of Lemma 2.7 since $n \geq 4$. Hence, just similarly as the beginning part of the proof of Lemma 2.8, it is enough to prove that, for $n - 2 \geq 2q > 0$ and $m_2 \geq t \geq 0$,

(1) $M(2q, m_2 - t) \oplus V(2q) \ni (W, x) \mapsto (AW{}^tD, \alpha^{-1}Ax)$

(2) $M(n-2q, t) \oplus V(n-2q) \ni (S, y) \mapsto (\alpha BS{}^tC, By)$

are FPs at the same time, where $\alpha \in GL_1, A \in Sp_q, D \in GL(m_2 - t), B \in T_u(n-2q), C \in GL_t, \det B = \alpha^{2q}$ and $(\det C)(\det D) = 1$. If $t = 0$, then $D \in SL(m_2)$ and (1) becomes a FP in our case by 2 of Lemma 2.6. (2) becomes just $y \mapsto By$ which is a FP even when $\alpha = 1$ since $n - 2q \geq 2$. If $t = m_2$, then $C \in SL(m_2)$ and (1) becomes just $x \mapsto \alpha^{-1}Ax$ which is a FP, and α always remains. In (2), put $(\alpha BS{}^tC, By) = (B'S{}^tC, \alpha'B'y)$. Then we have $\det B' = (\alpha')^{-n}$ so that (2) reduces to 5 of Lemma 2.7 with $r = -n$. So if $m_2 = 1$ and $n - 2q \geq 3$, it is a FP. If $m_2 = 1$ and $n - 2q = 2$, it is a FP just similarly as in the proof of Proposition 2.23. If $m_2 \geq 2$, it is a FP by 5. of Lemma 2.7. Finally assume that $m_2 > t > 0$. Then (1) is with full scalars and it is a FP. The restriction of scalars happens in the following 3 cases (a)-(c). (a) When $2q \geq m_2 - t =$ even, then $\det D = 1$ (and hence $\det C = 1$) in a generic isotropy subgroup. Then (2) reduces to 5 of Lemma 2.7 with $r = -n$. Hence just similarly as above, we see that (2) is a FP. (b) When $2q \geq m_2 - t + 1 =$ even, then $\alpha^{-1} \det D = 1$ and hence $\det C = \alpha^{-1}$ in a generic isotropy subgroup. Note that in this case, $t \geq 2$ since otherwise we have $2m_1 \geq m_2 =$ even, a contradiction. If we put $(\alpha BS{}^tC, By) = (B'S{}^tC', \alpha'B'y)$ with $C' \in SL_t$, we have $\det B' = (\alpha')^r$ with $r = (tn - n + 2q)/(1-t)$. Hence (2) reduces to 5 of Lemma 2.7. We have $r \neq -(n-2q)$ since otherwise $qt = 0$. If $n - 2q = t$, we have $r = t(n-1)/(1-t) \neq 0$. If $n - 2q = t + 1$, then $r \neq -1$ since otherwise $n = 2$, a contradiction. Hence (2) is a FP by 5 of Lemma 2.7. (c) When $2q \geq m_2 - t + 1 =$ even $(= 2(u+1))$, we have $\det D = \alpha^{-1} \in GL_1$ (and hence $\det C = \alpha$) in the isotropy subgroup at $(e_1, \ldots, e_{u+1}, e_{q+1}, \ldots, e_{q+u}, e_{u+1})$. Then (2) reduces to 5 of Lemma 2.7 with $r = (tn + n - 2q)/(-t-1)$. If $t = 1$, it is a FP just similarly in the proof of Proposition 2.23. For $t \geq 2$, we have $r \neq -(n-2q)$ since otherwise $qt = 0$, a contradiction. When $n - 2q = t$, we have $r = t(n+1)/(-t-1) \neq 0$. When $n - 2q = t + 1$, we have $r \neq -1$ since otherwise $tn = 0$. Thus by 5 of Lemma 2.7, it is a FP. ∎

Proposition 2.25. $((Sp_{m_1} \times GL_1 \times (SL_{m_2} \times SL_1)) \times SL_n, (\Lambda_1 \boxplus (\Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1))) \otimes$

Λ_1) $(m_1 \geq 2, n \geq 4)$ *is a FP if and only if one of the following conditions holds.*

1. $m_2 > n = even > 2m_1$,
2. $m_2 > n = odd$,
3. $n \geq m_2 \geq 2$ *and* $n > 2m_1 + 1$.

Proof. If $m_2 = 1$, then it is a non FP since $(GL_1 \times (SL_1 \times SL_1) \times SL_n, \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ is a non FP. So we assume that $m_2 \geq 2$. To prove the only if part, it is enough to show that it is a non FP when $2m_1 \geq n =$ even or $2m_1 + 1 \geq n(=$ odd$) \geq m_2$. If $2m_1 \geq n =$ even, it is a non FP since $(Sp_{m_1} \times SL_n, \Lambda_1 \otimes \Lambda_1)$ is a non FP in this case. Now assume that $2m_1 + 1 \geq n = 2n' + 1 \geq m_2$. Then the $(GL_1 \times SL(2n'+1))$-part of a generic isotropy subgroup of $((Sp_{m_1} \times GL_1) \times SL(2n'+1), (\Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ is $\{\{1\}, H\}$ with $H = \begin{pmatrix} Sp(n') & 0 \\ 0 & 1 \end{pmatrix}$ and $(H \times SL_{m_2}, \Lambda_1 \otimes \Lambda_1)$ is a non FP by 7 of Theorem 2.3. Now assume that $m_2 > n$. Then by Proposition 1.2, it reduces to the Proposition 2.15, and it is a FP if and only if $n > 2m_1$ or $n =$ odd, i.e., 1 and 2. Next assume that $m_2 = n$. For the orbits related with $M(n)'$, the $(GL_1 \times SL_n)$-part of an isotropy subgroup of $(GL_1 \times SL_{m_2} \times SL_n, \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1)$ is $\{1\} \times SL_n$ and $((Sp_{m_1} \times SL_1) \times SL_n, (\Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ is a FP if and only if $n > 2m_1 + 1$ by 7 of Theorem 2.3. For the orbits related with $M(n)''$, it reduce to Proposition 2.15, and it is a FP if and only if $n > m_1$ or $n =$ odd. Hence if $m_2 = n$, it is a FP if and only if $n > 2m_1 + 1$. Finally assume that $n > 2m_1 + 1$ and $n > m_2 \geq 2$. The SL_n-part of an isotropy subgroup of $(Sp_{m_1} \times SL_n, \Lambda_1 \otimes \Lambda_1)$ contains $ST_u(n)$ or $SH_{n,q}$ $(n - 2 \geq 2q > 0)$. By 2 of Lemma 2.7, $(GL_1 \times (SL_{m_2} \times SL_1) \times ST_u(n), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1) \cong (T_u(n) \times (SL_{m_2} \times SL_1), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1))$ with $m_2 \geq 2$ is a FP. When it contains $SH_{n,q}$, as in the proof of Lemma 2.8, it is enough to show that, for $n - 2 \geq 2q > 0$ and $m_2 \geq t \geq 0$,
(1) $M(2q, m_2 - t) \oplus V(2q) \ni (W, x) \mapsto (\alpha A W^t D, \alpha A x)$
(2) $M(n - 2q, t) \oplus V(n - 2q) \ni (S, y) \mapsto (\alpha B S^t C, \alpha B y)$
are FPs at the same time, where $\alpha \in GL_1, A \in Sp_q, D \in GL(m_2 - t), B \in ST_u(n - 2q), C \in GL_t$ and $(\det C)(\det D) = 1$. If $t = 0$, then $D \in SL(m_2)$ and (1) is a FP by 3 of Lemma 2.6 since $m_2 \geq 2$. (2) becomes just $y \mapsto \alpha B y$ which is a FP even when $\alpha = 1$ since $n - 2q \geq 2$. If $t = m_2$, then $C \in SL(m_2)$ and (1) becomes just $x \mapsto \alpha A x$ which is a FP where α does not vanish. So (2) is a FP by 2 of Lemma 2.7. Finally assume that $m_2 > t > 0$. First we deal with the case $m_2 \geq 3$. (1) is a FP (cf. 1 of Lemma 2.6) and the restriction of scalars occurs in the following 3 cases (a)-(c). (a) When $2q \geq m_2 - t =$ even, we have $\det(\alpha D) = 1$ (and hence $\det C = \alpha^{m_2 - t}$) in a generic isotropy subgroup. If we put $(\alpha B S^t C, \alpha B y) = (B' S^t C', \alpha' B' y)$ with $C' \in SL_t$, we have $\det B' = (\alpha')^r$ with $r = m_2(n - 2q)/(t - m_2)$. Hence (2) is reduced to 5 of Lemma 2.7. If $t = 1$, (2) is a FP for $n - 2q \geq 3$. If $t = 1$ and $n - 2q = 2$, as we see in the proof of Proposition 2.24, we can replace $T_u(2)$ to GL_2 with the same determinant, and hence (2) is a FP. Assume that $t \geq 2$. Then we have $r \neq -(n - 2q)$ since otherwise we have $t = 0$, a contradiction. When

$n-2q=t$, then clearly $r \neq 0$. When $n-2q=t+1$, then $r \neq -1$ since otherwise $m_2=-1$, a contradiction. Hence (2) is a FP by 5 of Lemma 2.7. (b) When $2q \geq m_2-t+1=$ even, we have $\alpha\det(\alpha D)=1$ (and hence $\det C=\alpha^{m_2-t+1}$) in a generic isotropy subgroup. Then (2) is reduced to 5 of Lemma 2.7 with $r=(m_2+1)(n-2q)/(t-m_2-1)$. When $t=1$, it is a FP by similar argument as (a). When $t \geq 2$, we have $r \neq -(n-2q)$ since otherwise we have $t=0$. When $n-2q=t$, clearly $r \neq 0$. When $n-2q=t+1$, we have $r \neq -1$ since otherwise $m_2=-2$. Hence (2) is a FP. (c) When $2q \geq m_2-t+1=$ even, we have $\det \alpha D=\alpha$ (and hence $\det C=\alpha^{m_2-t-1}$) in the isotropy subgroup at $(e_1,\ldots,e_{u+1},e_{q+1},\ldots,e_{q+u},e_{u+1}) \in M(2q,m_2-t)$ with $m_2-t=2u+1$. If $m_2-t=1$, we have $t \geq 2$ since $m_2 \geq 3$. Therefore (2) is a FP by 2 of Lemma 2.7. Assume that $m_2-t \geq 3$. Put $(\alpha BS^tC,\alpha By)=(B'S^tC',\alpha'B'y)$ with $C' \in SL_t$. Then we have $\det B'=(\alpha')^r$ with $r=(n-2q)(m_2-1)/(1+t-m_2)$. Hence (2) reduces to 5 of Lemma 2.7. When $t \geq 2$, we have $r \neq 0,-(n-2q)$ and if $r=-1$, we have $n-2q \neq t+1$ since otherwise $tm_2=0$, a contradiction. Hence (2) is a FP for $t \geq 2$. When $t=1$, (2) is a FP for $n-2q \geq 3$. If $n-2q=2$, we have $q=m_1$ and $B \in ST_u(2)$ can be replaced by $B \in SL_2$ by Proposition 1.11. Since $m_2-t \geq 3$, we have $\det C \neq 1$, and (2) is a FP. Finally consider the case $m_2=2>t>0$, i.e., $t=1$. Put $H_q=\{\left(\begin{smallmatrix}\alpha A & * \\ O & \alpha B\end{smallmatrix}\right) \mid A \in Sp_q, B \in ST_u(n-2q), \alpha \in GL_1\} \cong GL_1 \times SH_{n,q}$. It is enough to show that $(H_q \times (SL_2 \times SL_1), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1))$ is a FP. By a direct calculation of the isotropy subalgebra of $(H_q \times SL_2, \Lambda_1 \otimes \Lambda_1)$ at (e_i,e_1) with $n \geq i \geq 2q+1$, each H_q-part contains $\{\left(\begin{smallmatrix}d & * \\ 0 & -d\end{smallmatrix}\right) \oplus (aI_{2q-2}+A) \oplus \left(\begin{smallmatrix}2a-d & * \\ O & B\end{smallmatrix}\right) \mid A \in Lie(Sp_{q-1}), B \in Lie(T_u(n-2q-1))$ with $\operatorname{tr} B=(n-2q)a+d\}$. Hence one can easily see that it is a FP. ∎

3 A list

Theorem 3.1. *If we restrict the scalar multiplications of* $((GSp_{m_1} \times GL_{m_2} \times GL_1) \times GL_n, (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ *with* $m_1 \geq 2$ *and* $n \geq 4$, *then it is a FP if and only if it is one of the following case.*

1. $((GSp_{m_1} \times GL_{m_2} \times SL_1) \times SL_n, (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ *with* $m_1 \geq 2,\ n \geq 4$.
2. $((GSp_{m_1} \times SL_{m_2} \times GL_1) \times SL_n, (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ $(m_1 \geq 2,\ n \geq 4)$ *with* $m_2 > n$ *or* $n=$ *odd* $> m_2$ *or* $n > m_2 =$ *odd or* $n > \max\{2m_1, m_2\}$.
3. $((GSp_{m_1} \times SL_{m_2} \times SL_1) \times SL_n, (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ $(m_1 \geq 2,\ n \geq 4)$ *with* $m_2 > n$ *or* $n > \max\{2m_1+1, m_2+1(\geq 3)\}$.
4. $((Sp_{m_1} \times GL_{m_2} \times GL_1) \times SL_n, (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ $(m_1 \geq 2,\ n \geq 4)$ *with* $2m_1 < n$ *or* $n=$ *odd*.
5. $((Sp_{m_1} \times GL_{m_2} \times SL_1) \times SL_n, (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ $(m_1 \geq 2,\ n \geq 4)$ *with* $n > 2m_1+1$.
6. $((Sp_{m_1} \times GL_{m_2} \times SL_1) \times GL_n, (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ *with* $m_1 \geq 2,\ n \geq 4$.

7. $((Sp_{m_1} \times SL_{m_2} \times SL_1) \times GL_n, (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ $(m_1 \geq 2,\ n \geq 4)$ *with* $m_2 > n$ *or* $m_2 > 2m_1 + 1$.

8. $((Sp_{m_1} \times SL_{m_2} \times SL_1) \times SL_n, (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ $(m_1 \geq 2,\ n \geq 4)$ *with* $m_2 > n > 2m_1 + 1$ *or* $(n > 2m_1 + m_2 + 1$ *and* $m_2 > 2m_1 + 1)$.

9. $((Sp_{m_1} \times SL_{m_2} \times GL_1) \times GL_n, (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ $(m_1 \geq 2,\ n \geq 4)$ *with* $m_2 > n$ *or* $m_2 > 2m_1$ *or* $m_2 =$ *odd.*

10. $(((GL_1 \times Sp_{m_1}) \times SL_{m_2} \times SL_1) \times GL_n, ((\Lambda_1 \otimes \Lambda_1) \boxplus \Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ $(m_1 \geq 2,\ n \geq 4)$ *with* $m_2 \geq 2$.

11. $(((GL_1 \times Sp_{m_1} \times SL_1) \times SL_{m_2}) \times SL_n, (\Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1) \boxplus \Lambda_1) \otimes \Lambda_1)$ $(m_1 \geq 2,\ n \geq 4)$ *with* $m_2 > n$ *or* $(n > m_2$ *and* $n > 2m_1 + 1)$.

12. $((Sp_{m_1} \times SL_{m_2} \times GL_1) \times SL_n, (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ $(m_1 \geq 2,\ n \geq 4)$ *with one of the following conditions:*

    (a) $m_2 > n > 2m_1$ *or* $m_2 > n =$ *odd,*

    (b) $n > 2m_1 + m_2$ *and* $(m_2 > 2m_1$ *or* $m_2 =$ *odd),*

    (c) $2m_1 + m_2 > n > m_2$, *and* $n > 2m_1 + 1$, *and* $n \not\equiv m_2 \mod 2$, *and* $(m_2 > 2m_1$ *or* $m_2 =$ *odd).*

13. $((GL_1 \times (Sp_{m_1} \times SL_{m_2}) \times SL_1) \times SL_n, (\Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1) \boxplus \Lambda_1) \otimes \Lambda_1)$ $(m_1 \geq 2, n \geq 4)$ *with one of the following conditions:*

    (a) $m_2 > n$,

    (b) $m_2 = n > 2m_1 + 1$,

    (c) $n > m_2$ *and* $n > 2m_1 + 1$ *and* $(m_2 > 2m_1$ *or* $m_2 =$ *odd).*

14. $((Sp_{m_1} \times GL_1 \times (SL_{m_2} \times SL_1)) \times SL_n, (\Lambda_1 \boxplus (\Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1))) \otimes \Lambda_1)$ $(m_1 \geq 2, n \geq 4)$ *with one of the following conditions:*

    (a) $m_2 > n =$ *even* $> 2m_1$,

    (b) $m_2 > n =$ *odd,*

    (c) $n \geq m_2 \geq 2$ *and* $n > 2m_1 + 1$.

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