A classification of some prehomogeneous vector spaces related with hypergeometric functions

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Abstract

In this paper, we give the detailed proof of a classification of finite reductive prehomogeneous vector spaces of type \((\text{Sp}_m \times GL_{m_2} \times GL_{l_1}) \times GL_n, (\Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1)(m_1 \geq 2, n \geq 4)\) under various restricted scalar multiplications, which are omitted in [KKMOT]. They are related with hypergeometric functions [O].

Introduction

Let \(G\) be a connected linear algebraic group, \(V\) a finite dimensional vector space (\(\dim V \geq 1\)), and \(\rho\) a rational representation of \(G\) on \(V\), all defined over the complex number field \(\mathbb{C}\). If \(V\) has a Zariski-dense \(G\)-orbit, we call a triplet \((G, \rho, V)\) a prehomogeneous vector space (abbrev. PV). When there is no confusion, we sometimes write \((G, \rho)\) instead of \((G, \rho, V)\). When \(G\) is reductive, we call it a reductive PV. For any rational representation \(\rho : G \to GL(V)\) with finitely many orbits, \((G, \rho, V)\) must be a PV. Such a PV is called a finite PV (abbrev. FP). We would like to classify all reductive FPs of type \((G \times GL_n, \rho \otimes \Lambda_1)(n \geq 2)\) which are related with hypergeometric functions. All reductive FPs with full scalar multiplications are completely classified in [KKY]. However if we restrict the scalar multiplications, then the difficulty of different type arises, and only the special cases of the restriction of scalar multiplications are studied. In [KKMOT], all reductive FPs of \(((G \times GL_1) \times SL_n, (\rho \otimes \Lambda_1) \otimes \Lambda_1, (V(m) \otimes V(1)) \otimes V(n))\) with \(n \geq 2\) under various restricted scalar multiplications are completely classified, but the main part of the proof of the most complicated type \(((\text{Sp}_{m_1} \times GL_{m_2} \times GL_{l_1}) \times GL_n, (\Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1)\) with \(m_1 \geq 2\) and \(n \geq 4\) are not written in details. In this paper, we give the complete proof for this omitted case. Note that such FPs with \(m_1 = 1\) (i.e., \(\text{Sp}_1 = SL_2\)) (resp. \(n = 2, 3\)) are classified in [Ka] (resp. Theorem 3.11 in [KKMOT]). We denote the representation \(\Lambda_1 \otimes 1 \otimes 1 \oplus (1 \otimes \Lambda_1 \otimes 1) \oplus (1 \otimes 1 \otimes \Lambda_1)\) of \(\text{Sp}_{m_1} \times GL_{m_2} \times GL_l\) by \(\Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1\).

In Section 1, we give the preliminaries. In particular, we review some basic facts related with Grassmann variety and the orbits. We also give the orbital decomposition of
\((S_{p_m} \times GL_n, \Lambda_1 \otimes \Lambda_1)\) and the isotropy subalgebra of each orbit in the convenient form for later use.

In Section 2, we quote Theorems in \([KKMOT]\), by which we classify FPs of type \(((S_{p_m} \times GL_{m_2} \times GL_{l_1}) \times GL_n, (\Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1) \otimes \Lambda_1)\) with \(m_1 \geq 2\) and \(n \geq 4\) under various restricted scalar multiplications.

In Section 3, we give the list of finite prehomogeneous vector spaces of type \(((S_{p_m} \times GL_{m_2} \times GL_{l_1}) \times GL_n, (\Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1) \otimes \Lambda_1)\) with \(m_1 \geq 2\) and \(n \geq 4\) under various restricted scalar multiplications.

**Notation** We denote \(\mathbb{C}^n\) by \(V(n)\). As usual, \(\mathbb{C}\) stands for the field of complex numbers. We denote by \(e_i^{(n)}\) the \(i\)-th fundamental vector in \(\mathbb{C}^n\). We often write \(e_i\) for simplicity. For positive integers \(m, n\), we denote by \(M(m, n)\) the totality of \(m \times n\) matrices over \(\mathbb{C}\). If \(m = n\), we simply write \(M(n)\) instead of \(M(m, n)\). We also use the notations \(M(m, n)^r = \{X \in M(m, n) | \text{rank } X = \min\{m, n\}\}\) and \(M(m, n)^r = \{X \in M(m, n) | X \in M(m, n)^r\}\).

For \(r < n\), we put \(M_{m, n, r} = \{(X) \in M(m, n) | X \in M(m, r)\}\). We denote by \(I_n\) (or \(I(n)\)) the identity matrix of size \(n\). We denote by \(^tA\) the transposed matrix of a matrix \(A\). Two triplets are called isomorphic and denoted by \((G, \rho, V) \cong (G', \rho', V')\) if there exits a group isomorphism \(\sigma : \rho(G) \to \rho'(G)\) and an isomorphism \(\tau : V \to V'\) of vector spaces satisfying \(\tau(\rho(g)(v)) = (\sigma \rho(g)) \tau(v)\) for all \(g \in G\) and \(v \in V\).

We denote by \(GL_n\) (resp. \(SL_n, SO_n, Sp_n, Sp_{2n}, (G_2), E_6, E_7\)) the general linear group \(\{X \in M(n) | \det X \neq 0\}\) (resp. the special linear group \(\{X \in GL_n | \det X = 1\}\), the special orthogonal group \(\{X \in GL_n | X J_n X = J_n\}\) where \(J_n = (\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix})\), exceptional algebraic groups \((G_2), E_6, E_7\)\). When the expression of \(n\) is complicated, we also write \(GL(n)\) instead of \(GL_n\) etc. Further we denote by \(GSp_n\) the general symplectic group \(\{X \in GL_{2n} | ^t J_n X J_n X = x J_n\) with \(x \in GL_1\} = \{\alpha A | \alpha \in GL_1, A \in Sp_n\} \cong GL_1 \times Sp_n\). We denote by \(T_u(n)\) the group of all nonsingular upper matrices and put \(ST_u(n) = T_u(n) \cap SL_n\). Then we write \(H_{n, q} = \{(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}) \in GL_n | A \in Sp_n, B \in T_u(n - 2q), C \in M(2q, n - 2q)\}\) and \(SH_{n, q} = SL_n \cap H_{n, q}\) with \(2q \leq n\).

We denote by \(\Lambda_1\) the standard representation of \(GL_n\) on \(V(n)\). For a subgroup \(H\) of \(GL_n\), the restriction \(\Lambda_1 |_H\) is also simply denoted by \(\Lambda_1\). More generally, \(\Lambda_k (k = 1, \ldots, r)\) denotes the fundamental irreducible representation of a simple algebraic group of rank \(r\). We have \((GSp_n, \Lambda_1) \cong (GL_1 \times Sp_n, \Lambda_1 \otimes \Lambda_1)\). In general, we denote by \(\rho^*\) the dual representation of a rational representation \(\rho\). It is known that \((H, \sigma, V)\) is a FP if and only if \((H, \sigma^*, V^*)\) is a FP for any algebraic group \(H\), not necessarily reductive (see [P]). Hence \((G, \rho_1^{(s)} \oplus \cdots \oplus \rho_l^{(s)})\) is a FP if and only if \((G, \rho_1 \oplus \cdots \oplus \rho_l)\) is a FP where \(\rho^{(s)}\) implies \(\rho\) or its dual \(\rho^*\). Also if \(G_1\) and \(G_2\) are reductive, then we have \((G_1 \times G_2, \rho_1^{(s)} \otimes \rho_2^{(s)}) \cong (G_1 \times G_2, \rho_1 \otimes \rho_2)\). Using these facts and by the form of FPs (see [KKY]), it is not necessary to consider the dual representation as far as we deal with FPs. For a representation \(\rho : G \to GL(V)\) and a point \(v\) of \(V\), we denote by \(G_v\) the isotropy
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1 Preliminaries

Proposition 1.1. ([KKMOT, Proposition 1.1]) Assume that \((H \times GL_n, \rho \otimes \Lambda_1)\) is a FP. Then \((H \times SL_n, \rho \otimes \Lambda_1)\) is also a FP if and only if the \(GL_n\)-part of the connected component of the isotropy subgroup of each orbit is not contained in \(SL_n\). In this case, they have the same orbits.

Proposition 1.2. ([KKMOT, Proposition 1.2]) Let \(\sigma : H \rightarrow GL_m\) be a representation of an algebraic group \(H\).

1. If \(m < n\), then \((H \times SL_n, \sigma \otimes \Lambda_1, M(m, n))\) is a FP if and only if \((H \times GL_n, \sigma \otimes \Lambda_1, M(m, n))\) is a FP. In this case, they have the same orbits.

2. If \(m \geq n\) and the number of orbits of \(H \times SL_n\) in \(M(m, n)\) is finite, then \((H \times \sigma \otimes \Lambda_1, M(m, n))\) is a FP if and only if \((H \times GL_n, \sigma \otimes \Lambda_1, M(m, n))\) is a FP. In this case, they have the same orbits.

Next we shall review the relation between the Grassmann variety and finite prehomogeneity ([SK, Section 8]).

Definition 1.3. Let \(V\) be an \(m\)-dimensional vector space. For any \(n\) satisfying \(m \geq n \geq 0\), \(Grass_n(V) = \{W | W\) is an \(n\)-dimensional subspace of \(V\}\) is an \(n(m - n)\)-dimensional variety which is called the Grassmann variety.

Then the following assertion holds.

Proposition 1.4. ([SK, Proposition 1 in Section 8]) (Correspondence of orbits). Let \(G\) be any algebraic group. For \(m \geq n \geq 1\), and for any representation \(\rho : G \rightarrow GL_m\), consider a triplet \((G \times GL_n, \rho \otimes \Lambda_1, M(m, n))\) and a triplet \((G, \rho, \bigcup_{k=0}^{n} Grass_k(V(m)))\) without assuming the prehomogeneity. Then \(G \times GL_n\)-orbits in \(M(m, n)\) correspond bijectively to \(G\)-orbits in \(\bigcup_{k=0}^{n} Grass_k(V(m))\).

In particular, when we assume a number of \(G \times GL_n\)-orbits on \(M(m, n)\) is finite, also a number of \(G\)-orbits on \(\bigcup_{k=0}^{n} Grass_k(V(m))\) is finite. Moreover for any \(t\) satisfying \(n > t \geq 1\), a number of \(G\)-orbits on \(\bigcup_{k=0}^{t} Grass_k(V(m))\) is finite. Therefore a number of \(G \times GL_t\)-orbits on \(M(m, t)\) is finite. In general, if an irreducible algebraic variety \(W\) is decomposed into finitely many orbits by the action of a algebraic group \(H\), \(W\) has a Zariski dense \(H\)-orbit. Hence the following Lemma is obtained, which is fundamental for a classification of FPs.
Lemma 1.5. ([KKMOT, Lemma 1.3]) Let $G$ be any algebraic group, not necessarily reductive, and $\rho$ its representation, not necessarily irreducible.

1. For $m > n \geq 2$, if $(G \times GL_n, \rho \otimes \Lambda_1, V(m) \otimes V(n))$ is a FP, then a triplet $(G \times GL_k, \rho \otimes \Lambda_1, V(m) \otimes V(k))$ is also a FP for any $k$ satisfying $n \geq k \geq 1$.

2. For $n \geq m \geq 2$, if $(G \times GL_n, \rho \otimes \Lambda_1, V(m) \otimes V(n))$ is a FP, then a triplet $(G \times GL_k, \rho \otimes \Lambda_1, V(m) \otimes V(k))$ is also a FP for any $k$.

Remark 1.6. (Castling transform) ([SK, Proposition 7 in section 2]) Let $\rho$ be a representation of an algebraic group $H$ on an $m$-dimensional vector space $V$. For any $n$ satisfying $m > n \geq 1$, the following conditions are equivalent.

1. $(H \times GL_n, \rho \otimes \Lambda_1, V \otimes V(n))$ is a PV.

2. $(H \times GL_{m-n}, \rho^* \otimes \Lambda_1, V \otimes V(m - n))$ is a PV.

3. $(H \times GL_{m-n}, \rho \otimes \Lambda_1, V \otimes V(m - n))$ is a PV if $H$ is reductive.

We say the triplets 1, 2 (resp. 1, 3 if $H$ is reductive) in Remark 1.6 are castling transforms of each other. This castling transformation is essential for the classification of irreducible PVs. However, in general, a castling transform of a FP is not necessarily a FP although it is a PV. For example, a castling transform $(SL_2 \times GL_2, 3\Lambda_1 \otimes \Lambda_1, V(4) \otimes V(3))$ of a FP $(GL_2, 3\Lambda_1, V(4))$ is a PV, but it is not a FP. If it is a FP, then by 1 of Lemma 1.5, $(SL_2 \times GL_2, 3\Lambda_1 \otimes \Lambda_1, V(4) \otimes V(2))$ must be a PV, which is a contradiction by dimension reason.

Proposition 1.7. ([KKMOT, Proposition 1.4]) If $(G \times GL_n, \rho \otimes \Lambda_1)$ with $n \geq 2$ is a FP, then we have $\rho = \rho_1 + \cdots + \rho_k$ with $k = 1,2,3$ where $\rho_1, \ldots, \rho_k$ are irreducible representations.

Here we review the symplectic group $Sp_m$. The action $\Lambda_1$ of $Sp_m$ on $V(2m)$ is given by $x \mapsto Ax$ ($A \in Sp_m$, $x \in V(2m)$) which satisfies $\langle Ax, Ay \rangle = \langle x, y \rangle$ where $\langle x, y \rangle = ^txJy$. Note that this condition is equivalent to $A \in Sp_m$.

Lemma 1.8. ([K, Lemma 7.49]) Let $v_1, \ldots, v_r$ and $u_1, \ldots, u_r$ be linearly independent elements of $V(2m)$ satisfying $\langle v_i, v_j \rangle = \langle u_i, u_j \rangle$ for $i,j = 1, \ldots, r$. Then there exists $A \in Sp_m$ satisfying $u_i = Av_i$ ($i = 1, \ldots, r$).

Now consider the action $\Lambda_1 \otimes \Lambda_1$ of $Sp_m \times GL_n$ on $M(2m,n)$ given by $X \mapsto AX^tB$ for $(A,B) \in Sp_m \times GL_n$ and $X \in M(2m,n)$. Note that this is essentially the same as the action $\Lambda_1 \otimes \Lambda_1$ of $GSp_m \times SL_n$ on $M(2m,n)$ given by $X \mapsto AX^tB$ for $(A,B) \in GSp_m \times SL_n$ and $X \in M(2m,n)$. It is clear that rank $X$ is invariant under the action of the group. Since $^tXJX \mapsto (AX^tB)J(AX^tB) = B(^tXJX)^tB$, rank $^tXJX$ is also
invariant. Since $^tXJX$ is an alternating matrix, its rank is always even. The condition
(rank $X$, rank $^tXJX$) $\neq$ (rank $Y$, rank $^tYJY$) implies that $X$ and $Y$ do not belong to the
same orbit. We shall show the converse.

**Proposition 1.9.** ([KKMOT, Proposition 1.5]) (The orbital decomposition of $(Sp_m \times
GL_n, \Lambda_1 \otimes \Lambda_1)$) If $X, Y \in M(2m,n)$ satisfy rank $X$ = rank $Y$ and rank $^tXJX$ = rank $^tYJY$,
then we have $Y = AX^tB$ for some $(A, B) \in Sp_m \times GL_n$. Hence the orbits of $(Sp_m \times
GL_n, \Lambda_1 \otimes \Lambda_1, M(2m,n))$ are given by

$$O_{p,q} = \{X \in M(2m,n) \mid \text{rank } X = p + q, \text{ rank } ^tXJX = 2q\}$$

with $m \geq p \geq q \geq 0$ and $n \geq p + q$. The orbit $O_{p,q}$ is represented by $X_{p,q} = \begin{pmatrix} I'_p & O & O \\ O & I'_q & O \end{pmatrix} \in
M(2m,n)$ where $I'_p = \begin{pmatrix} I_p \\ O \end{pmatrix} \in M(m,p)$ and $I'_q = \begin{pmatrix} I_q \\ O \end{pmatrix} \in M(m,q)$.

Now we shall calculate the isotropy subalgebra at $X_{p,q}$. The Lie algebra of $Sp_m$ is
given by $Lie(Sp_m) = \{ (A, B) \mid A \in M(m), B, C \in Sym(m) \}$. We divide this matrix to
the block size $(q, p-q, m-p, q, p-q, m-p)$ as follows:

$$\tilde{A} = \begin{pmatrix} A_1 & A_{12} & A_{13} & B_1 & B_{12} & B_{13} \\ A_{21} & A_2 & A_{23} & tB_{12} & B_2 & B_{23} \\ A_{31} & A_{32} & A_3 & tB_{13} & tB_{23} & B_3 \\ C_1 & C_{12} & C_{13} & -tA_1 & -tA_{21} & -tA_{31} \\ tC_{12} & C_2 & C_{23} & -tA_{12} & -tA_2 & -tA_{32} \\ tC_{13} & tC_{23} & C_3 & -tA_{13} & -tA_{23} & -tA_3 \end{pmatrix} \in Lie(Sp_m).$$

Similarly we divide $X_{p,q}$ to the block size $(q, p-q, m-p, q, p-q, m-p) \times (q, p-q, q, n-p-q)$
and also divide $D \in Lie(GL_n)(= M(n))$ to the block size $(q, p-q, q, n-p-q)$ as follows:

$$X_{p,q} = \begin{pmatrix} I_q & O & O & O \\ O & I_{p-q} & O & O \\ O & O & O & O \\ O & O & I_q & O \\ O & O & O & O \end{pmatrix} \in M(2m,n), D = \begin{pmatrix} D_1 & D_{12} & D_{13} & D_{14} \\ D_{21} & D_2 & D_{23} & D_{24} \\ D_{31} & D_{32} & D_3 & D_{34} \\ D_{41} & D_{42} & D_{43} & D_4 \end{pmatrix} \in Lie(GL_n).$$
Then $\tilde{A}X_{p,q} + X_{p,q}^tD = 0$ if and only if $(\tilde{A}, D) = \begin{pmatrix}
A_1 & O & O & B_1 & B_{12} & O \\
A_{21} & A_2 & A_{23} & tB_{12} & B_2 & B_{23} \\
O & O & A_3 & O & tB_{23} & B_3 \\
C_1 & O & O & -tA_1 & -tA_{21} & O \\
O & O & O & O & -tA_2 & O \\
O & O & C_3 & O & -tA_{23} & -tA_3
\end{pmatrix} \begin{pmatrix}
-tA_1 & -tA_{21} & -tC_1 & D_{14} \\
O & -tA_2 & O & D_{24} \\
-tB_{1} & -B_{12} & A_1 & D_{34} \\
O & O & O & D_4
\end{pmatrix}$.

By changing the rows and columns from $(1, \ldots, 6)$ to $(2, 1, 4, 3, 6, 5)$ and from $(1, 2, 3, 4)$ to $(1, 3, 2, 4)$, we obtain the following result.

**Proposition 1.10.** (cf. [KKMOT, Proposition 1.6]) The isotropy subalgebra of $(Sp_m \times GL_n, A_1 \otimes A_1)$ at $X_{p,q} \in M(2m, n)(m \geq p \geq q \geq 0, n \geq p + q)$ is isomorphic to $g_{p,q} = \{(A, D)\}$ where

$$A = \begin{pmatrix}
A_2 & A_{21} & tB_{12} & A_{23} & B_{23} & B_2 \\
O & A_1 & B_1 & O & O & B_{12} \\
O & C_1 & -tA_1 & O & O & -tA_{21} \\
O & O & O & A_3 & B_3 & tB_{23} \\
O & O & O & C_3 & -tA_3 & -tA_{23} \\
O & O & O & O & O & -tA_2
\end{pmatrix}, \quad D = \begin{pmatrix}
-tA_1 & -tC_1 & -tA_{21} & D_{14} \\
-tB_1 & -B_{12} & D_{34} \\
O & O & O & D_4
\end{pmatrix}$$

with the block size $(p - q, q, q, m - p, m - p, p - q) \times (q, q, p - q, n - p - q)$. Hence the isotropy subgroup $G_{p,q}$ at $X_{p,q}$ is locally isomorphic to $(GL(p - q) \times GL(n - p - q) \times Sp_q \times Sp(p - m)) \cdot U(k)$ where $k = (p - q)(2m - 2p + 2q) + \frac{1}{2}(p - q)(p - q + 1) + (p + q)(n - p - q)$.

Similarly the isotropy subalgebra of $(GS_{p_m} \times SL_n, A_1 \otimes A_1, M(2m, n))$ at $X_{p,q}$ $(m \geq p \geq q \geq 0, n \geq p + q)$ is isomorphic to $g'_{p,q} = \{(A', D')\}$ where $A' = \alpha A_{2m} + A, D' = \begin{pmatrix}
-tA_1 & -tC_1 & -tA_{21} & D_{14} \\
-tB_1 & -A_1 & -B_{12} & D_{34} \\
O & O & -\alpha - tA_2 & D_{24} \\
O & O & O & D_4
\end{pmatrix}$ with $\alpha = \frac{1}{p+q} (\text{tr} D_4 - \text{tr} A_2)$.

Here we put $H_{n,q} = \begin{pmatrix}
Sp_q & M(2q, n - 2q) \\
O & T_u(n - 2q)
\end{pmatrix}$, $H^*_{n,m} = \begin{pmatrix}
Sp_m & M(2m, n - 2m) \\
O & GL(n - 2m)
\end{pmatrix}$,

$SH_{n,q} = \begin{pmatrix}
Sp_q & M(2q, n - 2q) \\
O & ST_u(n - 2q)
\end{pmatrix}$, $SH^*_{n,m} = \begin{pmatrix}
Sp_m & M(2m, n - 2m) \\
O & SL(n - 2m)
\end{pmatrix}$,

$H'_{n,q} = SL_{n, q} \cap \begin{pmatrix}
GS_{p_q} & M(2q, n - 2q) \\
O & T_u(n - 2q)
\end{pmatrix}$, and $(H'_{n,m})^* = SL_{n, q} \cap \begin{pmatrix}
GS_{p_m} & M(2m, n - 2m) \\
O & GL(n - 2m)
\end{pmatrix}$. 
**Proposition 1.11. ([KKMOT, Proposition 1.7])**

1. The $GL_n$ (resp. $SL_n$)-part of an isotropy subgroup of $(Sp_m \times GL_n, \Lambda_1 \otimes \Lambda_1)(resp. (Sp_m \times SL_n, \Lambda_1 \otimes \Lambda_1))$ of any orbit contains a subgroup isomorphic to $H_{n,q}$ (resp. $SH_{n,q}$) for some $q$ satisfying $m \geq q$ and $n \geq 2q \geq 0$. If $n > 2m = 2q$, we can replace $H_{n,m}$ (resp. $SH_{n,m}$) by $H_{n,m}^\ast$ (resp. $SH_{n,m}^\ast$).

2. The $(GL_1 \times SL_n)$-part of an isotropy subgroup of $(GL_1 \times Sp_m \times SL_n, \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1)$ contains a subgroup isomorphic to $\{ (\alpha, (\alpha^{-1} A \beta^T)) \mid \alpha \in GL_1, A \in Sp_q, B \in T_u(n - 2q), \det B = \alpha^{2q}, C \in M(2q, n - 2q) \}$ for some $q$ satisfying $m \geq q$ and $n > 2q > 0$.

3. The $SL_n$-part of an isotropy subgroup of $(GSp_m \times SL_n, \Lambda_1 \otimes \Lambda_1)$ contains $H_{n,q}^\ast$. If $n > 2m = 2q$, we can replace $H_{n,m}^\ast$ by $H_{n,m}^{\ast\ast}$.

## 2 A classification

In this section, we classify FPs of type $((Sp_m, GL_m, GL_n) \times GL_n, (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ with $m_1 \geq 2$ and $n \geq 4$ under various restricted scalar multiplications. In the following Theorem 2.1 to Theorem 2.3, we gather the known results which we will use for our classification.

**Theorem 2.1. ([Kac, Theorem 2; SK, Section 8])**

1. $(SL_m \times GL_n, \Lambda_1 \otimes \Lambda_1, V(m) \otimes V(n))$ with $m \geq 1$ and $n \geq 2$,

2. $(SL_m \times SL_n, \Lambda_1 \otimes \Lambda_1, V(m) \otimes V(n))$ with $m \neq n$ and $n \geq 2$,

3. $(Sp_m \times GL_n, \Lambda_1 \otimes \Lambda_1)$ is a FP if and only if $m \geq 1$ and $n \geq 1$.

4. $(Sp_m \times SL_n, \Lambda_1 \otimes \Lambda_1)$ is a FP if and only if $2m < n$ or $n$ is odd ($\geq 1$).

**Theorem 2.2. ([KKY])**

1. $((GL_{m_1} \times GL_{m_2}) \times GL_{n_1}, (\Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ is a FP if and only if $m_1 \geq 1$ and $n \geq 1$.

2. $((Sp_{m_1} \times GL_{m_2}) \times GL_{n_1}, (\Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ is a FP if and only if $m_1 \geq 1$ and $n \geq 1$.

**Theorem 2.3. ([KKMOT, Theorem 2.3])**

1. $((SL_{m_1} \times GL_{m_2}) \times SL_{n_1}, (\Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ $(n \geq 2)$ is a FP if and only if $m_1 \neq n$.

2. $((SL_{m_1} \times SL_{m_2}) \times GL_{n_1}, (\Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ $(n \geq 2)$ is a FP if and only if $m_1 \neq m_2$ or $m_1 = m_2 > n$. 
3. \(((SL_{m_1} \times SL_{m_2}) \times SL_n, (\Lambda_1 \otimes \Lambda_1) \otimes \Lambda_1) (n \geq 2)\) is a FP if and only if \((n \neq m_1, n \neq m_2, n \neq m_1 + m_2, m_1 \neq m_2)\) or with \(m_1 = m_2 > n\).

4. \(((GS_{pm_1} \times SL_{m_2}) \times SL_n, (\Lambda_1 \otimes \Lambda_1) \otimes \Lambda_1) (m_1 \geq 2, n \geq 2)\) is a FP if and only if \(m_2 > n\) or \(n = \text{odd} > m_2\) or \(n > m_2 = \text{odd}\) or \(n > \max\{2m_1, m_2\}\).

5. \(((Sp_{pm_1} \times GL_{m_2}) \times SL_n, (\Lambda_1 \otimes \Lambda_1) \otimes \Lambda_1) (m_1 \geq 2, n \geq 2)\) is a FP if and only if \(n > 2m_1\) or \(n = \text{odd}\).

6. \(((Sp_{pm_1} \times SL_{m_2}) \times GL_n, (\Lambda_1 \otimes \Lambda_1) \otimes \Lambda_1) (m_1 \geq 2, n \geq 2)\) is a FP if and only if \(m_2 > n\) or \(m_2 > 2m_1\) or \(m_2 = \text{odd}\).

7. \(((Sp_{pm_1} \times SL_{m_2}) \times SL_n, (\Lambda_1 \otimes \Lambda_1) \otimes \Lambda_1) (m_1 \geq 2, n \geq 2)\) is a FP if and only if one of the following conditions holds.
   
   (a) \(m_2 > n > 2m_1\) or \(m_2 > n = \text{odd}\),
   
   (b) \(n > 2m_1 + m_2\) and \((m_2 > 2m_1\) or \(m_2 = \text{odd}\)),
   
   (c) \(2m_1 + m_2 > n > m_2, (m_2 > 2m_1\) or \(m_2 = \text{odd}\)\), \(n > 2m_1 + 1\) and \(n \neq m_2\) mod 2.

Here we put \(S(i_1, \ldots, i_t) = \sum_{k=1}^t E_{i_k,k} \in M(n,m) (n \geq i_1 > \cdots > i_t \geq 1)\) where \(E_{i,j}\) denotes the matrix unit in \(M(n,m)\). We also write \(S(i_1, \ldots, i_t)' = \sum_{k=1}^t E_{i_k,k}' \in M(n,t) (n \geq i_1 > \cdots > i_t \geq 1)\) where \(E_{i,j}'\) denotes the matrix unit in \(M(n,t)\). Hence we have \(S(i_1, \ldots, i_t) = (S(i_1, \ldots, i_t)' \mid O) \in M(n,m)\).

Lemma 2.4. ([KKMOT, Lemma 2.4])

1. For any \(q\) and \(m\), \((Sp_q \times GL_m, \Lambda_1 \otimes \Lambda_1) \cong (GS_{p_q} \times SL_m, \Lambda_1 \otimes \Lambda_1)\) is a FP while \((Sp_q \times SL_m, \Lambda_1 \otimes \Lambda_1)\) is a FP if and only if \(2q < m\) or \(m = \text{odd}\).

2. For any \(m\) and \(n\), \((ST_u(n) \times GL_m, \Lambda_1 \otimes \Lambda_1, M(n,m)) \cong (T_u(n) \times SL_m, \Lambda_1 \otimes \Lambda_1, M(n,m))\) is a FP with the orbits represented by \(S(i_1, \ldots, i_t) \in M(n,m) (n \geq i_1 > \cdots > i_t \geq 1)\).

3. If \(m \neq n\), then a triplet \((ST_u(n) \times SL_m, \Lambda_1 \otimes \Lambda_1, M(n,m))\) is also a FP with the orbits represented by \(S(i_1, \ldots, i_t) \in M(n,m) (n \geq i_1 > \cdots > i_t \geq 1)\).

4. For any \(m, n\) and \(q\) with \(n > 2q > 0\), a triplet \((SH_{n,q} \times GL_m, \Lambda_1 \otimes \Lambda_1, M(n,m))\) is a FP where \(SH_{n,q} = \left(\begin{array}{cc} Sp_q & M(2q, n-2q) \\ O & ST_u(n-2q) \end{array}\right)\).

5. For any \(m, n\) and \(q\) with \(n > 2q > 0\) where \(2q < m\) or \(m = \text{odd}\), a triplet \((H_{n,q} \times SL_m, \Lambda_1 \otimes \Lambda_1, M(n,m))\) is a FP where \(H_{n,q} = \left(\begin{array}{cc} Sp_q & M(2q, n-2q) \\ O & T_u(n-2q) \end{array}\right)\).
6. For any $m, n$ and $q$ with $n > 2q > 0$ and $n \neq m$, a triplet $(H_{n,q}' \times SL_m, \Lambda_1 \otimes \Lambda_1, M(n,m))$ is a FP where $H_{n,q}' = SL_n \cap \left( GSp_{q} M(2q, n-2q) \begin{pmatrix} O & T_u(n-2q) \end{pmatrix} \right)$.

7. For $m, n$ and $q$ with $n > 2q > 0$ and $n \neq m \mod 2$ where $2q < m$ or $m = \text{odd}$, a triplet $(SH_{n,q} \times SL_m, \Lambda_1 \otimes \Lambda_1, M(n,m))$ is a FP.

**Theorem 2.5.** ([KKY])

\[
((Sp_m, GL_n, (\Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1)) \text{ is a FP if and only if } m_1 \geq 1 \text{ and } n \geq 1.
\]

When we classify FPs of type $(Sp_m, GL_n, (\Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1)$ with $m_1 \geq 2$ and $n \geq 4$ under various restricted scalar multiplications, the following lemmas are essential.

**Lemma 2.6.** ([KKMOT, Lemma 3.3])

1. $(Sp_m \times (GL_n \times SL_1), \Lambda_1 \otimes (\Lambda_1 \oplus \Lambda_1))$ is a FP if and only if $n > 2m$ or $n = \text{odd}$. More generally, let $S_k$ be a subgroup of $GSp_m \times (SL_n \times GL_1)$ defined by $S_k = \{(A, B, \alpha) | \alpha \in GL_1, A \in GSp_m, \det A = \alpha^k, B \in SL_n\}$. Then $(S_k, \Lambda_1 \otimes (\Lambda_1 \oplus \Lambda_1))$, i.e., $M(2m, n) \oplus V(2m) \ni (X, y) \mapsto (A^X B, \alpha Y) = (\alpha^{k/2m} A^X B, \alpha^{(2m+k)/2m} A^X y)$ with $(A, B, \alpha) \in S_k$ and $A' \in Sp_m$, is a FP if and only if $(n = 1; k \neq -m)$ or $(2m \geq n = \text{even}; k \neq 0)$ or $(2m > n = \text{odd} \geq 3; k \neq 2m/(n - 1), -2m/(n + 1))$ or $n > 2m$.

2. $(GSp_m \times (SL_n \times SL_1), \Lambda_1 \otimes (\Lambda_1 \oplus \Lambda_1))$ is a FP if and only if $n \geq 2$.

3. $(Sp_m \times (GL_n \times SL_1), \Lambda_1 \otimes (\Lambda_1 \oplus \Lambda_1))$ is a FP if and only if $m_1 \geq 2$.

**Lemma 2.7.** ([KKMOT, Lemma 3.4])

1. $(T_u(m) \times (GL_n \times GL_1), \Lambda_1 \otimes (\Lambda_1 \oplus \Lambda_1))$ is a FP if and only if $n > 2m$.

2. $(T_u(m) \times (SL_n \times SL_1), \Lambda_1 \otimes (\Lambda_1 \oplus \Lambda_1))$ is a FP if and only if $m \geq 3$.

3. $(ST_u(m) \times (GL_n \times SL_1), \Lambda_1 \otimes (\Lambda_1 \oplus \Lambda_1))$ is a FP if and only if $m_1 \geq 3, n \geq 2, m \neq n$ and $m \neq n + 1$. 


5. \((ST_u(m) \times (SL_n \times GL_1), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1))\) is a FP if and only if \((n = 1, m \geq 3)\) or 
\((n \geq 2, m \neq n)\). More generally, let \(G_r\) be a subgroup of \(T_u(m) \times (SL_n \times GL_1)\) defined 
by \(G_r = \{(A, B, \alpha) \mid \alpha \in GL_1, A \in T_u(m), \det A = \alpha^r, B \in SL_n\}\). Then \((G_r, \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1))\), i.e., 
\(M(m, n) \oplus V(m) \ni (X, y) \Rightarrow (AX^t B, \alpha Ay)\) with \(\det \alpha A = \alpha^{m+r}\) and 
\((A, B, \alpha) \in G_r\), is a FP if and only if \((n = 1, m \geq 3)\) or \((n \geq 2, r \neq 0, -1, -m)\) or 
\((n \geq 2, r = 0; m \neq n)\) or \((n \geq 2, r = -1; m \neq n + 1)\) or \((n \geq 2, r = -m; m \geq 3)\).

**Lemma 2.8. ([KKMOT, Lemma 3.5])**

1. \((H'_{n,q} \times (GL_m \times SL_1), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1)) (n \geq 2q \geq 0)\) is a FP.

2. \((H'_{n,q} \times (SL_m \times GL_1), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1)) (n > 2q > 0)\) is a FP if \(n > m\).

3. \((H'_{n,q} \times (SL_m \times SL_1), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1))\) is a FP if \(n > m + 2 \geq 5\) and \(n > n - 2q \geq 3\).

**Proposition 2.9.** \(((GSp_{m_1} \times GL_{m_2} \times SL_1) \times SL_n, (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1)) (m_1 \geq 2, n \geq 4)\) 
is a FP.

**Proof.** By Proposition 1.11, the \(SL_n\)-part of an isotropy subgroup of \((GSp_{m_1} \times SL_n, \Lambda_1 \otimes \Lambda_1)\) 
contains \(H'_{n,q}\). Hence by 1 of Lemma 2.8, we have our result.

**Proposition 2.10.** \(((GSp_{m_1} \times SL_{m_2} \times GL_1) \times SL_n, (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1)) (m_1 \geq 2, n \geq 4)\) 
is a FP if and only if \(m_2 > n\) or \(n = odd > m_2\) or \(n > m_2 = odd\) or \(n > max\{2m_1, m_2\}\).

**Proof.** By 4 of Theorem 2.3, these conditions are necessary. If \(m_2 > n\), then it is a FP 
by Proposition 1.2 and Theorem 2.5. So we may assume that \(n > m_2\). By Proposition 
1.10, the \(SL_n\)-part \(H\) of an isotropy subgroup of \((GSp_{m_1} \times SL_n, \Lambda_1 \otimes \Lambda_1)\) of any orbit 
contains \(Sp_{m_1} (2m_1 \geq n = 2n')\) or \(ST_u(n)\) or \(H'_{n,q} (n > 2q > 0)\). By 2 of Lemma 2.6, 
\((Sp_{m_1} \times (SL_{m_2} \times GL_1), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1))\) is a FP if and only if \(n > m_2 = odd\). By 5 of 
Lemma 2.7, \((ST_u(n) \times (SL_{m_2} \times GL_1), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1))\) is a FP in our case. By 2 of Lemma 
2.8, \((H'_{n,q} \times (SL_{m_2} \times GL_1), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1), M(n, m_2) \oplus V(n))\) is a FP for \(n > m_2\). Hence 
we obtain our result.

**Proposition 2.11.** \(((GSp_{m_1} \times SL_{m_2} \times SL_1) \times SL_n, (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1)) (m_1 \geq 2, n \geq 4)\) 
is a FP if and only if \(m_2 > n\) or \(n = max\{2m_1 + 1, m_2 + 1(\geq 3)\}\).

**Proof.** By 3 of Theorem 2.3, \(((SL_{m_2} \times SL_1) \times SL_n, (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1))\) is a FP if and only 
if \(n \neq 1, n \neq m_2, n \neq m_2 + 1\) and \(m_2 \neq 1\). Note that we deal with the case \(n \geq 4\). If 
m_2 > n, then it is a FP by Propositions 1.2 and 2.9. So we assume that \(n > m_2 + 1 \geq 3\).

If \(2m_1 \geq n = even (\leq 2n')\), it is a non FP since \((Sp_{m_1} \times (SL_{m_2} \times SL_1), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1))\) 
with \(n = 2n' > m_2 + 1\) is a non FP by Lemma 2.6. Now we show that it is a non FP 
when \(2m_1 + 1 \geq n = odd\). If we put \(n = 2q + 1\), the \(SL_n\)-part of a generic isotropy 
subgroup of \((GSp_{m_1} \times SL_n, \Lambda_1 \otimes \Lambda_1)\) is \(H = \{(\alpha A \quad \ast) \quad \beta) \mid \alpha \in GL_1, A \in Sp_{2q}\}\). We
show that \((H \times (SL_{m_2} \times SL_1), \Lambda_1 \otimes (\Lambda_1 \boxtimes \Lambda_1))\) is a non FP. First assume that \(m_2 = 1\). If \(((\chi, 1), (\frac{1}{i})) \in M(n, m_2) \oplus V(n)\) is transferred to \(((\chi', 1), (\frac{1}{i}))\) by \(H \times (SL_{m_2} \times SL_1)\), the action \(X \mapsto X'\) is \((SP_q \times SL(m_2), \Lambda_1 \otimes \Lambda_1)\) which is a non FP. When \(m_2 = 2\), we consider similarly an element of type \(((\chi, 1), (\frac{1}{i}))\) with \(X \in M(n - 1, m_2 - 1)\), then the action \(X \mapsto X'\) is \((SP_q \times SL(m_2 - 1), \Lambda_1 \otimes \Lambda_1)\) which is a non FP. If \(n = m_2 + 2\) (resp. \(m_2 = 2\)), see Lemma 2.12 (resp. Lemma 2.13). Hence we may assume \(n > \max\{2m_1 + 1, m_2 + 2\}\) with \(m_2 \geq 3\). In this case, the \(SL_n\)-part of an isotropy subgroup of \((GSP_{m_1} \times SL_n, \Lambda_1 \otimes \Lambda_1)\) of any orbit contains \(ST_u(n)\) or \(H'_{n,q} (n > n - 2q \geq 2)\) by Proposition 1.11. By 4 of Lemma 2.7, \((ST_u(n) \times (SL_{m_2} \times SL_1), \Lambda_1 \otimes (\Lambda_1 \boxtimes \Lambda_1))\) is a FP in our case. If \(n - 2q \geq 3\), \((H'_{n,q} \times SL_{m_2}, \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1, M(n, m_2) \oplus V(n))\) is a FP by Lemma 2.8. If \(n - 2q = 2\), we have \(q = m_1\) and we can replace \(H'_{n,q}\) by \(H'_{n,m_1}\) by Proposition 1.11 and hence it is a FP.

**Lemma 2.12.** \(((GSP_{m_1} \times SL_{m_2} \times SL_1) \times SL(m_2 + 2), (\Lambda_1 \boxtimes \Lambda_1 \boxtimes \Lambda_1) \otimes \Lambda_1)\) with \(m_2 \geq 2m_1\), is a FP.

**Proof.** The process of the proof is similar as that of Proposition 2.1. It is enough to show that \((H'_{n,q} \times (SL_{m_2} \times SL_1), \Lambda_1 \otimes (\Lambda_1 \boxtimes \Lambda_1))\) is a FP when \(m_2 > t > 0\) and \(2q = m_2 - t + 1\) since other cases are proved in Lemma 2.8. The number of orbits related with \(M(m_2 + 2, m_2)''\) is finite by Proposition 1.2. Any point in \(M(m_2 + 2, m_2)'\) is \(H'_{n,q} \times SL_{m_2}\)-equivalent to \(\begin{pmatrix} O & I'_{2q-1} \\ S(i_1, \ldots, i_t) & O \end{pmatrix}\) with \(S(i_1, \ldots, i_t) \in M(t + 1, t)\) and \(I'_{2q-1} = \begin{pmatrix} I_{2q-1} \\ O \end{pmatrix}\). Since the \(H'_{n,q}\)-part of the isotropy subalgebra at this point contains \((-a_{2q-2} + A) \oplus (-a_{-d} + d) \oplus (-a_{d} + * B) | A \in Lie(Sp_q), B \in Lie(T_{u}(n - 2q - 1)), tr B = (2q - 1)a - d\), it is a FP.

**Lemma 2.13.** \(((GSP_{m_1} \times SL_2 \times SL_1) \times SL_n, (\Lambda_1 \boxtimes \Lambda_1 \boxtimes \Lambda_1) \otimes \Lambda_1)\) with \(n > 2m_1 + 1\), is a FP.

**Proof.** Similarly as Lemma 2.12, it is enough to show that \((H'_{n,q} \times (SL_2 \times SL_1), \Lambda_1 \otimes (\Lambda_1 \boxtimes \Lambda_1))\) is a FP when \(m_2 = 2 > t = 1 > 0\). Any point in \(M(n, 2)'\) is transformed to \((e_i, e_1) (n \geq i \geq 2q + 1)\) by \(H'_{n,q} \times SL_2\) and the \(H'_{n,q}\)-part of the isotropy subalgebra contains \(\{(d_{-d} + A) \oplus (-a_{2q-2} + A) \oplus (-2a_{-d} + * B) | A \in Lie(Sp_q), B \in Lie(T_{u}(n - 2q - 1)), tr B = 2qa + d\}\), and hence it is a FP.

**Lemma 2.14.** Let \(SH_{n,q}, SH_{n,q}^*\) and \(H_{n,q}\) \((n > 2q > 0)\) be as in Proposition 1.11.

1. \((SH_{n,q} \times (GL_m \times GL_1), \Lambda_1 \otimes (\Lambda_1 \boxtimes \Lambda_1))\) is a FP.
2. (a) \((SH_{n,q} \times (GL_m \times SL_1), \Lambda_1 \otimes (\Lambda_1 \boxtimes \Lambda_1))\) is a FP if \(n - 2q \geq 3\).
   (b) \((SH_{n,q}^* \times (GL_m \times SL_1), \Lambda_1 \otimes (\Lambda_1 \boxtimes \Lambda_1))\) is a FP if \(n - 2q \geq 2\).
3. (a) \((SH_{n,q} \times (SL_m \times GL_1), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1))\) is a FP if \(n - 2q \geq 3, \ (n - 2q > m\ or\ n \neq m \mod 2\) and \((m > 2q \ or\ m = \text{odd})\).
(b) \((SH_{n,q}^* \times (SL_m \times GL_1), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1))\) is a FP if \(n - 2q \geq 2, \ (n - 2q > m\ or\ n \neq m \mod 2\) and \((m > 2q \ or\ m = \text{odd})\).

4. \((H_{n,q} \times (GL_m \times SL_1), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1))\) is a FP.

5. \((H_{n,q} \times (SL_m \times GL_1), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1))\) is a FP if \(m > 2q\ or\ m = \text{odd}.

6. \((H_{n,q} \times (SL_m \times SL_1), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1))\) is a FP if \(m > 2q + 1.

**Proof.** Just similarly as in the beginning part of the proof of Lemma 2.8, it is enough to show that, for \(m \geq t \geq 0,
(1) M(2q, m - t) \otimes V(2q) \ni (W, x) \mapsto (AW^tD, \alpha Ax),
(2) M(n - 2q, t) \otimes V(n - 2q) \ni (S, y) \mapsto (BS^tC, \alpha By)
are FPs at the same time where \(A \in SP_q, B \in SL_u(n - 2q)\) (resp. \(B \in SL(n - 2q)\) for \(SH_{n,q}, B \in T_u(n - 2q)\) for \(H_{n,q}\)) and the full subgroup of type \(\{(C \in GL_t, D \in GL(m - t))\} of GL_m \times GL_1\) (resp. \(GL_m \times SL_1, SL_m \times GL_1, SL_m \times SL_1\) acts. Hence we have 1 by 1 of Lemmas 2.6 and 2.7. We have 2 by 1 of Theorem 2.3, Lemma 2.6 and 3 of Lemma 2.7.

For 3, (1) and (2) are related with \((\det C)(\det D) = 1\ and\ \alpha \in GL_1.\) First assume that \(t = 0.\) Then \(D \in SL_m\) and (1) is a FP by 2 of Lemma 2.6. (2) becomes \(y \mapsto \alpha By\) which is a FP even when \(\alpha = 1\) since \(n - 2q \geq 2.\) Next assume that \(t = m.\) Then \(C \in SL_m\) and (1) becomes just \(x \mapsto \alpha Ax\) which is a FP even when \(\alpha = 1.\) If \(m = 1,\) (2) for \(SH_{n,q}\) (resp. \(SH_{n,q}^*\)) is a FP by 5 of Lemma 2.7 (resp. 1 of Theorem 2.3) since \(n - 2q \geq 3\) (resp. \(n - 2q \geq 2\)). If \(m \geq 2,\) (2) is a FP since \(m \neq n - 2q.\) Finally assume that \(m > t > 0.\) Then (1) is always a FP (cf. 1 of Lemma 2.6) and the restriction of scalars occurs in the following 3 cases (a)-(c). (a) When \(2q \geq m - t = \text{even},\) we have \(\det D = 1\) (and hence \(\det C = 1\)) in a generic isotropy subgroup of (1). Then (2) for \(SH_{n,q}\) with \(t = 1\) is a FP by 5 of Lemma 2.7 since \(n - 2q \geq 3.\) Since \((n - 2q > m(> t)\ or\ n \neq m \mod 2)\) and \(m \equiv t \mod 2\) implies that \(n - 2q \neq t,\) (2) for \(SH_{n,q}\) with \(t > 2\) (resp. \(SH_{n,q}^*\) with \(t \geq 1\)) is a FP by 5 of Lemma 2.7 (resp. 1 of Theorem 2.3). (b) When \(2q \geq m - t + 1 = \text{even} (= 2(u + 1))\), we have \(\alpha \det D = 1\) in a generic isotropy subgroup of (1). In this case, we have \(t \geq 2\) since \(m > 2q\ or\ m = \text{odd}.\) If we put \((BS^tC, \alpha By) = (B'S^tC', \alpha' B'y)\) with \(B' \in T_u(n - 2q),\ C' \in SL_t,\ \alpha' \in GL_1,\) we see easily that \(\det B' = (\alpha')^r\) with \(r = (n - 2q)/(t - 1).\) Hence this reduces to 5 of Lemma 2.7. We have \(r \neq - (n - 2q)\ since\ otherwise\ \(t = 0,\) a contradiction. When \(n - 2q = t,\) we have \(r \neq 0.\) When \(n - 2q = t + 1,\) we have \(r \neq - 1\ since\ otherwise\ \(t = 0,\) a contradiction. Hence (2) is a FP. (c) When \(2q \geq m - t + 1 = \text{even}(= 2(u + 1))\), we have \(\det D = \alpha\) in the isotropy subgroup of (1) at \((e_1, \ldots, e_{u+1}, e_{q+1}, \ldots, e_{q+u}, e_{u+1})\), and hence \(\det C = \alpha^{-1.}\) If we put \((BS^tC, \alpha By) = (B'S^tC', \alpha' B'y)\) with \(B' \in T_u(n - 2q),\ C' \in SL_t,\ \alpha' \in GL_1,\) we see easily that \(\det B' = (\alpha')^r\) with \(r = -(n - 2q)/(t + 1).\) Hence this reduces to 5 of Lemma 2.7.
When \( t = 1 \), (2) is a FP since \( n - 2q \geq 3 \) for \( SH_{n,q} \) (resp. \( n - 2q \geq 2 \) for \( SH^*_{n,q} \)). When \( t \geq 2 \), we have \( r \neq -(n - 2q) \) since otherwise \( t = 0 \), a contradiction. When \( n - 2q = t \), then clearly \( r \neq 0 \). Since \( n - 2q > m \) or \( n \not\equiv m \) mod 2, we have \( n - 2q \neq t + 1 = m \). Hence (2) is a FP.

For 4, (1) and (2) are FPs at the same time by 1 of Lemmas 2.6 and 2.7.

For 5, (1) and (2) are related with \((\text{det } C)(\text{det } D) = 1\) and \(\alpha \in GL_1\). If \( t = 0 \), then \(D \in SL_m\) and (1) is a FP by 2 of Lemma 2.6 since \( m > 2q \) or \( m = \text{odd} \). (2) becomes \( y \mapsto \alpha By \) which is a FP even when \( \alpha = 1 \). If \( t = m \), then \( C \in SL_m\) and (2) is a FP by 1 of Lemma 2.7. (1) becomes just \( x \mapsto \alpha Ax \) which is a FP even when \( \alpha = 1 \). Finally assume that \( m > t > 0 \). Then (1) is always a FP (cf. 1 of Lemma 2.6) and the restriction of scalars occurs in the following 3 cases (a)-(c). (a) When \( 2q \geq m - t = \text{even} \), we have \( \text{det } D = 1 \) (and hence \( \text{det } C = 1 \)) in a generic isotropy subgroup of (1). However \( \alpha \) remains and (2) is a FP by 1 of Lemma 2.7. (b) When \( 2q \geq m - t + 1 = \text{even} \), we have \( \alpha \text{det } D = 1 \) in a generic isotropy subgroup of (1). In this case, we have \( t \geq 2 \) since \( m > 2q \) or \( m = \text{odd} \). If we put \((BS^tC, \alpha By) = (BS^tC', \alpha' By')\) with \( B' \in T_u(n-2q) \), \( C' \in SL_t \), \( \alpha' \in GL_1 \), we see easily that \( \text{det } B' = (\text{det } B)(\alpha')^r \) with \( r = (n-2q)/(t-1) \). Hence \( \text{det } B' \) and \( \alpha' \) have no relation and (2) is a FP by Lemma 2.7. (c) When \( 2q \geq m - t + 1 = \text{even} \), we have \( \alpha \text{det } D = 1 \) in the isotropy subgroup of (1) at \((e_1, \ldots, e_{u+1}, e_{q+1}, \ldots, e_{q+u}, e_{u+1})\), and hence \( \text{det } C = \alpha^{-1} \). If we put \((BS^tC, \alpha By) = (BS^tC', \alpha' By')\) with \( B' \in T_u(n-2q) \), \( C' \in SL_t \), \( \alpha' \in GL_1 \), we see easily that \( \text{det } B' = (\text{det } B)(\alpha')^r \) with \( r = -(n-2q)/(t+1) \). Hence \( \text{det } B' \) and \( \alpha' \) have no relation, and (2) is a FP by 1 of Lemma 2.7.

For 6, (1) and (2) are related with \((\text{det } C)(\text{det } D) = 1\) and \(\alpha = 1\). If \( t = 0 \), (1) is a FP by 4 of Lemma 2.6 since \( m > 2q + 1 \). (2) becomes just \( y \mapsto By \) with \( B \in T_u(n-2q) \) which is a FP. If \( t = m \), (2) is a FP by 2 of Lemma 2.7 since \( m \geq 2 \). (1) becomes just \( x \mapsto Ax \) with \( A \in Sp_q \) which is a FP. Finally assume that \( m > t > 0 \). (1) is always a FP by 1 of Lemma 2.6, and the restriction of scalars occurs in the following 3 cases (a)-(c). When (a) \( 2q \geq m - t = \text{even} \) (resp. (b) \( 2q \geq m - t + 1 = \text{even} \)), then \( \text{det } D = 1 \) (and hence \( C \in SL_t \)) in a generic isotropy subgroup. However since \( t \geq m - 2q > 1 \) in our case, (2) is a FP by 2 of Lemma 2.7. (c) When \( 2q \geq m - t + 1 = \text{even} \), we have \( \text{det } D = 1 \) (and hence \( C \in SL_t \)) in the isotropy subgroup of (1) at \((e_1, \ldots, e_{u+1}, e_{q+1}, \ldots, e_{q+u}, e_{u+1})\). Since \( m - 1 > 2q \geq m - t + 1 \), we have \( t \geq 3 \), and hence (2) is a FP by 2 of Lemma 2.7.

**Proposition 2.15.** \(((Sp_{m_1} \times GL_{m_2} \times GL_1) \times SL_n, (\Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1)\) \((m_1 \geq 2, n \geq 4)\) is a FP if and only if \(2m_1 < n\) or \(n = \text{odd}\).

**Proof.** By 1 of Lemma 2.4, the condition is necessary. If \(2m_1 < n\) or \(n = \text{odd}\), the \(SL_n\)-part of an isotropy subgroup of \((Sp_{m_1} \times SL_n, \Lambda_1 \otimes \Lambda_1)\) contains \(SH_{n,q}\) \((n > 2q \geq 0)\) by Proposition 1.11. Hence we obtain our result by 1 of Lemmas 2.7 and 2.14.

**Proposition 2.16.** \(((Sp_{m_1} \times GL_{m_2} \times SL_1) \times SL_n, (\Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1)\) \((m_1 \geq 2, n \geq 4)\) is a FP if and only if \(n > 2m_1 + 1\).
Proof. \(((Sp_m_1 \times SL_1) \times SL_n, (\Lambda_1 \boxtimes \Lambda_1) \otimes \Lambda_1)\) is a FP if and only if \(n > 2m_1 + 1\) by 7 of Theorem 2.3. Under this condition, the \(SL_n\)-part of an isotropy subgroup of \((Sp_m_1 \times SL_n, \Lambda_1 \otimes \Lambda_1)\) contains \(SH_{n,q} (n - 2q \geq 3, m_1 \geq q)\) or \(SH_{n,m_1}^* (n - 2m_1 = 2, q = m_1)\) by Proposition 1.11. Hence we obtain our result by 2 of Lemma 2.14.

**Proposition 2.17.** \(((Sp_m_1 \times SL_{m_2} \times GL_1) \times SL_n, (\Lambda_1 \boxtimes \Lambda_1 \boxtimes \Lambda_1) \otimes \Lambda_1)\) \((m_1 \geq 2, n \geq 4)\) is a FP if and only if one of the following conditions holds.

1. \(m_2 > n > 2m_1\) or \(m_2 > n = \text{odd},\)
2. \(n > 2m_1 + m_2\) and \((m_2 > 2m_1\) or \(m_2 = \text{odd}),\)
3. \(2m_1 + m_2 > n > m_2,\) \((m_2 > 2m_1\) or \(m_2 = \text{odd}),\) \(n > 2m_1 + 1\) and \(n \neq m_2\) mod 2.

**Proof.** By 7 of Theorem 2.3, if it is a FP, these conditions are necessary. Assume that \(m_2 > n\). Then by Propositions 1.2 and 2.15, it is a FP if and only if \(n > 2m_1\) or \(n = \text{odd},\) Now assume that the condition 2 or 3 is satisfied. By Proposition 1.11, the \(SL_n\)-part of an isotropy subgroup of \((Sp_m_1 \times SL_n, \Lambda_1 \otimes \Lambda_1)\) contains \(ST_u(n)\) or \(SH_{n,q} (n > 2q > 0)\) or \(SH_{n,m}^* (n - 2q = 2\) and \(q = m_1)\). By 5 of Lemma 2.7, \((ST_u(n) \times (SL_{m_2} \times GL_1), \Lambda_1 \otimes (\Lambda_1 \boxtimes \Lambda_1))\) is a FP in our case. The condition 2 or 3 implies the condition in 3 of Lemma 2.14, and hence we have our result.

**Proposition 2.18.** \(((Sp_m_1 \times GL_{m_2} \times SL_1) \times GL_n, (\Lambda_1 \boxtimes \Lambda_1 \boxtimes \Lambda_1) \otimes \Lambda_1)\) \((m_1 \geq 2, n \geq 4)\) is a FP. Note that in this case, it is always FP without the condition on \(n\) by Lemma 1.5. This is isomorphic to \(((GL_1 \times (Sp_m_1 \times SL_1) \times GL_{m_2}) \times SL_n, (\Lambda_1 \otimes (\Lambda_1 \boxtimes \Lambda_1 \boxtimes \Lambda_1) \otimes \Lambda_1)).\)

**Proof.** If the \(GL_n\)-part of a generic isotropy subgroup contains \(Sp_{n'} (n = 2n')\) or \(T_u(n)\), it is a FP by 1 of Lemma 2.6 (resp. by the 3rd form of 1 of Lemma 2.7). Otherwise it contains \(H_{n,q} (n > 2q > 0)\) by Proposition 1.11. Then by 4 of Lemma 2.14, we have our result.

**Proposition 2.19.** \(((Sp_m_1 \times SL_{m_2} \times GL_1) \times GL_n, (\Lambda_1 \boxtimes \Lambda_1 \boxtimes \Lambda_1) \otimes \Lambda_1)\) \((m_1 \geq 2, n \geq 4)\) is a FP if and only if \(m_2 > n\) or \(m_2 > 2m_1\) or \(m_2 = \text{odd}.\) Note that this is isomorphic to \(((GL_1 \times (Sp_m_1 \times SL_{m_2}) \times GL_1) \times SL_n, (\Lambda_1 \otimes (\Lambda_1 \boxtimes \Lambda_1 \boxtimes \Lambda_1) \otimes \Lambda_1)).\)

**Proof.** These conditions are necessary by 6 of Theorem 2.3. If \(m_2 > n\), it is a FP by Proposition 1.2. So we may assume that \(n \geq m_2 > 2m_1\) or \(n \geq m_2 = \text{odd}.\) By Proposition 1.11, the \(GL_n\)-part of an isotropy subgroup of \((Sp_m_1 \times GL_n, \Lambda_1 \otimes \Lambda_1)\) contains \(Sp_{n'} (2m_1 \geq n = 2n'), T_u(n)\) or \(H_{n,q} (n > 2q > 0).\) Hence by 2 of Lemma 2.6, 1 of Lemma 2.7 and 5 of Lemma 2.14, we have our result.

**Proposition 2.20.** \(((Sp_m_1 \times SL_{m_2} \times SL_1) \times GL_n, (\Lambda_1 \boxtimes \Lambda_1 \boxtimes \Lambda_1) \otimes \Lambda_1)\) \((m_1 \geq 2, n \geq 4)\) is a FP if and only if \(m_2 > n\) or \(m_2 > 2m_1 + 1.\)
Proof. First assume that \( n \geq m_2 \) and \( 2m_1 + 1 \geq m_2 \). Then the \( GL_n \)-part of the isotropy subgroup of \((SL_{m_2} \times GL_n, \Lambda_1 \otimes \Lambda_1, M(m_2, n))\) at \((I_{m_2}, O)\) is \( H = (SL(m_2) \ast O)_Q\). Then \((Sp_{m_1} \times SL_1) \times H\) acts on \(\{(X, O) \in M(2m_1 + 1, n) \mid X \in M(2m_1 + 1, m_2)\}\) as \((Sp_{m_1} \times SL_1) \times SL_{m_2}, (\Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1)\) which is a non FP in our case by 7 of Theorem 2.3. If \( m_2 > n \), then by Propositions 1.2 and 2.18, it is a FP. So we may assume that \( n \geq m_2 > 2m_1 + 1 \). Then, by Proposition 1.11, the \( GL_n \)-part of an isotropy subgroup of \((Sp_{m_1} \times GL_n, \Lambda_1 \otimes \Lambda_1)\) contains \( T_u(n) \) or \( H_{n,q}\) \((n > 2q > 0)\). Since \( m_2 \geq 2\), \( (T_u(n) \times (SL_{m_2} \times SL_1), \Lambda_1 \otimes (\Lambda_1 \oplus \Lambda_1))\) is a FP by 2 of Lemma 2.7. Since \( m_2 > 2m_1 + 1 \geq 2q + 1\), we have our result by 6 of Lemma 2.14.

**Proposition 2.21.** \((Sp_{m_1} \times SL_{m_2} \times SL_1) \times SL_n, (\Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1)\) \((m_1 \geq 2, n \geq 4)\) is a FP if and only if \( m_2 > n \geq 2m_1 + 1 \) or \( n > 2m_1 + m_2 + 1 \) and \( m_2 > 2m_1 + 1 \).

Proof. Assume that \( m_2 > n \). Then by Propositions 1.2 and 2.16, it is a FP if and only if \( n > 2m_1 + 1 \). If \( n = m_2 \), it is a non FP since \((SL_{m_2} \times SL_n, \Lambda_1 \otimes \Lambda_1)\) is a non FP. Hence we assume that \( n > m_2 \). We shall show that if \( 2m_1 + m_2 + 1 \geq n > m_2 \), it is a non FP, or \( n = 2m_1 + m_2 \), it is a non FP since \((Sp_{m_1} \times SL_{m_2}) \subset SL_{2m_1 + m_2}\) etc. Hence we may assume that \( n > m_2 > n > 2m_1 \). Then there exists \( q \) satisfying \( n - 2q = m_2 \) or \( n - 2q = m_2 + 1 \) \((m_1 \geq q \geq 0)\). The \( SL_n \)-part of some isotropy subgroup of \((Sp_{m_1} \times SL_n, \Lambda_1 \otimes \Lambda_1)\) is contained in \( SH_{n,q}^* = (Sp_q \ast O \quad SL(n - 2q))\) by Proposition 1.10. Then \((SH_{n,q}^* \times (SL_{m_2} \times SL_1), \Lambda_1 \otimes (\Lambda_1 \oplus \Lambda_1))\) \((n - 2q = m_2 \) or \( m_2 + 1))\) is a non FP. Hence we may assume that \( n > 2m_1 + m_2 + 1 \). Then by Propositions 1.2 and 2.20, we obtain our result.

**Proposition 2.22.** \((GS_{p_{m_1}} \times SL_{m_2} \times SL_1) \times GL_n, (\Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1)\) \((m_1 \geq 2, n \geq 4)\) is a FP if and only if \( m_2 \geq 2 \).

Proof. If \( m_2 = 1 \), it is a non FP since \((SL_1 \times SL_1) \times GL_n, (\Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1)\) is a non FP by 2 of Theorem 2.3. Assume that \( m_2 \geq 2 \). The \( GL_n \)-part of an isotropy subgroup of \((GS_{p_{m_1}} \times GL_n, \Lambda_1 \otimes \Lambda_1)\) contains \( GS_{p_q}(n = 2q), T_u(n) \) or \( H = (GS_{p_q} M(2q, n - 2q))_O T_u(n - 2q)\) with \( n > 2q > 0 \). By 3 of Lemma 2.6, \((GS_{p_q} \times (SL_{m_2} \times SL_1), \Lambda_1 \otimes (\Lambda_1 \oplus \Lambda_1))\) is a FP. By 2 of Lemma 2.7, \((T_u(n) \times (SL_{m_2} \times SL_1), \Lambda_1 \otimes (\Lambda_1 \oplus \Lambda_1))\) is a FP. Hence it is enough to show that \((H \times (SL_{m_2} \times SL_1), \Lambda_1 \otimes (\Lambda_1 \oplus \Lambda_1))\) \((m_2 \geq 2)\) is a FP. For this, just by the same argument of the beginning part of the proof of Lemma 2.8, it is enough to show that, for any \( t \) satisfying \( m_2 \geq t \geq 0 \),

1. \( M(2q, m_2 - t) \oplus V(2q) \ni (W, x) \mapsto (AW^t D, Ax)\)
2. \( M(n - 2q, t) \ni (S, y) \mapsto (BS^t C, By)\)

are FPs at the same time, where \( A \in GS_{p_q}, D \in GL(m_2 - t), B \in T_u(n - 2q), C \in GL_4 \) and \((\det C)(\det D) = 1\). If \( t = 0 \), then \( D \in SL_{m_2} \) and (1) is a FP by 3 of Lemma 2.6. (2) becomes just \( y \mapsto By \) which is a FP. If \( t = m_2 \), then \( C \in SL_{m_2} \) and (2) is a FP by 2 of
Lemma 2.7. (2) becomes just \( x \mapsto Ax \) which is a FP. Finally assume that \( m_2 > t > 0 \).

Then (1) is a FP by 1 of Lemma 2.6. The restriction of scalars occurs in the following 3 cases (a)-(c).

(a) When \( 2q \geq m_2 - t = \text{even} \) (resp. \( 2q \geq m_2 - t + 1 = \text{even} \)), we have
\[
(\det A)^{m_2-t}(\det D)^{2q} = 1 \quad \text{and} \quad \det C = (\det A)^{(m-t)/2q} \quad \text{(resp.} \quad (\det A)^{(m-t+1)/2q})
\]
in a generic isotropy subgroup. Hence no restriction of scalars occurs in (2). So by 1 of Lemma 2.7, (2) is a FP.

(b) When \( 2q > m_2 - t + 1 = \text{even} \), we have
\[
\det D = (\det A)^{(1-(m_2-t))/2q}\quad \text{and hence} \quad \det C = (\det A)^{(m-t+1)/2q}
\]
in the isotropy subgroup at \((e_1,\ldots,e_{u+1},e_{q+1},\ldots,e_{q+u},e_{u+1}) \in M(2q,m_2-t+1)\) with \( m_2-t = 2u+1 \). Note that if we write \((AW^tD,Ax) = (A'W^tD',\alpha'A'x)\) with \( A' \in Sp_q \), the condition \( D' = \alpha' \) implies that \( D = (\det A)^{(1-(m_2-t))/2q} \). If \( m_2-t > 1 \), we have \( \det C \neq 1 \) and hence (2) is a FP by 1 of Lemma 2.7. If \( m_2-t = 1 \) and \( t \geq 2 \), then (2) is a FP by 2 of Lemma 2.7. Now assume that \( m_2 = 2 \) and \( t = 1 \). By the simple calculation of the isotropy subalgebra, we see that the \( H \)-part of the isotropy subgroup of \((H \times SL_2, \Lambda_1 \otimes \Lambda_1)\) at
\[
\begin{pmatrix}
0 & e^{(2q)}_1 \\
(e_{n-2q})_1 & 0
\end{pmatrix}
\]
contains \( \{(ab \ast \ast, 0 ab^{-1} \ast, 0 0 (ab)^{-1}) \} \), \( A \in Sp_{q-1}, a,b \in GL_1 \} \times T_u(n-2q-1) \subset GL_n \). Hence \((H \times (SL_2 \times SL_1), \Lambda_1 \otimes (\Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1) \) is a FP, and we obtain our result.

Proposition 2.23. \(((GL_1 \times (Sp_{m_1} \times SL_1) \times SL_{m_2}) \times SL_n, (\Lambda_1 \otimes (\Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1) \otimes \Lambda_1) \) \((m_1 \geq 2, n \geq 4)\) is a FP if and only if \( m_2 > n \) or \((n \geq m_2 \text{ and } n \geq 2m_1 + 1)\).

**Proof.** First we show that it is a non FP for \( 2m_1 + 1 \geq n \geq m_2 \). If \( n = m_2 \), it is clearly a non FP. If \( n = 2m_1 + 1 \), it is a non FP since the \( SL_n \)-part of a generic isotropy subgroup of \((GL_1 \times Sp_{m_1} \times SL_1 \times SL_n, \Lambda_1 \otimes (\Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1) \) is \((Sp_{m_1} \times SL_1, \Lambda_1 \oplus \Lambda_1) \subset (SL_n, \Lambda_1) \) and \((Sp_{m_1} \times SL_1) \times SL_{m_2}, (\Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1) \) with \( 2m_1 + 1 > m_2 \) is a non FP by 7 of Theorem 2.3. So we may assume that \( 2m_1 > n > m_2 \). If \( n = 2n' \), it is a non FP since \((Sp_{m_1} \times SL_{m_2} \times SL_1, \Lambda_1 \otimes (\Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1) \) with \( n = 2n' > m_2 \) is a non FP by 4 of Lemma 2.6. If \( n = 2n' + 1 \), it is a non FP since the \( SL_n \)-part of a generic isotropy subgroup of \((GL_1 \times Sp_{m_1} \times SL_1 \times SL_n, \Lambda_1 \otimes (\Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1) \) is \((Sp_{m_1} \times SL_1, \Lambda_1 \oplus \Lambda_1) \subset (SL_n, \Lambda_1) \) and \((Sp_{m_1} \times SL_1) \times SL_{m_2}, (\Lambda_1 \otimes (\Lambda_1 \oplus \Lambda_1)) \) \( (2n' + 1 > m_2) \) is a non FP by 7 of Theorem 2.3. If \( m_2 > n \), then by Propositions 1.2, it reduces to Proposition 2.18, and it is a FP. Finally assume that \( n > m_2 \) and \( n > 2m_1 + 1 \). The \((GL_1 \times SL_n)\)-part \( H \) of an isotropy subgroup of \((GL_1 \times Sp_{m_1} \times SL_1 \times SL_n, \Lambda_1 \otimes (\Lambda_1 \otimes \Lambda_1) \otimes \Lambda_1) \) contains \((GL_1 \times ST_u(n)) \) or \((\alpha, (\alpha^{-1}A \Gamma)) \) \( \alpha \in GL_1, A \in Sp_q, B \in T_u(n-2q), \det B = \alpha^{-q}, C \in M(2q,n-2q) \) with \( n > 2q > 0 \). By 5 of Lemma 2.7, \((ST_u \times (SL_{m_2} \times GL_1), \Lambda_1 \otimes (\Lambda_1 \otimes \Lambda_1)) \) is a FP in our case. Hence, to prove that \((H \times SL(m_2), \Lambda_1 \otimes \Lambda_1) \) is a FP, just similarly as the beginning part of the proof of Lemma 2.8, it is enough to show that for \( n > 2q > 0 \),

1. \( M(2q,m_2-t) \oplus V(2q) \ni (W,z) \mapsto (\alpha^{-1}AW^tD, Ax) \),
2. \( M(n-2q,t) \oplus V(n-2q) \ni (S,y) \mapsto (BS'C, \alpha By) \)

are FPs at the same time where \( \alpha \in GL_1, A \in Sp_q, D \in GL(m_2-t), B \in T_u(n-2q), C \in \text{GL}(n-2q) \).
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GL_t, (det C)(det D) = 1, det B = α^{2q}. If t = 0, then D ∈ SL(m_2) and (1) is a FP by 1 of Lemma 2.6. (2) becomes just y ↦ αBy which is a FP even when det(αB) = α^n = 1 since n > 2m_1 + 1 implies n - 2q ≥ 2. If t = m_2, then C ∈ SL(m_2) and (1) becomes just x ↦ Ax, which is always a FP. Now (2) reduces to 5 in Lemma 2.7 with r = 2q. So if m_2 = 1 and n - 2q ≥ 3, it is a FP. If m_2 = 1 and n - 2q = 2, the condition n > 2m_1 + 1(m_1 ≥ 1) implies q = m_1. In this case, the SL_n-part of a generic isotropy subgroup of (GL_1 × S_p m_1 × SL_n, Λ_1 ⊗ Λ_1 ⊗ Λ_1) is (GSp(m_1) * GL(2)) ∩ SL_n. Since V(2) ⊕ V(2) ⊆ (x, y) ↦ (Bx, αBy) with B ∈ GL_2, det B = α^{2m_1}, is a FP, (2) is a FP. If m_2 ≥ 2, it is a FP by 5 of Lemma 2.7 since r = 2q 7C O, -1, -(n - 2q). Finally assume that m_2 > t > 0. Then (1) is a FP by (1) of Lemma 2.6. The restriction of scalars occurs in the following cases (a)-(c). (a) When 2q ≥ m_2 - t = even (resp. (b),(c) When 2q ≥ m_2 - t + 1 = even), then we have α^{-2q(m_2-t)}(det D)^{2q} = 1 in a generic isotropy subgroup for (a),(b) (resp. in the isotropy subgroup at (e_1, ..., e_{u+1}, e_{q+1}, ..., e_{q+u}, e_{u+1}) ∈ M(2q, m_2 - t + 1) with m_2 - t = 2u + 1 for (c)). Hence we have det C = α^{-(m_2-t)} for (a)-(c). If we write (B'S^g C, αBy) = (B'S^g C', α'B'y) with C' ∈ SL_t, we have det B' = (α')^r with r = (tn - mqn + 2q^{m_2})/m_2. Hence (2) reduces to 5 of Lemma 2.7. If t = 1, then it is a FP for n - 2q ≥ 3. If n - 2q = 2, by the same argument as above, it is also a FP. Assume that t ≥ 2. Then r 7C -(n - 2q) since otherwise we have tn = 0. If n - 2q = t, then we have r 7C 0. If n - 2q = t + 1, then we see that r 7C -1. In both cases, otherwise we have t(n - m_2) = 0. Hence (2) is also a FP by 5 of Lemma 2.7.

Proposition 2.24. ((GL_1 × S_p m_1 × SL_m_2 × SL_1) × SL_n_1, (Λ_1 ⊗ (Λ_1 ⊕ Λ_1) ⊕ Λ_1) (m_1 ≥ 2, n ≥ 4) is a FP if and only if one of the following conditions holds.

1. m_2 > n,
2. m_2 = n > 2m_1 + 1,
3. n > m_2 and n > 2m_1 + 1 and (m_2 > 2m_1 or m_2 = odd).

Proof. First assume that n > m_2 and 2m_1 ≥ m_2 = even. Then it is a non FP since (GL_1 × (S_p m_1 × SL_m_2) × SL_n, Λ_1 ⊗ (Λ_1 ⊕ Λ_1) ⊕ Λ_1) is a non FP in this case by 6 of Theorem 2.3. Next assume that 2m_1 + 1 ≥ n ≥ m_2. If m_2 = n, then the SL_n-part of a generic isotropy subgroup of (GL_1 × SL_m_2 × SL_n, Λ_1 ⊗ Λ_1 ⊗ Λ_1) is SL_n and ((S_p m_1 × SL_1) × SL_n, (Λ_1 ⊕ Λ_1) ⊕ Λ_1) is a non FP in this case by 7 of Theorem 2.3. So it is a non FP. If n = 2m_1 + 1 > m_2, then the SL_n-part of a generic isotropy subgroup of (GL_1 × (S_p m_1 × SL_1) × SL_n, Λ_1 ⊗ (Λ_1 ⊕ Λ_1) ⊕ Λ_1) is S_p m_1 × SL_1 and ((S_p m_1 × SL_1) × SL_m_2, (Λ_1 ⊕ Λ_1) ⊕ Λ_1) is a non FP in this case by 7 of Theorem 2.3. Hence it is a non FP. If 2m_1 ≥ n = 2n' > m_2, it is a non FP since (S_p n' × (SL_m_2 × SL_1), Λ_1 ⊗ (Λ_1 ⊕ Λ_1)) is a non FP by 4 of Lemma 2.6. If 2m_1 ≥ n = 2n' + 1 > m_2, it is a non FP since the SL_n-part of a generic isotropy subgroup of ((GL_1 × S_p m_1 × SL_1) × SL_n, (Λ_1 ⊗ (Λ_1 ⊕ Λ_1) ⊕ Λ_1) is
$$(Sp(n'))^O_1$$ and $$(Sp_n' \times SL_1) \times SL_{m_2}, (\Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1$$ is a non FP in this case by 7 of Theorem 2.3. If $m_2 > n$, then by Propositions 1.2 and 2.9, it is a FP. If $m_2 = n > 2m_1 + 1$, for the orbits related with $M(n)'$, it reduces to $((Sp_{m_1} \times SL_1) \times SL_n, (\Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1)$ which is a FP in this case by 7 of Theorem 2.3. For the orbits related with $M(n)'$, by Proposition 1.2, it reduces to Proposition 2.9. Finally assume that $n > m_2$ and $n > 2m_1 + 1$ and $(m_2 > 2m_1 \text{ or } m_2 = \text{odd})$. Then the $(GL_1 \times SL_n)$-part of an isotropy subgroup of $(GL_1 \times Sp_{m_1} \times SL_n, \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1)$ contains $(GL_1 \times ST_{u}(n))$ or $H = \{(\alpha, (\alpha^{-1} A G')) | \alpha \in GL_1, A \in Sp_q, B \in T_u(n - 2q), det B = \alpha^{2q}, C \in M(2q, n - 2q)\}$ with $n - 2 \geq 2q > 0$. Note that $(ST_u(n) \times (GL_{m_2} \times SL_1), \Lambda_1 \otimes (\Lambda_1 \oplus \Lambda_1))$ is a FP by 3 of Lemma 2.7 since $n \geq 4$. Hence, just similarly as the beginning part of the proof of Lemma 2.8, it is enough to prove that, for $n - 2 \geq 2q > 0$ and $m_2 \geq t \geq 0$,

(1) $M(2q, m_2 - t) \otimes V(2q) \ni (W, x) \mapsto (AW^TD, \alpha^{-1}Ax)$

(2) $M(n - 2q, t) \otimes V(n - 2q) \ni (S, y) \mapsto (\alpha BS^tC, By)$

are FPs at the same time, where $\alpha \in GL_1, A \in Sp_q, D \in GL(m_2 - t), B \in T_u(n - 2q), C \in GL_t, det B = \alpha^{2q}$ and $(det C)(det D) = 1$. If $t = 0$, then $D \in SL(m_2)$ and (1) becomes a FP in our case by 2 of Lemma 2.6. (2) becomes just $y \mapsto By$ which is a FP even when $\alpha = 1$ since $n - 2q \geq 2$. If $t = m_2$, then $C \in SL(m_2)$ and (1) becomes just $x \mapsto \alpha^{-1}Ax$ which is a FP, and $\alpha$ always remains. In (2), put $(\alpha BS^tC, By) = (B'S^tC, \alpha'B'y)$. Then we have det $B' = (\alpha')^{-r}$ so that (2) reduces to 5 of Lemma 2.7 with $r = -n$. If $m_2 = 1$ and $n - 2q \geq 3$, it is a FP. If $m_2 = 1$ and $n - 2q = 2$, it is a FP just similarly as in the proof of Proposition 2.23. If $m_2 \geq 2$, it is a FP by 5. of Lemma 2.7. Finally assume that $m_2 > t > 0$. ThenId (1) is with full scalars and it is a FP. The restriction of scalars happens in the following 3 cases (a)-(c). (a) When $2q \geq m_2 - t = even$, then det $D = 1$ (and hence det $C = 1$) in a generic isotropy subgroup. Then (2) reduces to 5 of Lemma 2.7 with $r = -n$. Hence just similarly as above, we see that (2) is a FP. (b) When $2q \geq m_2 - t + 1 = even$, then $\alpha^{-1}det D = 1$ and hence det $C = \alpha^{-1}$ in a generic isotropy subgroup. Note that in this case, $t \geq 2$ since otherwise we have $2m_1 \geq m_2 = even$, a contradiction. If we put $(\alpha BS'C, By) = (B'S'^tC', \alpha'B'y')$ with $C' \in SL_t$, we have det $B' = (\alpha')^{-r}$ with $r = (tn - n + 2q)/(1 - t)$. Hence (2) reduces to 5 of Lemma 2.7. We have $r \neq -(n - 2q)$ since otherwise qt = 0. If $n - 2q = t$, we have $r = t(n - 1)/(1 - t) \neq 0$. If $n - 2q = t + 1$, then $r \neq -1$ since otherwise $n = 2$, a contradiction. Hence (2) is a FP by 5 of Lemma 2.7. (c) When $2q \geq m_2 - t + 1 = even (= 2(u + 1))$, we have det $D = \alpha^{-1} \in GL_1$ (and hence det $C = \alpha$) in the isotropy subgroup at $(e_1, \ldots, e_{u+1}, e_{q+1}, \ldots, e_{q+u}, e_{u+1})$. Then (2) reduces to 5 of Lemma 2.7 with $r = (tn + n - 2q)/(1 - t)$. If $t = 1$, it is a FP just similarly in the proof of Proposition 2.23. For $t \geq 2$, we have $r \neq -(n - 2q)$ since otherwise qt = 0, a contradiction. When $n - 2q = t$, we have $r = t(n + 1)/(1 - t) \neq 0$. When $n - 2q = t + 1$, we have $r \neq -1$ since otherwise tn = 0. Thus by 5 of Lemma 2.7, it is a FP.

Proposition 2.25. $((Sp_{m_1} \times GL_1 \times (SL_{m_2} \times SL_1)) \times SL_n, (\Lambda_1 \oplus (\Lambda_1 \otimes (\Lambda_1 \oplus \Lambda_1))) \otimes$
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\( \Lambda_1 \) \((m_1 \geq 2, n \geq 4)\) is a FP if and only if one of the following conditions holds.

1. \( m_2 > n = \text{even} > 2m_1 \),

2. \( m_2 > n = \text{odd} \),

3. \( n \geq m_2 \geq 2 \) and \( n > 2m_1 + 1 \).

**Proof.** If \( m_2 = 1 \), then it is a non FP since \((GL_1 \times (SL_1 \times SL_1) \times SL_n, \Lambda_1 \otimes (\Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1)\) is a non FP. So we assume that \( m_2 \geq 2 \). To prove the only if part, it is enough to show that it is a non FP when \( 2m_1 = n = \text{even} \) or \( 2m_1 + 1 = n = \text{odd} \). If \( 2m_1 = n = \text{even} \), it is a non FP since \((Sp_{m_1} \times SL_n, \Lambda_1 \otimes \Lambda_1)\) is a non FP in this case. Now assume that \( 2m_1 + 1 = n = 2n' + 1 \geq m_2 \). Then the \((GL_1 \times SL(2n' + 1))\)-part of a generic isotropy subgroup of \((Sp_{m_1} \times GL_1) \times SL(2n' + 1), (\Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1)\) is \{1, H\} with \( H = (Sp(n')^0_0) \) and \((H \times SL_{m_2}, \Lambda_1 \otimes \Lambda_1)\) is a non FP by 7 of Theorem 2.3. Now assume that \( m_2 > n \). Then by Proposition 1.2, it reduces to the Proposition 2.15, and it is a FP if and only if \( n > 2m_1 + 1 \) or \( n = \text{odd} \), i.e., 1 and 2. Next assume that \( m_2 = n \). For the orbits related with \( M(n)' \), the \((GL_1 \times SL_n)\)-part of an isotropy subgroup of \((GL_1 \times SL_{m_2} \times SL_n, \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1)\) is \{1\} \times SL_n and \((Sp_{m_1} \times SL_1) \times SL_n, (\Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1)\) is a FP if and only if \( n > 2m_1 + 1 \) by 7 of Theorem 2.3. Now assume that \( m_2 = n \). For the orbits related with \( M(n)'' \), it reduce to Proposition 2.15, and it is a FP if and only if \( n > m_1 \) or \( n = \text{odd} \). Hence if \( m_2 = n \), it is a FP if and only if \( n > 2m_1 + 1 \). Finally assume that \( m_2 > n \). Then by Proposition 1.2, it reduces to the Proposition 2.15, and it is a FP if and only if \( n > m_1 + 1 \) and \( n = \text{odd} \). The \( SL_n\)-part of an isotropy subgroup of \((Sp_{m_1} \times SL_n, \Lambda_1 \otimes \Lambda_1)\) contains \( ST_u(n) \) or \( SH_{n,q} \) \((n - 2 \geq 2q > 0)\). By 2 of Lemma 2.7, \((GL_1 \times (SL_{m_2} \times SL_1) \times ST_u(n), (\Lambda_1 \otimes (\Lambda_1 \oplus \Lambda_1)) \otimes \Lambda_1)\) \(\cong (T_u(n) \times (SL_{m_2} \times SL_1), (\Lambda_1 \otimes (\Lambda_1 \oplus \Lambda_1))\) with \( m_2 \geq 2 \) is a FP. When it contains \( SH_{n,q} \), as in the proof of Lemma 2.8, it is enough to show that, for \( n - 2 \geq 2q > 0 \) and \( m_2 = t \geq 0 \),

\begin{align*}
(1) \quad & M(2q, m_2 - t) \oplus V(2q) \ni (W, x) \mapsto (\alpha AW^t D, \alpha Ax) \\
(2) \quad & M(n - 2q, t) \oplus V(n - 2q) \ni (S, y) \mapsto (\alpha BS'C, \alpha By)
\end{align*}

are FPs at the same time, where \( \alpha \in GL_1, A \in Sp_q, B \in GL(m_2 - t), C \in GL_t \) and \((\det C)(\det D) = 1\). If \( t = 0 \), then \( D \in SL(m_2) \) and \((1)\) is a FP by 3 of Lemma 2.6 since \( m_2 \geq 2 \). \((2)\) becomes just \( y \mapsto \alpha By \) which is a FP even when \( \alpha = 1 \) since \( n - 2q \geq 2 \). If \( t = m_2 \), then \( C \in SL(m_2) \) and \((1)\) becomes just \( x \mapsto \alpha Ax \) which is a FP where \( \alpha \) does not vanish. So \((2)\) is a FP by 2 of Lemma 2.7. Finally assume that \( m_2 > t > 0 \). First we deal with the case \( m_2 \geq 3 \). \((1)\) is a FP (cf. 1 of Lemma 2.6) and the restriction of scalars occurs in the following 3 cases \((a)-(c)\). \((a)\) When \( 2q \geq m_2 - t = \text{even} \), we have \( \det(\alpha D) = 1 \) (and hence \( \det C = \alpha^{m_2-t} \)) in a generic isotropy subgroup. If we put \((\alpha BS'C, \alpha By) = (B'S'C', \alpha'B'y) \) with \( C' \in SL_t \), we have \( \det B' = (\alpha')^r \) with \( r = m_2(n - 2q) / (t - m_2) \). Hence \((2)\) is reduced to 5 of Lemma 2.7. If \( t = 1 \), \((2)\) is a FP for \( n - 2q \geq 3 \). If \( t = 1 \) and \( n - 2q = 2 \), as we see in the proof of Proposition 2.24, we can replace \( T_u(2) \) to \( GL_2 \) with the same determinant, and hence \((2)\) is a FP. Assume that \( t \geq 2 \). Then we have \( r \neq -(n - 2q) \) since otherwise we have \( t = 0 \), a contradiction. When
n − 2q = t, then clearly r ≠ 0. When n − 2q = t + 1, then r ≠ −1 since otherwise m₂ = −1, a contradiction. Hence (2) is a FP by 5 of Lemma 2.7. (b) When 2q ≥ m₂ − t + 1 = even, we have det(aD) = 1 (and hence det C = α⁻ᵐ²−t⁻¹) in a generic isotropy subgroup. Then (2) is reduced to 5 of Lemma 2.7 with r = (m₂ + 1)(n − 2q)/(t − m₂ − 1). When t = 1, it is a FP by similar argument as (a). When t ≥ 2, we have r ≠ −(n − 2q) since otherwise we have t = 0. When n − 2q = t, clearly r = 0. When n − 2q = t + 1, we have r ≠ −1 since otherwise m₂ = −2. Hence (2) is a FP. (c) When 2q ≥ m₂ − t + 1 = even, we have det αD = α (and hence det C = α⁻ᵐ²−t⁻¹) in the isotropy subgroup at (e₁, . . . , eₙ₊₁, eₙ₊₁, . . . , eₙ₊₁, eₙ₊₁) ∈ M(2q, m₂ − t) with m₂ − t = 2u + 1. When m₂ − t = 1, we have t ≥ 2 since m₂ ≥ 3. Therefore (2) is a FP by 2 of Lemma 2.7. Assume that m₂ − t ≥ 3. Put (αB'S'C, αB'y) = (B'S'C', α'B'y) with C' ∈ SL₂. Then we have det B' = (α')r with r = (n − 2q)(m₂ − 1)/(1 + t − m₂). Hence (2) reduces to 5 of Lemma 2.7. When t ≥ 2, we have r ≠ 0, −(−n − 2q) and if r = −1, we have n − 2q ≤ t + 1 since otherwise m₂ = 0, a contradiction. Hence (2) is a FP for t ≥ 2. When t = 1, (2) is a FP for n − 2q ≥ 3. If n − 2q = 2, we have q = m₁ and B ∈ STₜ(2) can be replaced by B ∈ SL₂ by Proposition 1.11. Since m₂ − t ≥ 3, we have det C ≠ 1, and (2) is a FP. Finally consider the case m₂ = 2 > t > 0, i.e., t = 1. Put H₉ = {(aA₀ αB) | A ∈ Spq, B ∈ STₜ(n − 2q), α ∈ GL₁} ≅ GL₁ × SHₙ,q. It is enough to show that (H₉ × (SL₂ × SL₁), A₁ ⊕ (A₁ ⊕ A₁)) is a FP. By a direct calculation of the isotropy subalgebra of (H₉ × (SL₂ × SL₁), A₁ ⊕ (A₁ ⊕ A₁)) at (e₁, e₁) with n ≥ i ≥ 2q + 1, each H₉-part contains {([d 0 ]* [0 *]}) A ∈ Lie(Spₙ−₁), B ∈ Lie(Tₜ(n − 2q − 1)) with tr B = (n − 2q)a + d}. Hence one can easily see that it is a FP.

3 A list

Theorem 3.1. If we restrict the scalar multiplications of ((GSpₘ₁ × GLₘ₂ × GL₁) × GLₙ, (A₁ ⊕ A₁ ⊕ A₁) ⊕ A₁) with m₁ ≥ 2 and n ≥ 4, then it is a FP if and only if it is one of the following case.

1. ((GSpₘ₁ × GLₘ₂ × SL₁) × SLₙ, (A₁ ⊕ A₁ ⊕ A₁) ⊕ A₁) with m₁ ≥ 2, n ≥ 4.

2. ((GSpₘ₁ × SLₘ₂ × GL₁) × SLₙ, (A₁ ⊕ A₁ ⊕ A₁) ⊕ A₁) (m₁ ≥ 2, n ≥ 4) with m₂ > n or n = odd > m₂ or n > m₂ = odd or n > max{2m₁, m₂}.

3. ((GSpₘ₁ × SLₘ₂ × SL₁) × SLₙ, (A₁ ⊕ A₁ ⊕ A₁) ⊕ A₁) (m₁ ≥ 2, n ≥ 4) with m₂ > n or n > max{2m₁ + 1, m₂ + 1(≥ 3)}.

4. ((Sₘ₁ × GLₘ₂ × GL₁) × SLₙ, (A₁ ⊕ A₁ ⊕ A₁) ⊕ A₁) (m₁ ≥ 2, n ≥ 4) with 2m₁ < n or n = odd.

5. ((Sₘ₁ × GLₘ₂ × SL₁) × SLₙ, (A₁ ⊕ A₁ ⊕ A₁) ⊕ A₁) (m₁ ≥ 2, n ≥ 4) with n > 2m₁ + 1.

6. ((Sₘ₁ × GLₘ₂ × SL₁) × GLₙ, (A₁ ⊕ A₁ ⊕ A₁) ⊕ A₁) with m₁ ≥ 2, n ≥ 4.
7. \((Sp_{m_1} \times SL_{m_2} \times SL_1) \times GL_n, (\Lambda_1 \ominus \Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1) (m_1 \geq 2, n \geq 4)\) with \(m_2 > n\) or \(m_2 > 2m_1 + 1\).

8. \((Sp_{m_1} \times SL_{m_2} \times SL_1) \times SL_n, (\Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1) (m_1 \geq 2, n \geq 4)\) with \(m_2 > n > 2m_1 + 1\) or \(n > 2m_1 + m_2 + 1\) and \(m_2 > 2m_1 + 1\).

9. \((Sp_{m_1} \times SL_{m_2} \times GL_1) \times GL_n, (\Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1) (m_1 \geq 2, n \geq 4)\) with \(m_2 > n\) or \(m_2 > 2m_1\) or \(m_2 = \text{odd}\).

10. \(((GL_1 \times Sp_{m_1}) \times SL_{m_2} \times SL_1) \times GL_n, ((\Lambda_1 \otimes \Lambda_1) \oplus \Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1) (m_1 \geq 2, n \geq 4)\) with \(m_2 \geq 2\).

11. \(((GL_1 \times Sp_{m_1}) \times SL_{m_2}) \times SL_n, (\Lambda_1 \otimes (\Lambda_1 \oplus \Lambda_1) \oplus \Lambda_1) \otimes \Lambda_1) (m_1 \geq 2, n \geq 4)\) with \(m_2 > n\) or \((n > m_2\) and \(n > 2m_1 + 1\).

12. \((Sp_{m_1} \times SL_{m_2} \times GL_1) \times SL_n, (\Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1) (m_1 \geq 2, n \geq 4)\) with one of the following conditions:

(a) \(m_2 > n > 2m_1\) or \(m_2 > n = \text{odd}\),

(b) \(n > 2m_1 + m_2\) and \((m_2 > 2m_1\) or \(m_2 = \text{odd}\)),

(c) \(2m_1 + m_2 > n > m_2,\) and \(n > 2m_1 + 1,\) and \(n \neq m_2\) mod 2, and \((m_2 > 2m_1\) or \(m_2 = \text{odd}\)).

13. \((GL_1 \times (Sp_{m_1} \times SL_{m_2}) \times SL_1) \times SL_n, (\Lambda_1 \otimes (\Lambda_1 \oplus \Lambda_1) \oplus \Lambda_1) \otimes \Lambda_1) (m_1 \geq 2, n \geq 4)\) with one of the following conditions:

(a) \(m_2 > n,\)

(b) \(m_2 = n > 2m_1 + 1,\)

(c) \(n > m_2\) and \(n > 2m_1 + 1\) and \((m_2 > 2m_1\) or \(m_2 = \text{odd}\)).

14. \((Sp_{m_1} \times GL_1 \times (SL_{m_2} \times SL_1)) \times SL_n, (\Lambda_1 \oplus (\Lambda_1 \otimes (\Lambda_1 \oplus \Lambda_1))) \otimes \Lambda_1) (m_1 \geq 2, n \geq 4)\) with one of the following conditions:

(a) \(m_2 > n = \text{even} > 2m_1,\)

(b) \(m_2 > n = \text{odd},\)

(c) \(n \geq m_2 \geq 2\) and \(n > 2m_1 + 1.\)
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