Josai Mathematical Monographs
vol 6 (2013), pp. 109-124

Upper bounds for the integral moments of Dirichlet L-functions

Keiju Sono

Abstract

In this paper, we apply Soundararajan's method [12] to the Dirichlet L-functions associated to primitive Dirichlet characters and evaluate their $2k$-th integral moments on the critical line under the assumption of the generalized Riemann hypothesis.

1 Introduction

Evaluating the moments of the Riemann zeta function or other L-function is an important problem in analytic number theory. For the Riemann zeta function, we generally consider the following integral:

$$M_k(T) := \int_T^{2T} |\zeta(1/2 + it)|^{2k} \, dt. \tag{1.1}$$

In most cases, we assume that k is positive, but several mathematicians consider the case of k is negative (see [3], [9]). For positive k, it is generally expected that $M_k(T)$ is asymptotic to $C_k T(\log T)^{k^2}$ as $T \to \infty$ for some $C_k > 0$, but in spite of many efforts, such asymptotic formulas have been established only for $k = 1$ (Hardy and Littlewood, [4]) and $k = 2$ (Ingham, [7]).

However, there are many results on the lower or upper bounds for $M_k(T)$. For instance, the lower bound $M_k(T) \gg_k T(\log T)^{k^2}$ was obtained by Ramachandra [10] for positive integers $2k$, by Heath-Brown [5] for all positive rational numbers k, and by Ramachandra [11] for all positive real numbers k under the assumption of the Riemann hypothesis (or simply RH). On the other hand, Heath-Brown ([5], [6]) proved that the estimate $M_k(T) \ll_k T(\log T)^{k^2}$ holds for all $k \in [0, 2]$, assuming RH. Moreover, he showed that the same upper bound holds unconditionally for $k = \frac{1}{v}$, $v \in \mathbf{N}$.

Recently a remarkable advance was accomplished by Soundararajan [12]. He proved that under the assumption of RH, the estimate

$$M_k(T) \ll_{k,\epsilon} T(\log T)^{k^2+\epsilon} \quad (\forall \epsilon > 0) \tag{1.2}$$

holds for all positive k. He started from Hadamard's factorization for $\zeta(s)$, and obtained a good inequality bounding $\log\left|\zeta(\frac{1}{2} + it)\right|$, $t \in \mathbf{R}$ on RH. Using this inequality, he gives certain upper bounds for the measure of the set

$$S(T,V) := \{t \in [T,2T] \mid \log |\zeta(1/2+it)| \geq V\}$$

for each $V > 0$. Since $M_k(T)$ is expressed by

$$\int_T^{2T} |\zeta(1/2+it)|^{2k} dt = 2k \int_{-\infty}^{\infty} e^{2kV} \text{meas}(S(T,V)) dV$$

and the contribution of negative V is relatively small, his upper bounds for $S(T,V)$ enables one to give the estimate (1.2). The techniques used in the argument above are very complicated (at least to the author), but fortunately there is a good survey about his paper (Koltes, [8]), in which many detail computations omitted in the original paper are supplemented.

In this paper, we apply Soundararajan's argument above to the moments of Dirichlet L-functions on the critical line. Assuming the generalized Riemann hypothesis (or simply GRH), we construct the inequality bounding $\log|L(\frac{1}{2}+it,\chi)|$ (Proposition 2.1). This inequality gives an order estimate for $|L(\frac{1}{2}+it,\chi)|$ in Corollary 2.2, which is stronger than the Lindelöf hypothesis for Dirichlet L-functions in t-aspect (we note that the Lindelöf hypothesis for Dirichlet L-functions is a consequence of GRH). In Theorem 4.1, by using this inequality, we evaluate the measure of the set

$$S_\chi(T,V) := \{t \in [T,2T] \mid \log |L(1/2+it,\chi)| \geq V\}$$

for each $V > 0$, where χ is always a primitive character modulo q which is not quadratic (for some technical reason, we do not deal with the quadratic characters, see the proof of Lemma 3.2). These estimates enables us to evaluate the sum of the integral moments of Dirichlet L-functions. Precisely, under the assumption of GRH, the following upper bound is obtained for all $k > 0$ (Theorem 5.1):

$$\sideset{}{^*}\sum_{\substack{\chi (\mathrm{mod} q) \\ \chi^2 \neq \chi_0}} \int_T^{2T} \left| L\left(\frac{1}{2}+it,\chi\right)\right|^{2k} dt \ll_{k,\epsilon} \phi(q) T (\log qT)^{k^2+\epsilon} \quad (\forall \epsilon > 0).$$

Here, the sum above is over all non-quadratic primitive characters modulo q. The implied constant above depends only on k and ϵ. The arguments and computations are almost the same as those in the original paper [12] (in many parts we only have to replace T to qT and multiply c_χ). We omit many computations, since the similar ones are demonstrated in Koltes' survey [8] in detail. The only differences are that we slightly generalize the asymptotic formula introduced in [2] to obtain an upper bound for certain sum over primes involving Dirichlet characters in Lemma 3.1 (the difference arises from the point that $L(s,\chi)$ does not have a pole at $s = 1$), and that we use the orthogonality of Dirichlet characters in Lemma 3.3.

2 The main proposition and an order estimate for $L(\frac{1}{2}+it,\chi)$

Firstly, we prepare the following main proposition, which is an analogous result of Soundararajan's inequality for the Riemann zeta function [12].

Proposition 2.1. *Assume GRH. Let χ be a primitive Dirichlet character modulo q, $T>0$ be sufficiently large, $t\in[T,2T]$, $x\geq 2$. Let $\lambda_0=0.4912\cdots$ be the number which satisfies $e^{-\lambda_0}=\lambda_0+\frac{\lambda_0^2}{2}$. Then, for any $\lambda\geq\lambda_0$, we have*

$$\log\left|L\left(\frac{1}{2}+it,\chi\right)\right|\leq \mathrm{Re}\sum_{n\leq x}\frac{\chi(n)\Lambda(n)}{n^{\frac{1}{2}+\frac{\lambda}{\log x}+it}\log n}\frac{\log(\frac{x}{n})}{\log x}+\frac{1+\lambda}{2}\cdot\frac{\log qT}{\log x}+O\left(\frac{1}{\log x}\right). \tag{2.1}$$

Proof. Let Z_χ be the set of all non-trivial zeros of $L(s,\chi)$ and $\mathfrak{a}=0$ or 1 be the number such that $\chi(-1)=(-1)^{\mathfrak{a}}$. Then, by Hadamard's factorization, we have

$$\frac{L'}{L}(s,\chi)=-\frac{1}{2}\log\frac{q}{\pi}-\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s+\mathfrak{a}}{2}\right)+B(\chi)+\sum_{\rho\in Z_\chi}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right). \tag{2.2}$$

Here, $B(\chi)$ is a constant dependent on χ, whose the real part is given by

$$\mathrm{Re}B(\chi)=-\sum_{\rho\in Z_\chi}\mathrm{Re}\frac{1}{\rho}.$$

We put

$$F_\chi(s)=\mathrm{Re}\sum_{\rho=\frac{1}{2}+i\gamma\in Z_\chi}\frac{1}{s-\rho}=\sum_{\rho=\frac{1}{2}+i\gamma\in Z_\chi}\frac{\sigma-\frac{1}{2}}{(\sigma-\frac{1}{2})^2+(t-\gamma)^2}$$

for $s=\sigma+it$. If $\sigma=\mathrm{Re}(s)\geq\frac{1}{2}$, $F_\chi(s)$ is non-negative. We take the real parts of the both sides of (2.2). Then, by Stirling's formula,

$$-\mathrm{Re}\frac{L'}{L}(s,\chi)=\frac{1}{2}\log(qT)-F_\chi(s)+O(1) \tag{2.3}$$

holds for $t\in[T,2T]$. By integrating both sides of (2.3) with respect to $\sigma=\mathrm{Re}(s)$ from $\frac{1}{2}$ to σ_0 $(>\frac{1}{2})$, we obtain

$$\begin{aligned}
&\log\left|L\left(\frac{1}{2}+it,\chi\right)\right|-\log\left|L\left(\sigma_0+it,\chi\right)\right|\\
&=\left(\frac{1}{2}\log(qT)+O(1)\right)\left(\sigma_0-\frac{1}{2}\right)-\frac{1}{2}\sum_{\rho=\frac{1}{2}+i\gamma\in Z_\chi}\log\frac{(\sigma_0-\frac{1}{2})^2+(t-\gamma)^2}{(t-\gamma)^2}\\
&\leq\left(\sigma_0-\frac{1}{2}\right)\left(\frac{1}{2}\log(qT)-\frac{1}{2}F_\chi(s_0)+O(1)\right)
\end{aligned} \tag{2.4}$$

for $s_0=\sigma_0+it$. On the other hand, as an analogue of Perron's formula, the following identity holds for $x\geq 2$, $c=\max\{1,2-\sigma\}$:

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}-\frac{L'}{L}(s+w,\chi)\frac{x^w}{w^2}dw=\sum_{n\leq x}\frac{\chi(n)\Lambda(n)}{n^s}\log\frac{x}{n}.$$

By moving the path of integration to the left and picking up the residues, the left hand side equals

$$-\frac{L'}{L}(s,\chi)\log x-\left(\frac{L'}{L}(s,\chi)\right)'-\sum_{\rho\in Z_\chi}\frac{x^{\rho-s}}{(\rho-s)^2}-\sum_{k=0}^{\infty}\frac{x^{-2k-s-\mathfrak{a}}}{(2k+s+\mathfrak{a})^2}.$$

Hence the following identity holds:

$$\begin{aligned}-\frac{L'}{L}(s,\chi)=\sum_{n\leq x}\frac{\chi(n)\Lambda(n)}{n^s}\frac{\log\frac{x}{n}}{\log x}+\frac{1}{\log x}\left(\frac{L'}{L}(s,\chi)\right)'\\+\frac{1}{\log x}\sum_{\rho\in Z_\chi}\frac{x^{\rho-s}}{(\rho-s)^2}+\frac{1}{\log x}\sum_{k=0}^{\infty}\frac{x^{-2k-s-\mathfrak{a}}}{(2k+s+\mathfrak{a})^2}.\end{aligned}\tag{2.5}$$

By integrating both sides of (2.5) with respect to $\sigma=\mathrm{Re}(s)$ from σ_0 to ∞, and taking their real parts, we have

$$\begin{aligned}&\log|L(s_0,\chi)|\\&=\mathrm{Re}\left(\sum_{n\leq x}\frac{\chi(n)\Lambda(n)}{n^{s_0}\log n}\frac{\log\frac{x}{n}}{\log x}-\frac{1}{\log x}\frac{L'}{L}(s_0,\chi)\right.\\&\left.+\frac{1}{\log x}\sum_{\rho\in Z_\chi}\int_{\sigma_0}^{\infty}\frac{x^{\rho-s}}{(\rho-s)^2}d\sigma+\frac{1}{\log x}\sum_{k=0}^{\infty}\int_{\sigma_0}^{\infty}\frac{x^{-2k-s-\mathfrak{a}}}{(2k+s+\mathfrak{a})^2}d\sigma\right).\end{aligned}\tag{2.6}$$

The last two integrals are evaluated as follows:

$$\left|\sum_{\rho\in Z_\chi}\int_{\sigma_0}^{\infty}\frac{x^{\rho-s}}{(\rho-s)^2}d\sigma\right|\leq\sum_{\rho\in Z_\chi}\int_{\sigma_0}^{\infty}\frac{x^{\frac{1}{2}-\sigma}}{|\rho-s_0|^2}d\sigma=\frac{x^{\frac{1}{2}-\sigma_0}}{\log x}\cdot\frac{F_\chi(s_0)}{\sigma_0-\frac{1}{2}},$$

and

$$\sum_{k=0}^{\infty}\int_{\sigma_0}^{\infty}\frac{x^{-2k-s-\mathfrak{a}}}{(2k+s+\mathfrak{a})^2}d\sigma=O\left(\frac{1}{\log x}\right),$$

since

$$\int_{\sigma_0}^{\infty}\frac{x^{-s-\mathfrak{a}}}{(s+\mathfrak{a})^2}d\sigma=O\left(\frac{1}{\log x}\right),\quad\left|\int_{\sigma_0}^{\infty}\frac{x^{-2k-s-\mathfrak{a}}}{(2k+s+\mathfrak{a})^2}d\sigma\right|\leq\frac{1}{4k^2\log x}\quad(k\geq 1).$$

In addition, by (2.3), we have

$$\mathrm{Re}\frac{L'}{L}(s_0,\chi) = F_\chi(s_0) - \frac{1}{2}\log(qT) + O(1).$$

Therefore, from (2.6), we have

$$\begin{aligned}\log|L(s_0,\chi)| \le &\mathrm{Re}\sum_{n\le x}\frac{\chi(n)\Lambda(n)}{n^{s_0}\log n}\cdot\frac{\log\frac{x}{n}}{\log x} - \frac{F_\chi(s_0)}{\log x} + \frac{\log(qT)}{2\log x}\\ &+ \frac{x^{\frac{1}{2}-\sigma_0}}{(\log x)^2}\cdot\frac{F_\chi(s_0)}{\sigma_0-\frac{1}{2}} + O\left(\frac{1}{\log x}\right).\end{aligned} \tag{2.7}$$

Combining (2.4) and (2.7), we obtain

$$\begin{aligned}&\log\left|L\left(\frac{1}{2}+it,\chi\right)\right|\\ &\le \mathrm{Re}\sum_{n\le x}\frac{\chi(n)\Lambda(n)}{n^{s_0}\log n}\cdot\frac{\log\frac{x}{n}}{\log x} + \frac{1}{2}(\log qT)\left(\sigma_0 - \frac{1}{2} + \frac{1}{\log x}\right)\\ &+ F_\chi(s_0)\left\{\frac{x^{\frac{1}{2}-\sigma_0}}{(\log x)^2}\cdot\frac{1}{\sigma_0-\frac{1}{2}} - \frac{1}{2}(\sigma_0-\frac{1}{2}) - \frac{1}{\log x}\right\} + O\left(\frac{1}{\log x}\right).\end{aligned} \tag{2.8}$$

We put $\sigma_0 = \frac{1}{2} + \frac{\lambda}{\log x}$ for $\lambda > 0$. Then, the term involving $F_\chi(s_0)$ becomes negative when $\lambda \ge \lambda_0$. Therefore, for such λ, the inequality of this proposition holds. □

As a consequence of Proposition 2.1, we obtain the following order estimates for $L(\frac{1}{2}+it,\chi)$ under the assumption of GRH:

Corollary 2.2. *Assume GRH. Let χ be a primitive Dirichlet character modulo q and $\lambda_0 = 0.4912\cdots$ be the number given in Proposition 2.1. Then, for any $\epsilon > 0$, the inequality*

$$\left|L\left(\frac{1}{2}+it,\chi\right)\right| \le \exp\left(\left(\frac{1+\lambda_0}{4}+\epsilon\right)\frac{\log qT}{\log\log qT}\right) \tag{2.9}$$

holds for $t\in[T,2T]$ when qT is sufficiently large. In particular, we have

$$\left|L\left(\frac{1}{2}+it,\chi\right)\right| \le \exp\left(\frac{3\log qT}{8\log\log qT}\right) \tag{2.10}$$

for $t\in[T,2T]$ when qT is sufficiently large.

Proof. The proof is almost the same as the one in [8], in which the case of the Riemann zeta function is treated. We put $\lambda=\lambda_0$, $x=(\log qT)^{2-\delta}$ $(\delta>0)$ in (2.1). Then, by taking $\delta>0$ sufficiently small, we have

$$\log\left|L\left(\frac{1}{2}+it,\chi\right)\right|$$

$$\leq \sum_{n \leq (\log qT)^{2-\delta}} \frac{1}{\sqrt{n}} + \frac{1+\lambda_0}{2(2-\delta)} \cdot \frac{\log qT}{\log\log qT} + O\left(\frac{1}{\log\log qT}\right)$$
$$\leq \left(\frac{1+\lambda_0}{4} + \epsilon\right) \frac{\log qT}{\log\log qT}.$$

Thus we obtain (2.9). Moreover, since $\lambda_0 < \frac{1}{2}$, the inequality (2.10) holds. □

3 Auxiliary lemmas

In this section, we prepare several lemmas to prove the main theorem in the next section.

Lemma 3.1. *Assume GRH. Let χ be a non-principal character modulo q, $T > 1$ be sufficiently large, and $t \in [T, 2T]$. Then, the following estimate holds:*

$$\sum_{n \leq x} \frac{\chi(n)\Lambda(n)}{n^{2it}} \ll \sqrt{x}\log(qxT). \tag{3.1}$$

Proof. We denote the left hand side of (3.1) by $\psi(x,t;\chi)$, and define $\psi_0(x,t;\chi)$ by

$$\psi_0(x,t;\chi) = \begin{cases} \psi(x,t;\chi) & \text{(if } x \text{ is not power of primes)} \\ \psi(x,t;\chi) - \frac{\chi(x)\Lambda(x)}{2x^{2it}} & (x = p^m, p : \text{prime}, m \in \mathbf{N}). \end{cases}$$

By almost the same argument given in [2] (in this book, the case $t = 0$ is considered), $\psi_0(x,t;\chi)$ is approximated by the integral

$$J(x,T^{'};\chi) := \frac{1}{2\pi i}\int_{c-iT'}^{c+iT^{'}} -\frac{L^{'}(z+2it,\chi)}{L(z+2it,\chi)} \cdot \frac{x^z}{z} dz$$

for sufficiently large $T^{'} > 1$ (if necessary, we slightly change the value of this $T^{'}$ so that $L(z+2it,\chi)$ does not have any zero on $\mathrm{Im}(z) = \pm T^{'}$), and by computing this integral, we have

$$\psi_0(x,t;\chi) = -\sum_{\substack{\rho \in Z_\chi \\ |\mathrm{Im}\rho - 2t| < T'}} \frac{x^{\rho-2it}}{\rho - 2it} - \frac{L^{'}(2it,\chi)}{L(2it,\chi)} \left\{ + \sum_{n \geq 0} \frac{x^{-2n-\mathfrak{a}-2it}}{2n+\mathfrak{a}+2it} \right\}$$
$$+ O\left(\frac{x(\log x)^2}{T'} + \frac{x(\log q(2t+T^{'}))^2}{T'\log x} + \log x\right).$$

Here, the third term of the right hand side appears if and only if $2t < T^{'}$. Easily we have

$$\frac{L^{'}(2it,\chi)}{L(2it,\chi)} \ll (\log qT)^2,$$

$$\sum_{n\geq 0}\frac{x^{-2n-\mathfrak{a}-2it}}{2n+\mathfrak{a}+2it}\ll 1 \quad \text{(uniformly)}.$$

Moreover, under the assumption of GRH, we have

$$\sum_{\substack{\rho=\frac{1}{2}+i\gamma\in Z_\chi\\ |\gamma-2t|<T'}}\left|\frac{x^{\rho-2it}}{\rho-2it}\right|\ll x^{\frac{1}{2}}(\log q(2t+T^{'}))\log T^{'},$$

since $\mathrm{Re}(\rho)=\frac{1}{2}$ for all $\rho\in Z_\chi$. Therefore, we get

$$\begin{aligned}\sum_{n\leq x}\frac{\chi(n)\Lambda(n)}{n^{2it}} \sim &\psi_0(x,t;\chi)\\ &\ll x^{\frac{1}{2}}(\log q(2t+T^{'}))\log T^{'}+(\log qT)^2+\frac{x(\log x)^2}{T'}\\ &+\frac{x(\log q(2t+T^{'}))^2}{T'\log x}+\log x.\end{aligned}$$

By putting $T^{'}=\sqrt{x}$, we obtain (3.1). □

Lemma 3.2. *Let χ be a Dirichlet character modulo q such that $\chi^2\neq\chi_0$ (χ_0 denotes the principal character). Then, under the assumption of GRH, for $T\leq t\leq 2T$, $2\leq x\leq (qT)^2$, $\sigma\geq\frac{1}{2}$, the following estimate holds:*

$$\left|\sum_{\substack{n\leq x\\ n\neq\text{prime}}}\frac{\chi(n)\Lambda(n)}{n^{\sigma+it}\log n}\cdot\frac{\log\frac{x}{n}}{\log x}\right|\ll\log_3(qT).$$

Here, $\log_3 x:=\log\log\log x$.

Proof. The left hand side is the sum over $n=p^k$, where p runs the set of prime numbers and $k\geq 2$. In these, the sum over $n=p^k$, $k\geq 3$ is at most

$$\sum_{\substack{n=p^k\\ p:\text{prime}\\ k\geq 3}}\frac{1}{n^\sigma}\leq\sum_{k\geq 3}\sum_{n=2}^{\infty}\frac{1}{n^{\frac{k}{2}}}=\sum_{k\geq 3}\left(\zeta\left(\frac{k}{2}\right)-1\right)=O(1).$$

Hence our main problem is to evaluate the sum

$$\sum_{\substack{n\leq x\\ n=p^2\\ p:\text{prime}}}\frac{\chi(n)\Lambda(n)}{n^{\sigma+it}\log n}\cdot\frac{\log\frac{x}{n}}{\log x}. \tag{3.2}$$

One can easily see that the difference between $\sum_{p\le z}\frac{\chi(p^2)\log p}{p^{2it}}$ and $\sum_{n\le z}\frac{\chi(n^2)\Lambda(n)}{n^{2it}}$ is at most

$$\left|\sum_{p\le z}\frac{\chi(p^2)\log p}{p^{2it}}-\sum_{n\le z}\frac{\chi(n^2)\Lambda(n)}{n^{2it}}\right|\ll\sqrt{z}(\log z)^2$$

for $z\ge 2$. Therefore, if $\chi^2\ne\chi_0$, by Lemma 3.1 we have

$$\sum_{p\le z}\frac{\chi(p^2)\log p}{p^{2it}}\ll\sqrt{z}\log^2(qzT). \tag{3.3}$$

Moreover, by Abel's summation formula, if $z\le qT$, we have

$$\begin{aligned}\sum_{p\le z}\frac{\chi(p^2)}{p^{2it}}&=\sum_{p\le z}\frac{\chi(p^2)\log p}{p^{2it}}\cdot\frac{1}{\log p}\\&\ll\frac{\sqrt{z}\log^2(qzT)}{\log z}+\int_2^z\frac{\sqrt{u}\log^2(quT)}{u\log^2 u}du\\&\ll\frac{\sqrt{z}}{\log z}\log^2(qT).\end{aligned}\tag{3.4}$$

We return to the sum (3.2). We write $x=y^2$, $\sqrt{2}\le y\le qT$. If $y\le\log^4 qT$,

$$\left|\sum_{p\le y}\frac{\chi(p^2)}{p^{2\sigma+2it}}\cdot\frac{\log\frac{y}{p}}{\log y}\right|\le\sum_{p\le\log^4 qT}\frac{1}{p}=O\left(\log_3 qT\right),$$

since $\sum_{p\le x}\frac{1}{p}\ll\log\log x$. Next, we assume $y>\log^4 qT$. Then

$$\left|\sum_{p\le y}\frac{\chi(p^2)}{p^{2\sigma+2it}}\cdot\frac{\log\frac{y}{p}}{\log y}\right|\le O(\log_3 qT)+\left|\sum_{\log^4 qT<p\le y}\frac{\chi(p^2)}{p^{2\sigma+2it}}\cdot\frac{\log\frac{y}{p}}{\log y}\right|. \tag{3.5}$$

By Abel's summation formula, the second term of the right hand side of (3.5) is the absolute value of

$$\begin{aligned}&\sum_{p\le y}\frac{\chi(p^2)\log\frac{y}{p}}{p^{2it}\log y}\cdot\frac{1}{y^{2\sigma}}-\sum_{p\le\log^4 qT}\frac{\chi(p^2)\log\frac{y}{p}}{p^{2it}\log y}\cdot\frac{1}{(\log^4 qT)^{2\sigma}}\\&+\int_{\log^4 qT}^y\frac{2\sigma}{u^{2\sigma+1}}\sum_{p\le u}\frac{\chi(p^2)\log\frac{y}{p}}{p^{2it}\log y}du.\end{aligned}\tag{3.6}$$

The first and second term of (3.6) is clearly $O(1)$, and the third term becomes

$$2\sigma\int_{\log^4 qT}^y\frac{1}{u^{2\sigma+1}}\left\{\sum_{p\le u}\frac{\chi(p^2)}{p^{2it}}-\frac{1}{\log y}\sum_{p\le u}\frac{\chi(p^2)\log p}{p^{2it}}\right\}du,$$

which is also evaluated by $O(1)$, by using the estimates (3.3) and (3.4). Thus we obtain the result. □

The following lemma holds unconditionally:

Lemma 3.3. *Let $T > 1$ be sufficiently large, $2 \le x \le qT$, and $k \in \mathbf{Z}_{\ge 0}$ satisfies $x^k \le \frac{qT}{\log qT}$. Then, for any sequence $(a_p) \subset \mathbf{C}$, we have*

$$\sum_{\chi (\mathrm{mod} q)} \int_T^{2T} \left| \sum_{p \le x} \frac{\chi(p) a_p}{p^{\frac{1}{2}+it}} \right|^{2k} dt \ll \phi(q) T k! \left(\sum_{p \le x} \frac{|a_p|^2}{p} \right)^k .$$

In particular, there exists certain constants $c_\chi \ge 0$ such that $\sum_{\chi (\mathrm{mod} q)} c_\chi = \phi(q)$ and the estimate

$$\int_T^{2T} \left| \sum_{p \le x} \frac{\chi(p) a_p}{p^{\frac{1}{2}+it}} \right|^{2k} dt \ll c_\chi T k! \left(\sum_{p \le x} \frac{|a_p|^2}{p} \right)^k$$

holds.

Proof. We write

$$\left(\sum_{p \le x} \frac{\chi(p) a_p}{p^{\frac{1}{2}+it}} \right)^k = \sum_{n \le x^k} \frac{\chi(n) a_{k,x}(n)}{n^{\frac{1}{2}+it}}.$$

Then, by the orthogonality of characters, we have

$$\begin{aligned} &\sum_{\chi (\mathrm{mod} q)} \int_T^{2T} \left| \sum_{p \le x} \frac{\chi(p) a_p}{p^{\frac{1}{2}+it}} \right|^{2k} dt \\ &= \phi(q) \sum_{\substack{n,m \le x^k \\ n \equiv m (\mathrm{mod} q) \\ (nm,q)=1}} \frac{a_{k,x}(n) \overline{a_{k,x}(m)}}{\sqrt{nm}} \int_T^{2T} \left(\frac{m}{n} \right)^{it} dt \\ &\le \phi(q) T \sum_{n \le x^k} \frac{|a_{k,x}(n)|^2}{n} + O\left(\phi(q) \sum_{\substack{n,m \le x^k, n \ne m \\ n \equiv m (\mathrm{mod} q)}} \frac{|a_{k,x}(n) a_{k,x}(m)|}{\sqrt{nm} |\log \frac{m}{n}|} \right). \end{aligned} \tag{3.7}$$

Let us evaluate the O-term of (3.7). Since

$$\frac{|a_{k,x}(n) a_{k,x}(m)|}{\sqrt{nm}} \le \frac{1}{2} \left(\frac{|a_{k,x}(n)|^2}{n} + \frac{|a_{k,x}(m)|^2}{m} \right),$$

we have

$$\sum_{\substack{n,m \le x^k, n \ne m \\ n \equiv m (\mathrm{mod} q)}} \frac{|a_{k,x}(n) a_{k,x}(m)|}{\sqrt{nm} |\log \frac{n}{m}|} \le \frac{1}{2} \sum_{\substack{n,m \le x^k, n \ne m \\ n \equiv m (\mathrm{mod} q)}} \left(\frac{|a_{k,x}(n)|^2}{n |\log \frac{m}{n}|} + \frac{|a_{k,x}(m)|^2}{m |\log \frac{m}{n}|} \right)$$

$$= \sum_{n \le x^k} \frac{|a_{k,x}(n)|^2}{n} \sum_{\substack{m \le x^k, m \ne n \\ m \equiv n (\mathrm{mod} q)}} \frac{1}{|\log \frac{m}{n}|}.$$

We decompose the sum over m into $m < n$ and $m > n$. Then the former is

$$\sum_{\substack{m<n \\ m \equiv n (\mathrm{mod} q)}} \frac{1}{|\log \frac{m}{n}|} = \sum_{\substack{m<n \\ m \equiv n (\mathrm{mod} q)}} \frac{1}{|\log (1 - \frac{n-m}{n})|} \le n \sum_{\substack{m<n \\ m \equiv n (\mathrm{mod} q)}} \frac{1}{n-m}$$

$$= n \sum_{\substack{m'<n \\ m' \equiv 0 (\mathrm{mod} q)}} \frac{1}{m'} = n \sum_{m''<\frac{n}{q}} \frac{1}{qm''} \ll \frac{n}{q} \log \frac{n}{q} \ll \frac{x^k}{q} \log \frac{x^k}{q} \ll T.$$

The latter is

$$\sum_{\substack{n<m \le x^k \\ m \equiv n (\mathrm{mod} q)}} \frac{1}{\log \frac{m}{n}} = \sum_{\substack{n<m \le x^k \\ m \equiv n (\mathrm{mod} q)}} \frac{1}{\log (1 + \frac{m-n}{n})} \le \sum_{\substack{1 \le m' \le x^k \\ m' \equiv 0 (\mathrm{mod} q)}} \frac{1}{\log (1 + \frac{m'}{n})}.$$

Further, we decompose this sum into $1 \le m' \le n$ and $n < m' \le x^k$. Since $\frac{1}{\log(1+x)} \le \frac{2}{x} \quad (0 < x \le 1)$, the former is

$$\sum_{\substack{1 \le m' \le n \\ m' \equiv 0 (\mathrm{mod} q)}} \frac{1}{\log (1 + \frac{m'}{n})} \le \sum_{\substack{1 \le m' \le n \\ m' \equiv 0 (\mathrm{mod} q)}} \frac{2n}{m'} \ll \frac{x^k}{q} \log x^k \ll T.$$

If $m' > n$, then $\frac{1}{\log(1+\frac{m'}{n})} \le \frac{1}{\log 2}$ holds. Therefore, the latter is simply evaluated by

$$\sum_{\substack{n<m' \le x^k \\ m' \equiv 0 (\mathrm{mod} q)}} \frac{1}{\log (1 + \frac{m'}{n})} \ll \frac{\frac{x^k}{q}}{\log 2} \ll T.$$

Consequently we have

$$\sum_{\substack{m \le x^k, m \ne n \\ m \equiv n (\mathrm{mod} q)}} \frac{1}{|\log \frac{m}{n}|} \ll T.$$

Therefore, the O-term in (3.7) is evaluated by

$$\ll \phi(q) T \sum_{n \le x^k} \frac{|a_{k,x}(n)|^2}{n}.$$

Finally, by applying Soundararajan's argument [12] to the sum for $\frac{|a_{k,x}(n)|^2}{n}$, we obtain

$$\sum_{\chi(\mathrm{mod} q)} \int_T^{2T} \left| \sum_{p \le x} \frac{\chi(p) a_p}{p^{\frac{1}{2}+it}} \right|^{2k} dt \ll \phi(q) T \sum_{n \le x^k} \frac{|a_{k,x}(n)|^2}{n}$$
$$\ll \phi(q) T k! \left(\sum_{p \le x} \frac{|a_p|^2}{p} \right)^k .$$

□

4 Large values of $L(\frac{1}{2}+it,\chi)$

Now we have prepared all the lemmas to prove the main theorems (Theorem 4.1 and Theorem 5.1). To describe them, we introduce several notations. For a Dirichlet character χ, sufficiently large $T > 1$, and any $V \in \mathbf{R}$, we define the set $S_\chi(T,V)$ by

$$S_\chi(T,V) := \{t \in [T,2T] \,|\, \log |L\,(1/2+it,\chi)| \ge V\}$$

and let $\mu(S_\chi(T,V))$ be its Lebesgue measure. We establish upper bounds for this $\mu(S_\chi(T,V))$ when V is sufficiently large.

Theorem 4.1. *Assume GRH. Let χ be a primitive Dirichlet character modulo q such that $\chi^2 \ne \chi_0$. Then, for each V, the following estimate for $\mu(S_\chi(T,V))$ holds*:
1) *If* $10\sqrt{\mathrm{loglog} qT} < V \le \mathrm{loglog} qT$,

$$\mu(S_\chi(T,V)) \ll \frac{c_\chi T V}{\sqrt{\mathrm{loglog} qT}} \exp\left(-\frac{V^2}{\mathrm{loglog} qT}\left(1 - \frac{4}{\log_3 qT}\right)\right).$$

2) *If* $\mathrm{loglog} qT < V \le \frac{1}{2}(\mathrm{loglog} qT)\log_3 qT$,

$$\mu(S_\chi(T,V)) \ll \frac{c_\chi T V}{\sqrt{\mathrm{loglog} qT}} \exp\left(-\frac{V^2}{\mathrm{loglog} qT}\left(1 - \frac{7V}{4\mathrm{loglog} qT \log_3 qT}\right)^2\right).$$

3) *If* $\frac{1}{2}(\mathrm{loglog} qT)\log_3 qT < V \le \frac{3\log qT}{8\mathrm{loglog} qT}$,

$$\mu(S_\chi(T,V)) \ll c_\chi T \exp\left(-\frac{V}{129} \log V\right).$$

4) *If* $V > \frac{3\log qT}{8\mathrm{loglog} qT}$,

$$\mu(S_\chi(T,V)) = 0.$$

Here, c_χ are the constants given in Lemma 3.3.

Proof. The case 4) directly follows from Corollary 2.2. Henceforth we assume $V \leq \frac{3\log qT}{8\log\log qT}$. For such V, we define $A \geq 1$ by

$$A := \begin{cases} \frac{1}{2}\log_3 qT & (10\sqrt{\log\log qT} < V \leq \log\log qT) \\ \frac{\log\log qT}{2V}\log_3 qT & (\log\log qT < V \leq \frac{1}{2}(\log\log qT)\log_3 qT) \\ 1 & (V > \frac{1}{2}(\log\log qT)\log_3 qT) \end{cases}$$

and put $x = (qT)^{\frac{A}{V}} (< qT)$, $z = x^{\frac{1}{\log\log qT}}$. Combining Proposition 2.1 and Lemma 3.2, we have

$$\begin{aligned} \log\left|L\left(\frac{1}{2}+it,\chi\right)\right| &\leq \left|\sum_{p\leq x}\frac{\chi(p)}{p^{\frac{1}{2}+\frac{\lambda_0}{\log x}+it}}\cdot\frac{\log\frac{x}{p}}{\log x}\right| + \frac{(1+\lambda_0)V}{2A} + O(\log_3(qT)) \\ &\leq S_1(t)+S_2(t)+\frac{(1+\lambda_0)V}{2A}+O(\log_3(qT)), \end{aligned} \tag{4.1}$$

where

$$S_1(t) := \left|\sum_{p\leq z}\frac{\chi(p)}{p^{\frac{1}{2}+\frac{\lambda_0}{\log x}+it}}\cdot\frac{\log\frac{x}{p}}{\log x}\right|, \quad S_2(t) := \left|\sum_{z<p\leq x}\frac{\chi(p)}{p^{\frac{1}{2}+\frac{\lambda_0}{\log x}+it}}\cdot\frac{\log\frac{x}{p}}{\log x}\right|.$$

Now, if $t \in S_\chi(T,V)$ (i.e. $\log|L(\frac{1}{2}+it,\chi)| \geq V$), at least either one of

$$S_1(t) \geq V\left(1-\frac{7}{8A}\right) =: V_1 \quad \text{or} \quad S_2(t) \geq \frac{V}{8A} =: V_2$$

holds. In fact, if both of these are not valid, then $\frac{V}{A}$ is evaluated by $O(\log_3 qT)$, which is a contradiction in any case. We put

$$\mu_i := \mu(\{t \in [T,2T] | S_i(t) \geq V_i\})$$

for $i = 1,2$. Then

$$\mu(S_\chi(T,V)) \leq \mu_1 + \mu_2.$$

First, let us evaluate μ_2. Let k be the largest integer which satisfies $k \leq \frac{V}{A} - 1$. Then one can easily check that the condition $x^k \leq \frac{qT}{\log qT}$ is satisfied, so we can apply Lemma 3.3 to

$$a_p = \begin{cases} 0 & (p \leq z) \\ \frac{1}{p^{\frac{\lambda_0}{\log x}}}\cdot\frac{\log\frac{x}{p}}{\log x} & (z < p \leq x). \end{cases}$$

We have

$$\int_T^{2T} |S_2(t)|^2 dt = \int_T^{2T} \left|\sum_{p\leq x}\frac{\chi(p)a_p}{p^{\frac{1}{2}+it}}\right|^{2k} dt$$

$$\ll c_\chi T k! \left(\sum_{z<p\leq x} \frac{1}{p^{1+\frac{2\lambda_0}{\log x}}} \cdot \frac{\log^2 \frac{x}{p}}{\log^2 x} \right)^k$$
$$\ll c_\chi T (k(\log_3(qT) + O(1)))^k.$$

By combining this and trivial inequality

$$\int_T^{2T} |S_2(t)|^{2k} dt \geq \mu_2 V_2^{2k},$$

we obtain

$$\mu_2 \ll c_\chi T V_2^{-2k} (2k\log_3(qT))^k$$
$$\ll c_\chi T \exp\left(-\frac{V}{2A} \log V \right). \tag{4.2}$$

Next, we evaluate μ_1. If $k \in \mathbf{N}$ satisfies

$$1 \leq k \leq \frac{\log(\frac{qT}{\log qT})}{\log z}, \tag{4.3}$$

then $z^k \leq \frac{qT}{\log qT}$ holds. Therefore, we can apply Lemma 3.3 to

$$a_p = \begin{cases} \frac{1}{p^{\frac{\lambda_0}{\log x}}} \cdot \frac{\log \frac{x}{p}}{\log x} & (p \leq z) \\ 0 \quad (p > z) \end{cases}$$

and we have

$$\int_T^{2T} |S_1(t)|^{2k} dt \ll c_\chi T k! \left(\sum_{p \leq z} \frac{1}{p^{1+\frac{2\lambda_0}{\log x}}} \right)^k$$
$$\leq c_\chi T k! \left(\sum_{p \leq z} \frac{1}{p} \right)^k$$
$$\ll c_\chi T \sqrt{k} \left(\frac{k \log\log qT}{e} \right)^k.$$

By combining this and trivial inequality

$$\int_T^{2T} |S_1(t)|^{2k} dt \geq \mu_1 V_1^{2k},$$

we obtain

$$\mu_1 \ll c_\chi T \sqrt{k} \left(\frac{k \log\log qT}{e V_1^2} \right)^k. \tag{4.4}$$

We take $k \in \mathbf{N}$ by

$$k = \begin{cases} [\frac{V_1^2}{\mathrm{loglog}qT}] & (V \le (\mathrm{loglog}qT)^2) \\ [10V] & (V > (\mathrm{loglog}qT)^2). \end{cases}$$

Then the condition (4.3) is satisfied in any case and the estimate (4.4) becomes

$$\mu_1 \ll \begin{cases} c_\chi T \frac{V_1}{\sqrt{\mathrm{loglog}qT}} (\frac{1}{e})^{\frac{V_1^2}{\mathrm{loglogqT}}} & (V \le (\mathrm{loglog}qT)^2) \\ c_\chi T\sqrt{V} \left(\frac{10V\mathrm{loglog}qT}{eV^2(1-\frac{7}{8A})^2}\right)^{10V} & (V > (\mathrm{loglog}qT)^2). \end{cases} \tag{4.5}$$

Finally, by evaluating the right hand sides of (4.2) and (4.5) carefully in each case (this process is almost the same as the one in [12] or [8]. We only have to replace log$T$ to logqT and multiply the values by c_χ), we obtain the estimates of the theorem. □

5 Moments of the Dirichlet L-functions

We can apply Theorem 4.1 to the evaluation of the integral moments of the Dirichlet L-functions on $\mathrm{Re}(s) = \frac{1}{2}$.

Theorem 5.1. *Assume GRH. Then, for any fixed $k > 0$, we have*

$$\sum_{\substack{\chi(\mathrm{mod}q) \\ \chi^2 \ne \chi_0}}^{*} \int_T^{2T} \left| L\left(\frac{1}{2} + it, \chi\right)\right|^{2k} dt \ll_\epsilon \phi(q)T(\mathrm{log}qT)^{k^2+\epsilon} \quad (\forall \epsilon > 0).$$

Here, the sum above is over all primitive Dirichlet characters modulo q such that $\chi^2 \ne \chi_0$.

Proof. By evaluating the right hand sides of Theorem 4.1 roughly in the cases $10\sqrt{\mathrm{loglog}qT} \le V \le 4k\mathrm{loglog}qT$ and $V > 4k\mathrm{loglog}qT$, we obtain the following estimates:
1) If $10\sqrt{\mathrm{loglog}qT} \le V \le 4k\mathrm{loglog}qT$,

$$\mu(S_\chi(T,V)) \ll c_\chi T(\mathrm{log}qT)^{o(1)}\mathrm{exp}\left(-\frac{V^2}{\mathrm{loglog}qT}\right). \tag{5.1}$$

2) If $V > 4k\mathrm{loglog}qT$,

$$\mu(S_\chi(T,V)) \ll c_\chi T(\mathrm{log}qT)^{o(1)}\mathrm{exp}\left(-4kV\right). \tag{5.2}$$

3) If $V < 10\sqrt{\mathrm{loglog}qT}$, we use the trivial estimate

$$\mu(S_\chi(T,V)) \ll T. \tag{5.3}$$

We rewrite the integral moment as follows. For $T > 0$,

$$\begin{aligned}
&\int_T^{2T} \left|L\left(\frac{1}{2}+it,\chi\right)\right|^{2k} dt \\
&\quad = \int_T^{2T} e^{2k\log|L(\frac{1}{2}+it,\chi)|} dt \\
&\quad = \int_T^{2T} \int_{-\infty}^{\log|L(\frac{1}{2}+it,\chi)|} 2ke^{2kV}\,dV\,dt \\
&\quad = 2k\int_T^{2T} \int_{-\infty}^{\infty} \mathbf{1}_{\{t|\log|L(\frac{1}{2}+it,\chi)|\geq V\}}(t)e^{2kt}dV\,dt \\
&\quad = 2k\int_{-\infty}^{\infty} e^{2kV}\mu(S_\chi(T,V))dV \\
&\quad = 2k\left(\int_{-\infty}^{10\sqrt{\log\log qT}} + \int_{10\sqrt{\log\log qT}}^{4k\log\log qT} + \int_{4k\log\log qT}^{\infty}\right) e^{2kV}\mu(S_\chi(T,V))dV \\
&\quad =: 2kI_1 + 2kI_2 + 2kI_3,
\end{aligned}$$

say. Here, $\mathbf{1}_{\{t|\log|L(\frac{1}{2}+it,\chi)|\geq V\}}(t)$ is the characteristic function on the set $\{t \mid \log|L(\frac{1}{2}+it,\chi)| \geq V\}$. First, by (5.3),

$$I_1 \ll T\int_{-\infty}^{10\sqrt{\log\log qT}} e^{2kV}dV \ll Te^{20k\sqrt{\log\log qT}} \ll T(\log qT)^{k^2}.$$

Next, by (5.1),

$$\begin{aligned}
I_2 &\ll c_\chi T(\log qT)^{o(1)} \int_{10\sqrt{\log\log qT}}^{4k\log\log qT} e^{2kV-\frac{V^2}{\log\log qT}}dV \\
&\leq c_\chi T(\log qT)^{o(1)} \int_{10\sqrt{\log\log qT}}^{4k\log\log qT} e^{k^2\log\log qT}dV \\
&\ll c_\chi T(\log qT)^{o(1)} \cdot 4k(\log\log qT)(\log qT)^{k^2} \\
&\ll_\epsilon c_\chi T(\log qT)^{k^2+\epsilon}.
\end{aligned}$$

Finally, by (5.2),

$$I_3 \ll c_\chi T(\log qT)^{o(1)} \int_{4k\log\log qT}^{\infty} e^{-2kV}dV \ll_\epsilon c_\chi T(\log qT)^{-8k^2+\epsilon}.$$

Combining these, we obtain

$$\int_T^{2T} \left|L\left(\frac{1}{2}+it,\chi\right)\right|^{2k} dt \ll_\epsilon c_\chi T(\log qT)^{k^2+\epsilon}.$$

Since the constants c_χ satisfy

$$0 \leq \sideset{}{^*}\sum_{\substack{\chi(\mathrm{mod}\, q)\\ \chi^2\neq\chi_0}} c_\chi \leq \sum_{\chi(\mathrm{mod}\, q)} c_\chi = \phi(q),$$

by taking the sum over the Dirichlet characters satisfying the conditions, we obtain the result. □

References

[1] J.B. Conrey, A. Ghosh *On mean values of the zeta-function*, Mathematika, **31** (1984), 159-161

[2] H. Davenport *Multiplicative Number Theory*, Springer, New York (1980)

[3] S.M. Gonek *On negative moments of the Riemann zeta-function*, Mathematika **36** (1989), No.1, 71-88

[4] G.H. Hardy, J.E. Littlewood *Contributions to the theory of the Riemann zeta-function and the theory of the distribution of primes* , Acta Math. **41** (1918), 119-196

[5] D.R. Heath-Brown *Fractional moments of the Riemann zeta function*, J. London Math. Soc. **24** (1981), 65-78

[6] D.R. Heath-Brown *Fractional moments of the Riemann zeta function II*, Quart. J. Math. Oxford Ser. **44** (1993), 185-197

[7] A.E. Ingham *Mean-value theorems in the theory of the Riemann zeta-function*, Proc. London Math. Soc. **27** (1926), 273-300

[8] T. Koltes *Soundararajan's upper bound on moments of the Riemann zeta function*, Bachelor Thesis, Swiss Federal Institute of Technology Zurich, Department of Mathematics

[9] A. Laurinčikas *A remark on negative moments of the Riemann zeta-function*, Lith. Math. J. **40**, No.1 (2000), 23-28

[10] K. Ramachandra *Some remarks on the mean value of the Riemann zeta function and other Dirichlet series. I* , Hardy-Ramanujan J. **1** (1978), 15pp

[11] K. Ramachandra *Some remarks on the mean value of the Riemann zeta function and other Dirichlet series. II* , Hardy-Ramanujan J. **3** (1980), 1-24

[12] K. Soundararajan *Moments of the Riemann zeta function*, Ann. of Math. (2) **170** (2009), No.2, 981-993

Graduate School of Mathematical Sciences,
University of Tokyo,
Komaba, Meguro,
Tokyo, Japan
E-mail address: souno@ms.u-tokyo.ac.jp