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On a certain class of cuspidal prehomogeneous vector spaces and its basic relative invariants

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Abstract

In this note, we give a certain class of cuspidal prehomogeneous vector spaces and determine explicitly two basic relative invariants of a cuspidal prehomogeneous vector space $(GL(4) \times GL(3) \times SL(2), \Lambda_1^* \otimes 1 \otimes 1 + \Lambda_1 \otimes \Lambda_1^* \otimes \Lambda_1^*, M(1,4) \oplus M(4,3) \oplus M(4,3))$ which is a special case of the class. We consider everything over the complex number field $\mathbb{C}$.

Introduction

Let G be a linear algebraic group and ρ its rational representation on a finite dimensional vector space V, all defined over the complex number field $\mathbb{C}$. If V has a Zariski-dense G-orbit $\mathbb{O}$, we call the triplet (G, ρ, V) a *prehomogeneous vector space*. In this case, we call $v \in \mathbb{O}$ a *generic point*, and the isotropy subgroup $G_v = \{g \in G \mid \rho(g)v = v\}$ at v is called a *generic isotropy subgroup*. We call a prehomogeneous vector space (G, ρ, V) a *reductive* prehomogeneous vector space if G is reductive.

Let $\rho : G \to GL(V)$ be a rational representation of a linear algebraic group G on an m-dimensional vector space V and let n be a positive integer with $m > n$. A triplet $\mathcal{C}_1 := (G \times GL(n), \rho \otimes \Lambda_1, V \otimes V(n))$ is a prehomogeneous vector space if and only if a triplet $\mathcal{C}_2 := (G \times GL(m-n), \rho^* \otimes \Lambda_1, V^* \otimes V(m-n))$ is a prehomogeneous vector space. We say that $\mathcal{C}_1$ and $\mathcal{C}_2$ are the *castling transforms* of each other. Two triplets are said to be *castling equivalent* if one is obtained from the other by a finite number of successive castling transformations.

Assume that (G, ρ, V) is a prehomogeneous vector space with a Zariski-dense G-orbit $\mathbb{O}$. A non-zero rational function $f(v)$ on V is called a *relative invariant* if there exists a rational character $\chi : G \to GL(1)$ satisfying $f(\rho(g)v) = \chi(g)f(v)$ for $g \in G$. In this case, we write $f \leftrightarrow \chi$. Let

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$S_i = \{v \in V \mid f_i(v) = 0\}$ $(i = 1, \dots, l)$ be irreducible components of $S := V \setminus \mathbb{O}$ with codimension one. When G is connected, these irreducible polynomials $f_i(v)$ $(i = 1, \dots, l)$ are algebraically independent relative invariants and any relative invariant $f(v)$ can be expressed uniquely as $f(v) = cf_1(v)^{m_1} \cdots f_l(v)^{m_l}$ with $c \in \mathbb{C}^\times$ and $m_1, \dots, m_l \in \mathbb{Z}$. These $f_i(v)$ $(i = 1, \dots, l)$ are called the *basic relative invariants* of (G, ρ, V).

Prehomogeneous vector spaces (G, ρ, V) with $\dim G = \dim V$ are called *cuspidal* prehomogeneous vector spaces (cf. [CoMc]). Cuspidal prehomogeneous vector spaces are important in the sense of *contraction* (cf. [Gy]) of prehomogeneous vector spaces and include arithmetical interesting examples such as the space of binary cubic forms and $(SL(5) \times GL(4), \Lambda_2 \otimes \Lambda_1, V(10) \otimes V(4))$. However it is very difficult to determine the structures of the basic relative invariants and its b-function. For example, the microlocal structure of $(SL(5) \times GL(4), \Lambda_2 \otimes \Lambda_1, V(10) \otimes V(4))$ is most complicated in all irreducible prehomogeneous vector spaces.

In this note, we give a certain class of cuspidal prehomogeneous vector spaces and determine explicitly two basic relative invariants of a cuspidal prehomogeneous vector space $(GL(4) \times GL(3) \times SL(2), \Lambda_1^* \otimes 1 \otimes 1 + \Lambda_1 \otimes \Lambda_1^* \otimes \Lambda_1^*, M(1,4) \oplus M(4,3) \oplus M(4,3))$ which is a special case of the class. This is an interesting example of the cuspidal prehomogeneous vector spaces.

In Section 1, we give a certain class of cuspidal prehomogeneous vector spaces which was observed in [Kas, Theorem 3.22]. In Section 2, we construct two basic relative invariants of the cuspidal prehomogeneous vector space $(GL(4) \times GL(3) \times SL(2), \Lambda_1^* \otimes 1 \otimes 1 + \Lambda_1 \otimes \Lambda_1^* \otimes \Lambda_1^*, M(1,4) \oplus M(4,3) \oplus M(4,3))$ which is a special case of the class in Section 1.

Notation

Let V be an n-dimensional vector space spanned by $u_1, \dots, u_n$. For $G = GL(n)$, $SL(n)$, we denote by Λ_1 a representation of G on V which is defined by $(u_1, \dots, u_n) \mapsto (u_1, \dots, u_n)g$ for $g \in G$. Let $\bigwedge^k V$ $(1 \leq k \leq n-1)$ be a vector space spanned by exterior products $u_{i_1} \wedge \cdots \wedge u_{i_k}$ $(1 \leq i_1 < \cdots < i_k \leq n)$. We denote by Λ_k $(1 \leq k \leq n-1)$ a representation of $SL(n)$ on $\bigwedge^k V$ which is defined by $u_{i_1} \wedge \cdots \wedge u_{i_k} \mapsto \Lambda_1(g)u_{i_1} \wedge \cdots \wedge \Lambda_1(g)u_{i_k}$ for $g \in SL(n)$. Let $S^k V$ $(k \geq 1)$ be a vector space spanned by symmetric tensor products $u_{i_1} \cdots u_{i_k}$ $(1 \leq i_1 \leq \cdots \leq i_k \leq n)$. We denote by $k\Lambda_1$ $(k \geq 1)$ a representation of $SL(n)$ on $S^k V$ which is defined by $u_{i_1} \cdots u_{i_k} \mapsto \Lambda_1(g)u_{i_1} \cdots \Lambda_1(g)u_{i_k}$ for $g \in SL(n)$. We denote by ρ^* the contragredient representation of a rational representation ρ. For a rational representation ρ, $\rho^{(*)}$ stands for ρ or ρ^*. We denote by $V(n)$ an n-dimensional vector space. If $V(n)$ and $V(n)^*$ appear at the same time, $V(n)^*$ denotes the dual space of $V(n)$. We use $+$ instead of $\oplus$ if $\otimes$ and $\oplus$ appear at the same time.

1 A certain class of cuspidal prehomogeneous vector spaces

In [Kas, Theorem 3.22], a certain class of cuspidal prehomogeneous vector spaces was observed.

Theorem 1.1 (S. Kasai). *Let $\rho : G \longrightarrow GL(V(m))$ be an irreducible rational representation of a connected semisimple linear algebraic group G with the finite kernel. Assume that a triplet $\mathcal{P} := (G \times SL(2) \times GL(l), \rho \otimes 3\Lambda_1 \otimes \Lambda_1, V(m) \otimes V(4) \otimes V(l))$ is castling equivalent to $(SL(2) \times GL(1), 3\Lambda_1 \otimes \Lambda_1, V(4) \otimes V(1))$. Then $\mathcal{T} := (G \times GL(4l) \times GL(3l) \times SL(2), \rho \otimes \Lambda_1^{(*)} \otimes 1 \otimes 1 + 1 \otimes \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1, V(m) \otimes V(4l)^{(*)} + V(4l) \otimes V(3l) \otimes V(2))$ is a reductive cuspidal prehomogeneous vector space.*

Remark 1.2. *The triplet $(GL(4l) \times GL(3l) \times SL(2), \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1, V(4l) \otimes V(3l) \otimes V(2))$ is obtained from the regular trivial prehomogeneous vector space $(GL(2l) \times GL(l) \times SL(2), \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1, V(2l) \otimes V(l) \otimes V(2))$ by applying a castling transformation two times.*

Remark 1.3. *A correction to [Kas, Theorem 3.22] is given in [Ku, Correction 1.2].*

Example 1.4. *If $G = \{1\}$ and $l = 1$, then $\mathcal{P} = (SL(2) \times GL(1), 3\Lambda_1 \otimes \Lambda_1, V(4) \otimes V(1))$. By Theorem 1.1, $\mathcal{T} = (GL(4) \times GL(3) \times SL(2), \Lambda_1^{(*)} \otimes 1 \otimes 1 + \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1, V(4)^{(*)} + V(4) \otimes V(3) \otimes V(2))$ is a reductive cuspidal prehomogeneous vector space.*

Example 1.5. *If $(G, \rho) = (SL(3), \Lambda_1)$ and $l = 11$, then $\mathcal{P} = (SL(3) \times SL(2) \times GL(11), \Lambda_1 \otimes 3\Lambda_1 \otimes \Lambda_1, V(3) \otimes V(4) \otimes V(11))$ is castling equivalent to $(SL(2) \times GL(1), 3\Lambda_1 \otimes \Lambda_1, V(4) \otimes V(1))$. By Theorem 1.1, $\mathcal{T} = (SL(3) \times GL(44) \times GL(33) \times SL(2), \Lambda_1 \otimes \Lambda_1^{(*)} \otimes 1 \otimes 1 + 1 \otimes \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1, V(3) \otimes V(44)^{(*)} + V(44) \otimes V(33) \otimes V(2))$ is a reductive cuspidal prehomogeneous vector space.*

Example 1.6. *We define a sequence $\{a_i\}_{i \geq 0}$ by $a_0 = a_1 = 1$ and $a_{i+2} = 4a_{i+1} - a_i$ $(i \geq 0)$. Put $A_i := (SL(a_i) \times SL(2) \times GL(a_{i+1}),\ \Lambda_1 \otimes 3\Lambda_1 \otimes \Lambda_1,\ V(a_i) \otimes V(4) \otimes V(a_{i+1}))$ $(i \geq 0)$. Then we see that $A_0 = (SL(2) \times GL(1),\ 3\Lambda_1 \otimes \Lambda_1,\ V(4) \otimes V(1))$ and A_{i+1} is a castling transform of A_i. By Theorem 1.1, $\mathcal{T} = (SL(a_i) \times GL(4a_{i+1}) \times GL(3a_{i+1}) \times SL(2), \Lambda_1 \otimes \Lambda_1^{(*)} \otimes 1 \otimes 1 + 1 \otimes \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1, V(a_i) \otimes V(4a_{i+1})^{(*)} + V(4a_{i+1}) \otimes V(3a_{i+1}) \otimes V(2))$ $(i \geq 0)$ is a reductive cuspidal prehomogeneous vector space.*

By Theorem 1.1, we can obtain infinitely many reductive cuspidal prehomogeneous vector spaces. From here, we shall give the preliminaries for the proof of Theorem 1.1.

Proposition 1.7 (cf. [K]). *Let $\rho_i : G \longrightarrow GL(V_i)$ $(i = 1, 2)$ be a rational representation of a linear algebraic group G on a finite dimensional vector space V_i. Assume that (G, ρ_2, V_2) is a prehomogeneous vector space with a generic isotropy subgroup H and $(H, \rho_1|_H, V_1)$ is a prehomogeneous vector space. Then $(G, \rho_1 \oplus \rho_2, V_1 \oplus V_2)$ is a prehomogeneous vector space.*

Lemma 1.8. *Let $\rho : G \longrightarrow GL(V)$ be a rational representation of a linear algebraic group G on an m-dimensional vector space V and let n be a positive integer with $m > n$. Assume that $\mathcal{Q} := (G \times GL(n), \rho \otimes \Lambda_1, V \otimes V(n))$ is a prehomogeneous vector space and the G-part of its generic isotropy subgroup is reductive. When the representation space $V \otimes V(n)$ is identified with $\overbrace{V \oplus \cdots \oplus V}^{n}$, the*

representation $\rho \otimes \Lambda_1$ is given by $(\rho \otimes \Lambda_1)(g, A)(v_1, \dots, v_n) = (\rho(g)v_1, \dots, \rho(g)v_n)^t A$ for $(g, A) \in G \times GL(n)$ and $v_1, \dots, v_n \in V$. Then we have the following assertions.
(1) Let $v_0 = (v_1^{(0)}, \dots, v_n^{(0)})$ be a generic point of $\mathcal{Q}$ and let H be the G-part of the generic isotropy subgroup $(G \times GL(n))_{v_0}$ at v_0. Then $v_1^{(0)}, \dots, v_n^{(0)}$ are linearly independent and there exists the rational representation $\phi : H \longrightarrow GL(n)$ such that $(G \times GL(n))_{v_0} = \{(h, \phi(h)) \in G \times GL(n) | h \in H\}$.
(2) Let $\{f_1^{(0)}, \dots, f_{m-n}^{(0)}\}$ be a basis of $\langle v_1^{(0)}, \dots, v_n^{(0)} \rangle^{\perp} := \{f \in V^* | f(v) = 0$ for all $v \in \langle v_1^{(0)}, \dots, v_n^{(0)} \rangle\}$ as vector spaces, where $\langle v_1^{(0)}, \dots, v_n^{(0)} \rangle$ denotes the n-dimensional subspace of V generated by $v_1^{(0)}, \dots, v_n^{(0)}$. Then $f_0 := (f_1^{(0)}, \dots, f_{m-n}^{(0)}) \in \overbrace{V^* \oplus \cdots \oplus V^*}^{m-n}$ is a generic point of $(G \times GL(m-n), \rho^* \otimes \Lambda_1, V^* \otimes V(m-n))$ which is a castling transform of $\mathcal{Q}$. Furthermore, there exists the rational representation $\psi : H \longrightarrow GL(m-n)$ such that $(G \times GL(m-n))_{f_0} = \{(h, \psi(h)) \in G \times GL(m-n) | h \in H\}$ and $\rho|_H = \phi^* \oplus \psi$.

Proof. (1) Put $W = \{(v_1, \dots, v_n) \in \overbrace{V \oplus \cdots \oplus V}^{n} | v_1, \dots, v_n$ are linearly independent$\}$. Note that W is a nonempty open subset in $\overbrace{V \oplus \cdots \oplus V}^{n}$ and $G \times GL(n)$ acts on W by $\rho \otimes \Lambda_1$. Let $\mathbb{O}$ be the open orbit of $\mathcal{Q}$. Since $\overbrace{V \oplus \cdots \oplus V}^{n}$ is irreducible, we have $\mathbb{O} \subset W$. Since $v_1^{(0)}, \dots, v_n^{(0)}$ are linearly independent, for $h \in H$, there exists a unique $A \in GL(n)$ such that $(\rho(h)v_1^{(0)}, \dots, \rho(h)v_n^{(0)})^t A = (v_1^{(0)}, \dots, v_n^{(0)})$. Hence we can define a map $\phi : H \longrightarrow GL(n)$ by $(\rho(h)v_1^{(0)}, \dots, \rho(h)v_n^{(0)})^t \phi(h) = (v_1^{(0)}, \dots, v_n^{(0)})$ for $h \in H$. Since $\rho : G \longrightarrow GL(V)$ is a rational representation, we see that ϕ is a rational representation. Thus we obtain (1).
(2) Since H is reductive and $\rho|_H : H \longrightarrow GL(V)$ is a rational representation, there exist $v_{n+1}^{(0)}, \dots, v_m^{(0)} \in V$ and the rational representation $\psi : H \longrightarrow GL(m-n)$ such that $\{v_1^{(0)}, \dots, v_m^{(0)}\}$ is a basis of V as vector spaces and $(\rho(h)v_1^{(0)}, \dots, \rho(h)v_m^{(0)}) = (v_1^{(0)}, \dots, v_m^{(0)}) \begin{pmatrix} {}^t\phi(h)^{-1} & 0 \\ 0 & \psi(h) \end{pmatrix}$ for $h \in H$. Let $\{w_1^{(0)}, \dots, w_m^{(0)}\}$ be the dual basis of $\{v_1^{(0)}, \dots, v_m^{(0)}\}$. Since $\{w_{n+1}^{(0)}, \dots, w_m^{(0)}\}$ is a basis of $\langle v_1^{(0)}, \dots, v_n^{(0)} \rangle^{\perp}$, there exists an element $P \in GL(m-n)$ such that $(w_{n+1}^{(0)}, \dots, w_m^{(0)}) = (f_1^{(0)}, \dots, f_{m-n}^{(0)})P$. Since $(\rho^*(h)w_{n+1}^{(0)}, \dots, \rho^*(h)w_m^{(0)}) = (w_{n+1}^{(0)}, \dots, w_m^{(0)})^t \psi(h)^{-1}$ for $h \in H$, we have $(\rho^*(h)f_1^{(0)}, \dots, \rho^*(h)f_{m-n}^{(0)})^t ({}^tP^{-1}\psi(h)^tP) = (f_1^{(0)}, \dots, f_{m-n}^{(0)})$ for $h \in H$. Since $\rho(g)(\langle v_1^{(0)}, \dots, v_n^{(0)} \rangle) = \langle v_1^{(0)}, \dots, v_n^{(0)} \rangle$ if and only if $\rho^*(g)(\langle v_1^{(0)}, \dots, v_n^{(0)} \rangle^{\perp}) = \langle v_1^{(0)}, \dots, v_n^{(0)} \rangle^{\perp}$, we see that the G-part of the isotropy subgroup $(G \times GL(m-n))_{f_0}$ at f_0 coincides with H. Then we have $(G \times GL(m-n))_{f_0} = \{(h, {}^tP^{-1}\psi(h)^tP) \in G \times GL(m-n) | h \in H\}$. Since $\dim(G \times GL(m-n))_{f_0} = \dim H = \dim(G \times GL(m-n)) - \dim(V^* \otimes V(m-n))$, we see that f_0 is a generic point. Thus we obtain (2). ∎

Lemma 1.9. *For irreducible rational representations $i\Lambda_1$ $(i = 1, 2)$ and Λ_1 of $SL(2)$, the tensor product representation $i\Lambda_1 \otimes \Lambda_1$ decomposes to the direct sum representation of two irreducible* *representations as follows:*

$$i\Lambda_1 \otimes \Lambda_1 = \begin{cases} 2\Lambda_1 \oplus 1 & (i=1) \\ 3\Lambda_1 \oplus \Lambda_1 & (i=2) \end{cases} \tag{1.1}$$

Here $1 : SL(2) \longrightarrow GL(\mathbb{C})$ *is the unit representation.*

Proof. Let $V_j = \{F(u,v) = \sum_{m=0}^{j} x_{m+1}u^{j-m}v^m \mid x_1,\dots,x_{j+1} \in \mathbb{C}\}$ $(j = 1,2,3)$ be the vector space of all homogeneous polynomials in two variables u, v of degree j. When a representation space of $j\Lambda_1$ $(j = 1,2,3)$ is identified with V_j, the representation $j\Lambda_1$ is given by $j\Lambda_1(A)F(u,v) = F((u,v)A) = F(au+cv, bu+dv)$ for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2)$ and $F(u,v) \in V_j$. $T := \left\{ t(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in SL(2) \mid a \in \mathbb{C}^\times \right\}$ is a maximal torus of $SL(2)$ and its character group $X(T)$ is given by $X(T) = \{\varepsilon^n : T \longrightarrow \mathbb{C}^\times | n \in \mathbb{Z}\}$, where $\varepsilon^n(t(a)) = a^n$ for $t(a) \in T$. Since $j\Lambda_1(t(a))u^{j-m}v^m = (au)^{j-m}(a^{-1}v)^m = \varepsilon^{j-2m}(t(a))u^{j-m}v^m$ for $0 \le m \le j$, we see that the set of all the weights of $j\Lambda_1$ $(j = 1,2,3)$ is $\{\varepsilon^{j-2m} \mid m \in \mathbb{Z},\ 0 \le m \le j\}$, where the multiplicity of the weight ε^{j-2m} $(0 \le m \le j)$ is one. Since $i\Lambda_1 \otimes \Lambda_1(t(a))u^{i-m}v^m \otimes u = (au)^{i-m}(a^{-1}v)^m \otimes (au) = \varepsilon^{i+1-2m}(t(a))u^{i-m}v^m \otimes u$ and $i\Lambda_1 \otimes \Lambda_1(t(a))u^{i-m}v^m \otimes v = (au)^{i-m}(a^{-1}v)^m \otimes (a^{-1}v) = \varepsilon^{i-1-2m}(t(a))u^{i-m}v^m \otimes v$ for $0 \le m \le i$, we see that the set of all the weight of $i\Lambda_1 \otimes \Lambda_1$ $(i = 1,2)$ is $\{\varepsilon^{i+1-2l} \mid l \in \mathbb{Z},\ 0 \le l \le i+1\}$, where the multiplicity of the weight ε^{i+1-2l} $(l = 0, i+1)$ (resp. $(0 < l < i+1)$) is one (resp. two). We shall show the case $i = 1$. Since $2\Lambda_1 \oplus 1(t(a))(u^{2-m}v^m, 0) = ((au)^{2-m}(a^{-1}v)^m, 0) = \varepsilon^{2-2m}(t(a))(u^{2-m}v^m, 0)$ and $2\Lambda_1 \oplus 1(t(a))(0,1) = (0,1) = \varepsilon^0(t(a))(0,1)$ for $0 \le m \le 2$, we see that all the weights of $2\Lambda_1 \oplus 1$ coincide with those of $\Lambda_1 \otimes \Lambda_1$, including weight multiplicities. Thus we obtain $\Lambda_1 \otimes \Lambda_1 = 2\Lambda_1 \oplus 1$. We shall show the case $i = 2$. Since $3\Lambda_1 \oplus \Lambda_1(t(a))(u^{3-m}v^m, 0) = ((au)^{3-m}(a^{-1}v)^m, 0) = \varepsilon^{3-2m}(t(a))(u^{3-m}v^m, 0)$ and $3\Lambda_1 \oplus \Lambda_1(t(a))(0, u^{1-m'}v^{m'}) = (0, (au)^{1-m'}(a^{-1}v)^{m'}) = \varepsilon^{1-2m'}(t(a))(0, u^{1-m'}v^{m'})$ for $0 \le m \le 3$ and $0 \le m' \le 1$, we see that all the weights of $3\Lambda_1 \oplus \Lambda_1$ coincide with those of $2\Lambda_1 \otimes \Lambda_1$, including weight multiplicities. Thus we obtain $2\Lambda_1 \otimes \Lambda_1 = 3\Lambda_1 \oplus \Lambda_1$. ∎

Proposition 1.10. $\{(3\Lambda_1 \otimes \Lambda_1^*(A,B), 2\Lambda_1 \otimes \Lambda_1(A,B), \Lambda_1(A)) | A \in SL(2), B \in GL(l)\}$ *is a generic isotropy subgroup of* $(GL(4l) \times GL(3l) \times SL(2), \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1, V(4l) \otimes V(3l) \otimes V(2))$.

Proof. Recall Remark 1.2. We shall calculate a generic isotropy subgroup of $\mathcal{R}_0 := (GL(2l) \times SL(2) \times GL(l),\ \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1,\ V(2l) \otimes V(2) \otimes V(l))$. When its representation space is identified with $M(2l)$, the representation $\Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1$ is given by $\Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1(C,A,B)X = CX^t(\Lambda_1 \otimes \Lambda_1(A,B))$ for $(C,A,B) \in GL(2l) \times SL(2) \times GL(l), X \in M(2l)$. Then $I_{2l} \in M(2l)$ is a generic point and the generic isotropy subgroup at I_{2l} is $\{(\Lambda_1^* \otimes \Lambda_1^*(A,B), \Lambda_1(A), \Lambda_1(B)) | A \in SL(2), B \in GL(l)\}$. Since two representations Λ_1 and Λ_1^* of $SL(2)$ are equivalent, $\{(\Lambda_1 \otimes \Lambda_1^*(A,B), \Lambda_1(A), \Lambda_1(B)) \mid A \in SL(2), B \in GL(l)\}$ is a generic isotropy subgroup of $\mathcal{R}_0$. Since $\mathcal{R}_1 := (GL(2l) \times SL(2) \times GL(3l), \Lambda_1^* \otimes \Lambda_1^* \otimes$

$\Lambda_1, V(2l)^* \otimes V(2)^* \otimes V(3l))$ is a castling transform of $\mathcal{R}_0$, by Lemmas 1.8 and 1.9 , we see that $\{(\Lambda_1 \otimes \Lambda_1^*(A,B), \Lambda_1(A), 2\Lambda_1 \otimes \Lambda_1^*(A,B)) | A \in SL(2), B \in GL(l)\}$ is a generic isotropy subgroup of $\mathcal{R}_1$. Then $\{(2\Lambda_1 \otimes \Lambda_1^*(A,B), \Lambda_1(A), \Lambda_1^* \otimes \Lambda_1(A,B)) \mid A \in SL(2),\ B \in GL(l)\}$ is a generic isotropy subgroup of $\mathcal{R}_2 := (GL(3l) \times SL(2) \times GL(2l), \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1, V(3l) \otimes V(2) \otimes V(2l))$. Since $\mathcal{R}_3 := (GL(3l) \times SL(2) \times GL(4l), \Lambda_1^* \otimes \Lambda_1^* \otimes \Lambda_1, V(3l)^* \otimes V(2)^* \otimes V(4l))$ is a castling transform of $\mathcal{R}_2$, by Lemmas 1.8 and 1.9 , we see that $\{(2\Lambda_1 \otimes \Lambda_1^*(A,B), \Lambda_1(A), 3\Lambda_1 \otimes \Lambda_1^*(A,B) | A \in SL(2), B \in GL(l)\}$ is a generic isotropy subgroup of $\mathcal{R}_3$. Since two representations $2\Lambda_1$ and $(2\Lambda_1)^*$ of $SL(2)$ are equivalent, we obtain our assertion. ∎

proof of Theorem 1.1. Thus we can prove Theorem 1.1. By Propositions 1.7 and 1.10 , we see that $\mathcal{T}$ is a prehomogeneous vector space. Since $\dim G = -l^2 + 4ml - 3$, we see that the dimension of a group of $\mathcal{T}$ is equal to that of a representation space of $\mathcal{T}$. Thus we obtain our assertion. ∎

2 Basic relative invariants of a cuspidal prehomogeneous vector space $(GL(4)\times GL(3)\times SL(2), M(1,4)\oplus M(4,3)\oplus M(4,3))$

In this section, we will construct two basic relative invariants of the cuspidal prehomogeneous vector space $(G, \rho, V) = (GL(4) \times GL(3) \times SL(2), \Lambda_1^* \otimes 1 \otimes 1 + \Lambda_1 \otimes \Lambda_1^* \otimes \Lambda_1^*, M(1,4) \oplus M(4,3) \oplus M(4,3))$ which is a special case of the class in §1 (See Example 1.4). This example is related to parabolic type (not necessarily irreducible) associated an $\mathfrak{sl}_2$-triple (cf. [R]) and Dynkin-Kostant type for the exceptional groups, that is, E_8-type (cf. [Uk]).
The group action on the space is the following:

$$M(1,4) \oplus M(4,3) \oplus M(4,3) \ni (p, X, Y) \mapsto (pg_4^{-1}, (g_4 X g_3^{-1}, g_4 Y g_3^{-1}) g_2^{-1}) \tag{2.1}$$

for an element $g = (g_4, g_3, g_2) \in GL(4) \times GL(3) \times SL(2)$. This space is a cuspidal prehomogeneous vector space with $\dim G = \dim V = 28$. Therefore a generic isotropy subgroup is finite. Here we chose an element $(p_0, X_0, Y_0) = ((1001), {}^t(I_3|\ 0), {}^t(0\ |I_3))$ as a generic point of the prehomogeneous vector space.

Lemma 2.1. *For elements $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij}) \in M(3)$, we define a polynomial $\alpha(A,B,C)$ on $M(3) \oplus M(3) \oplus M(3)$ as follows:*

$$\alpha(A,B,C) := \det(A+B+C) - \{\det(B+C) + \det(A+C) + \det(A+B)\} + \{\det A + \det B + \det C\} \tag{2.2}$$

Then $\alpha(A,B,C)$ is a symmetric trilinear form on $M(3) \oplus M(3) \oplus M(3)$.

Proof. By direct calculation, we have

$$\begin{aligned}\alpha(A,B,C) &= \sum_{\sigma\in S_3}\mathrm{sgn}(\sigma)a_{1\sigma(1)}b_{2\sigma(2)}c_{3\sigma(3)}+\sum_{\sigma\in S_3}\mathrm{sgn}(\sigma)a_{1\sigma(1)}c_{2\sigma(2)}b_{3\sigma(3)}\\ &\quad+\sum_{\sigma\in S_3}\mathrm{sgn}(\sigma)b_{1\sigma(1)}a_{2\sigma(2)}c_{3\sigma(3)}+\sum_{\sigma\in S_3}\mathrm{sgn}(\sigma)b_{1\sigma(1)}c_{2\sigma(2)}a_{3\sigma(3)}\\ &\quad+\sum_{\sigma\in S_3}\mathrm{sgn}(\sigma)c_{1\sigma(1)}a_{2\sigma(2)}b_{3\sigma(3)}+\sum_{\sigma\in S_3}\mathrm{sgn}(\sigma)c_{1\sigma(1)}b_{2\sigma(2)}a_{3\sigma(3)}.\end{aligned}$$

Thus we obtain our assertion. ∎

For an element $X\in M(4,3)$, define $X(i)\in M(3)$ $(1\le i\le 4)$ by the matrix obtained by deleting the i-th row from X. Then, for $X,Y,Z\in M(4,3)$, we put

$$\alpha_i(X,Y,Z) := (-1)^{i-1}\alpha(X(i),Y(i),Z(i)) \quad (1\le i\le 4) \tag{2.3}$$

and $\mathfrak{A}(*,*,*) := {}^t(\alpha_1(*,*,*)\ \alpha_2(*,*,*)\ \alpha_3(*,*,*)\ \alpha_4(*,*,*))\in M(4,1)$. Thus we define the mapping Φ from $M(4,3)\oplus M(4,3)$ to $M(4)$ as follows:

$$\Phi(X,Y) := (\frac{1}{3}\mathfrak{A}(X,X,X)\mid\mathfrak{A}(X,Y,X)\mid\mathfrak{A}(Y,X,Y)\mid\frac{1}{3}\mathfrak{A}(Y,Y,Y))\in M(4) \tag{2.4}$$

Here we remark the followings:

Lemma 2.2. *For* $F^i_{(X,Y)}(u,v) := \frac{1}{3}\alpha_i(X,X,X)u^3+\alpha_i(X,Y,X)u^2v+\alpha_i(Y,X,Y)uv^2+\frac{1}{3}\alpha_i(Y,Y,Y)v^3$ $(1\le i\le 4)$ *, we have*
$F^i_{(X,Y)g_2^{-1}}(u,v)=F^i_{(X,Y)}((u,v)^tg_2^{-1})$ *for* $g_2\in SL(2)$.

Proof. We may check the compatibility for elements $\begin{pmatrix}a&0\\0&a^{-1}\end{pmatrix},\begin{pmatrix}1&\varepsilon\\0&1\end{pmatrix},\begin{pmatrix}0&1\\-1&0\end{pmatrix}\in SL(2)$.

$$\begin{aligned}&F^i_{(X,Y)^t\begin{pmatrix}a&0\\0&a^{-1}\end{pmatrix}}(u,v)\\ &=\tfrac13\alpha_i(aX,aX,aX)u^3+\alpha_i(aX,a^{-1}Y,aX)u^2v+\alpha_i(a^{-1}Y,aX,a^{-1}Y)uv^2+\tfrac13\alpha_i(a^{-1}Y,a^{-1}Y,a^{-1}Y)v^3\\ &=\tfrac13a^3\alpha_i(X,X,X)u^3+a\alpha_i(X,Y,X)u^2v+a^{-1}\alpha_i(Y,X,Y)uv^2+\tfrac13a^{-3}\alpha_i(Y,Y,Y)v^3\\ &=\tfrac13\alpha_i(X,X,X)(au)^3+\alpha_i(X,Y,X)(au)^2(a^{-1}v)+\alpha_i(Y,X,Y)(au)(a^{-1}v)^2+\tfrac13\alpha_i(Y,Y,Y)(a^{-1}v)^3\\ &=F^i_{(X,Y)}((u,v)\begin{pmatrix}a&0\\0&a^{-1}\end{pmatrix}),\end{aligned}$$

$$\begin{aligned}&F^i_{(X,Y)^t\begin{pmatrix}1&\varepsilon\\0&1\end{pmatrix}}(u,v)\\ &=\tfrac13\alpha_i(X+\varepsilon Y,X+\varepsilon Y,X+\varepsilon Y)u^3+\alpha_i(X+\varepsilon Y,Y,X+\varepsilon Y)u^2v\\ &+\alpha_i(Y,X+\varepsilon Y,Y)uv^2+\tfrac13\alpha_i(Y,Y,Y)v^3\\ &=\tfrac13\{\alpha_i(X,X,X)+3\varepsilon\alpha_i(X,Y,X)+3\varepsilon^2\alpha_i(Y,X,Y)+\varepsilon^3\alpha_i(Y,Y,Y)\}u^3\\ &+\{\alpha_i(X,Y,X)+2\varepsilon\alpha_i(Y,X,Y)+\varepsilon^2\alpha_i(Y,Y,Y)\}u^2v\\ &+\{\alpha_i(Y,X,Y)+\varepsilon\alpha_i(Y,Y,Y)\}uv^2+\tfrac13\alpha_i(Y,Y,Y)v^3\end{aligned}$$

$= \frac{1}{3}\alpha_i(X,X,X)u^3 + \alpha_i(X,Y,X)\{\varepsilon u^3 + u^2 v\}$
$+\alpha_i(Y,X,Y)\{\varepsilon^2 u^3 + 2\varepsilon u^2 v + uv^2\}$
$+\frac{1}{3}\alpha_i(Y,Y,Y)\{\varepsilon^3 u^3 + 3\varepsilon^2 u^2 v + 3\varepsilon uv^2 + v^3\}$
$= \frac{1}{3}\alpha_i(X,X,X)u^3 + \alpha_i(X,Y,X)u^2(\varepsilon u + v)$
$+\alpha_i(Y,X,Y)u(\varepsilon u + v)^2 + \frac{1}{3}\alpha_i(Y,Y,Y)(\varepsilon u + v)^3$
$= F^i_{(X,Y)}((u,v)\begin{pmatrix}1 & \varepsilon\\ 0 & 1\end{pmatrix}),$
$F^i_{(X,Y)^t\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}}(u,v)$
$= \frac{1}{3}\alpha_i(Y,Y,Y)u^3 - \alpha_i(Y,X,Y)u^2 v + \alpha_i(X,Y,X)uv^2 - \frac{1}{3}\alpha_i(X,X,X)v^3$
$= F^i_{(X,Y)}(-v,u) = F^i_{(X,Y)}((u,v)\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}).$

Thus we can prove the the compatibility of $F^i_{(X,Y)}(u,v)$ for the action of $SL(2)$. ∎

We need the following lemma to prove Proposition 2.4.

Lemma 2.3 (Cauchy-Binet). *Let A be an m by n matrix and B an n by m matrix. We denote by a_i $(1 \le i \le n)$ (resp. b_i $(1 \le i \le n)$) the i-th column (resp. row) of A (resp. B). Then we have the following assertions.*
(1) If $m > n$, then $\det(AB) = 0$.
(2) If $m = n$, then $\det(AB) = \det A \det B$.
(3) If $m < n$, then $\det(AB) = \sum_{1 \le i_1 < \cdots < i_m \le n} \det(a_{i_1}|\cdots|a_{i_m}) \det {}^t({}^t b_{i_1}|\cdots|{}^t b_{i_m})$.

Proposition 2.4. *For $g_2 \in SL(2), g_3 \in GL(3)$ and $g_4 \in GL(4)$, we have*

$$(1)\quad \Phi((X,Y)g_2^{-1}) = \Phi(X,Y)(3\Lambda_1(g_2^{-1})),$$
$$(2)\quad \Phi(g_4 X g_3^{-1}, g_4 Y g_3^{-1}) = (\det g_3)^{-1}(\det g_4)({}^t g_4^{-1})\Phi(X,Y).$$

Proof. (1) follows directly from Lemma 2.2. We will prove (2). For $X \in M(4,3)$, we put $S(X) := {}^t(s(1)\ s(2)\ s(3)\ s(4))$, where $s(i) := (-1)^{i-1}\det X(i)$ $(1 \le i \le 4)$. By Lemma 2.3, we have $S(g_4 X) = (\det g_4){}^t g_4^{-1} S(X)$ for $g_4 \in GL(4)$. Then we see that $\frac{1}{3}\mathfrak{A}(g_4X, g_4X, g_4X) = (\det g_4){}^t g_4^{-1}\frac{1}{3}\mathfrak{A}(X,X,X)$ and $\frac{1}{3}\mathfrak{A}(g_4Y, g_4Y, g_4Y) = (\det g_4){}^t g_4^{-1}\frac{1}{3}\mathfrak{A}(Y,Y,Y)$ for $g_4 \in GL(4)$. Note that $\alpha_i(X,Y,X) = (-1)^{i-1}\{\det((2X+Y)(i)) - 2\det((X+Y)(i)) - \det((2X)(i)) + 2\det(X(i)) + \det(Y(i))\}$ $(1 \le i \le 4)$. Then we see that $\mathfrak{A}(g_4X, g_4Y, g_4X) = (\det g_4){}^t g_4^{-1}\mathfrak{A}(X,Y,X)$ and $\mathfrak{A}(g_4Y, g_4X, g_4Y) = (\det g_4){}^t g_4^{-1}\mathfrak{A}(Y,X,Y)$ for $g_4 \in GL(4)$. Thus we obtain (2). ∎

Since $\det\Phi(X_0, Y_0) = 16$, we see that $\det\Phi(X,Y)$ is not identically zero. Thus we see that

$$\begin{aligned} &\det\Phi((X,Y)g_2^{-1}) = \det\Phi(X,Y)\\ &\det\Phi(g_4 X g_3^{-1}, g_4 Y g_3^{-1}) = (\det g_3)^{-4}(\det g_4)^3\det\Phi(X,Y)\end{aligned} \tag{2.5}$$

and hence we have the following Theorem

Theorem 2.5. $f_1(X,Y) = \det \Phi(X,Y)$ *is a basic relative invariant of the prehomogeneous vector space* $(GL(4)\times GL(3)\times SL(2), \Lambda_1^*\otimes 1\otimes 1+\Lambda_1\otimes\Lambda_1^*\otimes\Lambda_1^*, M(1,4)\oplus M(4,3)\oplus M(4,3))$ *corresponding to the rational character* $\chi_1(g_4, g_3, g_2) = (\det g_3)^{-4}(\det g_4)^3$.

Next we consider the construction of the second basic relative invariant of the prehomogeneous vector space $(GL(4) \times GL(3) \times SL(2), \Lambda_1^* \otimes 1 \otimes 1 + \Lambda_1 \otimes \Lambda_1^* \otimes \Lambda_1^*, M(1,4) \oplus M(4,3) \oplus M(4,3))$. For $\Phi(X,Y)$ in (2.4) , we put $\Phi(X,Y) = (\alpha_{ij})_{1\le i,j\le 4}$. Then we have the following proposition:

Proposition 2.6. *We define polynomials* ψ_{ij} $(1 \le i < j \le 4)$ *in* 16 *variables* $\alpha_{\ell k}$ $(1 \le \ell, k \le 4)$ *as follows:*

$$\psi_{ij} := 3\alpha_{i1}\alpha_{j4} - 3\alpha_{j1}\alpha_{i4} - \alpha_{i2}\alpha_{j3} + \alpha_{j2}\alpha_{i3}. \tag{2.7}$$

Then we have the following assertions.
(1) ψ_{ij} *are* $SL(2)$*-invariant and* $GL(3)$*-relative invariant polynomials.*
(2) Define

$$\Psi(X,Y) := \begin{pmatrix} 0 & \psi_{12} & \psi_{13} & \psi_{14} \\ -\psi_{12} & 0 & \psi_{23} & \psi_{24} \\ -\psi_{13} & -\psi_{23} & 0 & \psi_{34} \\ -\psi_{14} & -\psi_{24} & -\psi_{34} & 0 \end{pmatrix} \in \mathrm{Alt}(4). \tag{2.8}$$

Then we have

$$\Psi(g_4Xg_3^{-1}, g_4Yg_3^{-1}) = (\det g_3)^{-2}(\det g_4)^2({}^tg_4^{-1})\Psi(X,Y)g_4^{-1}.$$

We need the following Lemma 2.7 to prove Proposition 2.6

Lemma 2.7. *For two binary cubic forms*

$$\begin{aligned} F_s(u,v) &= s_1u^3 + s_2u^2v + s_3uv^2 + s_4v^3, \\ F_t(u,v) &= t_1u^3 + t_2u^2v + t_3uv^2 + t_4v^3, \end{aligned} \tag{2.9}$$

we put $G(s,t) := 3(s_1t_4 - s_4t_1) - (s_2t_3 - s_3t_2)$. *Then* $G(s,t)$ *is an* $SL(2)$*-invariant polynomial, that is,* $G(3\Lambda_1(g_2^{-1})s, 3\Lambda_1(g_2^{-1})t) = G(s,t)$ *for* $g_2 \in SL(2)$.

Proof. Since $SL(2)$ is generated by $t(a) := \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}$, $h := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $u(\varepsilon) := \begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix}$, it is enough to show the case $g_2 = t(a),\ h,\ u(\varepsilon)$. It is obvious to show the case $g_2 = t(a)$ (resp. $g_2 = h$). For $g_2^{-1} = u(\varepsilon)$,

$$\begin{aligned} &G(3\Lambda_1(g_2^{-1})s, 3\Lambda_1(g_2^{-1})t) \\ &= 3\{(s_1 + \varepsilon s_2 + \varepsilon^2 s_3 + \varepsilon^3 s_4)t_4 - s_4(t_1 + \varepsilon t_2 + \varepsilon^2 t_3 + \varepsilon^3 t_4)\} \\ &-\{(s_2 + 2\varepsilon s_3 + 3\varepsilon^2 s_4)(t_3 + 3\varepsilon t_4) - (s_3 + 3\varepsilon s_4)(t_2 + 2\varepsilon t_3 + 3\varepsilon^2 t_4)\} \\ &= G(s,t) \end{aligned}$$

Thus we obtain our assertion. ∎

[proof of Proposition 2.6]

By Lemma 2.7, we have (1). We shall show $\Psi(g_4X, g_4Y) = (\det g_4)^2({}^tg_4^{-1})\Psi(X,Y)g_4^{-1}$ for $g_4 \in GL(4)$. For $\sigma \in S_4$, we put $g(\sigma) := (\delta_{i\sigma(j)})_{1\le i,j\le 4}$, where $\delta_{st} = \begin{cases} 1 & (s=t) \\ 0 & (s\neq t) \end{cases}$. Since $GL(4)$ is generated by $\mathrm{diag}(a,b,c,d)$, $g(\sigma)$, $u(\varepsilon) := \begin{pmatrix} 1 & \varepsilon & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, it is enough to show the case $g_4 = \mathrm{diag}(a,b,c,d)$, $g(\sigma)$, $u(\varepsilon)$. It is obvious to show the case $g_4 = \mathrm{diag}(a,b,c,d)$ (resp. $g_4 = g(\sigma)$). For $g_4^{-1} = u(\varepsilon)$,

$$\begin{aligned}\Psi(g_4X, g_4Y) &= \begin{pmatrix} 0 & \psi_{12} & \psi_{13} & \psi_{14} \\ -\psi_{12} & 0 & \varepsilon\psi_{13}+\psi_{23} & \varepsilon\psi_{14}+\psi_{24} \\ -\psi_{13} & -\varepsilon\psi_{13}-\psi_{23} & 0 & \psi_{34} \\ -\psi_{14} & -\varepsilon\psi_{14}-\psi_{24} & -\psi_{34} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ \varepsilon & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \psi_{12} & \psi_{13} & \psi_{14} \\ -\psi_{12} & 0 & \psi_{23} & \psi_{24} \\ -\psi_{13} & -\psi_{23} & 0 & \psi_{34} \\ -\psi_{14} & -\psi_{24} & -\psi_{34} & 0 \end{pmatrix} \begin{pmatrix} 1 & \varepsilon & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= {}^tg_4^{-1}\Psi(X,Y)g_4^{-1}.\end{aligned}$$

Thus we have (2). □

Here we remark $\mathrm{Pf}(\Psi(X,Y)) = -3\det(\Phi(X,Y)) = -3f_1(X,Y)$. We have the following lemma from Proposition 2.6.

Lemma 2.8. *Put $R(X,Y) :={}^t X\Psi(X,Y)Y$, we have*

$$R(g_4Xg_3^{-1}, g_4Yg_3^{-1}) = (\det g_3)^{-2}(\det g_4)^2({}^tg_3^{-1})R(X,Y)g_3^{-1}. \tag{2.10}$$

Here if we consider a 2×3 matrix $H(p,X,Y) := \begin{pmatrix} pX \\ pY \end{pmatrix}$, then $H(pg_4^{-1}, (g_4Xg_3^{-1}, g_4Yg_3^{-1})g_2^{-1}) = {}^tg_2^{-1}\begin{pmatrix} pXg_3^{-1} \\ pYg_3^{-1} \end{pmatrix}$. We put $H(p,X,Y) = (\ell_1 \mid \ell_2 \mid \ell_3)$, with ℓ_1, ℓ_2, $\ell_3 \in \mathbb{C}^2$ and we make the following 3×3 alternating matrix : for $z_{ij} := \det(\ell_i \mid \ell_j)$, $Z(p,X,Y) := \begin{pmatrix} 0 & z_{12} & z_{13} \\ -z_{12} & 0 & z_{23} \\ -z_{13} & -z_{23} & 0 \end{pmatrix}$.

Then

$$Z(pg_4^{-1}, (g_4Xg_3^{-1}, g_4Yg_3^{-1})g_2^{-1}) = {}^tg_3^{-1}Z(p,X,Y)g_3^{-1}. \tag{2.11}$$

Thus if we put $\Delta(Z(p,X,Y)) := \begin{pmatrix} (z_{23})^2 & -z_{13}z_{23} & z_{12}z_{23} \\ -z_{13}z_{23} & (z_{13})^2 & -z_{12}z_{23} \\ z_{12}z_{23} & -z_{12}z_{23} & (z_{12})^2 \end{pmatrix}$, we have

$$\Delta(Z(pg_4^{-1},(g_4Xg_3^{-1},g_4Yg_3^{-1})g_2^{-1})) = (\det g_3)^{-2} g_3 \Delta(Z(p,X,Y))^t g_3. \tag{2.12}$$

Thus we have

$$\begin{aligned} &\Delta(Z(pg_4^{-1},(g_4Xg_3^{-1},g_4Yg_3^{-1})g_2^{-1}))R((g_4Xg_3^{-1},g_4Yg_3^{-1})g_2^{-1}) \\ &= (\det g_3)^{-4}(\det g_4)^2 g_3 \Delta(Z(p,X,Y))R(X,Y)g_3^{-1}. \end{aligned} \tag{2.13}$$

Hence we have

$$\begin{aligned} &\mathrm{tr}(\Delta(Z(pg_4^{-1},(g_4Xg_3^{-1},g_4Yg_3^{-1})g_2^{-1}))R((g_4Xg_3^{-1},g_4Yg_3^{-1})g_2^{-1})) \\ &= (\det g_3)^{-4}(\det g_4)^2 \mathrm{tr}(\Delta(Z(p,X,Y))R(X,Y)). \end{aligned} \tag{2.14}$$

We put $f_2(p,X,Y) := \mathrm{tr}(\Delta(Z(p,X,Y))R(X,Y))$. For the generic point (p_0,X_0,Y_0), we have $f_2(p_0,X_0,Y_0) = -4 \neq 0$, that is, f_2 is not identically zero. Here we remark $SL(2)$-invariance of f_2 as follows: for $g_2^{-1} = \begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix}$,

$f_2(p,(X,Y)g_2^{-1}) = \mathrm{tr}(\Delta(Z(p,X,Y))^t X\Psi(X,Y)(\varepsilon X + Y))$
$= \mathrm{tr}(\Delta(Z(p,X,Y))^t X\Psi(X,Y)Y) + \varepsilon \mathrm{tr}(\Delta(Z(p,X,Y))^t X\Psi(X,Y)X)$
$= \mathrm{tr}(\Delta(Z(p,X,Y))^t X\Psi(X,Y)Y)$.

For another generators $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}$ of $SL(2)$, we can easily check $SL(2)$-invariance of $f_2(p,X,Y)$.

We summarize the argument above as follows:

Theorem 2.9. *Two basic relative invariants of the prehomogeneous vector space* $(GL(4)\times GL(3)\times SL(2), \Lambda_1^* \otimes 1 \otimes 1 + \Lambda_1 \otimes \Lambda_1^* \otimes \Lambda_1^*, M(1,4) \oplus M(4,3) \oplus M(4,3))$ *are the followings:*

(2.15)
$$f_1(X,Y) = \det \Phi(X,Y) = -\frac{1}{3}\mathrm{Pf}(\Psi(X,Y)) \leftrightarrow (\det g_3)^{-4}(\det g_4)^3, \quad \deg_{(X,Y)} f_1(X,Y) = 12,$$

(2.16)
$$f_2(p,X,Y) = \mathrm{tr}(\Delta(Z(p,X,Y))^t X\Psi(X,Y)Y) \leftrightarrow (\det g_3)^{-4}(\det g_4)^2 \quad \deg_{(p,(X,Y))} f_2 = (4,12).$$

where $f \leftrightarrow \chi$ means that the rational character χ of the algebraic group corresponds to the polynomial f on the representation space.

Remark 2.10. *It is the open problem to give an explicit construction of basic relative invariants of another cuspidal prehomogeneous vector spaces of the type in §1 (See Theorem1.1).*

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