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ON THE SIGNATURE OF AREA FORM ON THE POLYGON SPACE

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1. introduction

The space of Euclidean polygons with prescribed exterior angles in the Euclidean plane up to similarities is a subspace of the moduli space $\mathcal{C}$ of Euclidean cone structures on the 2-sphere with prescribed cone angles. The latter space is known to be homeomorphic to the configuration space of points on $\mathbb{C}P^1$ ([10]), where the subspace consisting of configurations of all points on $\mathbb{R}P^1$ corresponds to the space of Euclidean polygons.

In [9], Thusrton exhibits a complex hyperbolic structure on $\mathcal{C}$ by using the area form. It gives an alternative way to get complex hyperbolic orbifolds obtained by Deligne and Mostow in [2] where monodoromy of hypergeometric functions induces their lattice. The metric completion of the complex hyperbolic structure on $\mathcal{C}$ is identical with a partial compactification of the configuration space by adding α-stable points modulo $PGL(2)$ in [2]. Here $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a sequence of n real numbers with

$$0 < \alpha_i < 1 \text{ and } \sum_{i=1}^{n} \alpha_i = 2, \tag{1}$$

and a point $y = (y_1, \ldots, y_n)$ in $(\mathbb{C}P^1)^n$ is α-stable if for all $x \in \mathbb{C}P^1$, $\sum_{y_i = x} \alpha_i < 1$. We note that the data (1) of α_i's exactly corresponds to the apex curvatures of cone points of Euclidean cone structures on the 2-sphere $\mathbb{S}^2$ by multiplying 2π, and also the exterior angles of Euclidean polygons by π.

In this note, we shall address on the signature of Hermitian form given by the area function on the space of Euclidean cone metric on $\mathbb{S}^2$ with cone points of possibly negative curvatures, that is with some α's negative. A particular case is given in [4] and in general in [11]. We also restate the signature of the area form on the space of Euclidean polygons givne in [1] in termes of exterior angles.

2. Spaces of Euclidean polygons

Let $n \geq 3$ and P be a Euclidean polygon with n cyclically ordered verticies $p_1, \ldots, p_n$ in $\mathbb{E}^2$. We call the i-th side of P the vector $p_{i+1} - p_i$ and denote by s_i its lentgh. The exterior angle θ_i at the vertex p_i is the oriented angle between the $(i-1)$-th side and i-th side. We say two polygons P an Q are congruent if there exists a congruent transformation of $\mathbb{E}^2$ which sends the set of verticies of P to those of Q with preserving their indices.

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Let $\theta = (\theta_1, \dots, \theta_n)$ be an n-tuple with $n \geq 3$ of real numbers satisfying $\sum_{i=1}^n \theta_i = 2\pi$. We assume $\theta_i \not\equiv 0 \mod \pi$. Let $\mathcal{P}_\theta$ denote the set of congruence classes of Euclidean n-gons with prescribed exterior angles $\theta = (\theta_1, \dots, \theta_n)$. We note that the space $\mathcal{P}_\theta$ is parametrised by the side lenghts s_i satisfying the equation

$$\sum_{k=1}^{n} s_k \exp\left(\sqrt{-1}\sum_{j=1}^{k}\theta_j\right) = 0. \tag{2}$$

Let $\mathcal{V}_\theta$ be the codimension two subspace in $\mathbb{R}^n$ whose elements $(s_1, \dots, s_n)$ satisfy the euqation (2). Then $\mathcal{P}_\theta$ lies in the space $\mathcal{V}_\theta$ satisfying $s_i > 0$ for $1 \leq i \leq n$ and its closure $\bar{\mathcal{P}}_\theta$ forms a polyhedral cone in $\mathcal{V}_\theta$ satisfying $s_i \geq 0$. Let Area be the function on $\mathcal{P}_\theta$ assigning each n-gon P its signed area $\mathrm{Area}(P)$. Obviously Area is quadratic on $\mathcal{P}_\theta$ and extends to a quadratic form $\mathcal{A}_\theta$ on $\mathcal{V}_\theta$.

A trivial example. In the case of $n = 3$, the element $T = (s_1, s_2, s_3)$ in $\mathcal{P}_\theta$ is a trianlge with prescribed exterior angles $\theta = (\theta_1, \theta_2, \theta_3)$. By the sine rule,

$$\mathrm{Area}(T) = \frac{\kappa^2}{2}\sin\theta_1\sin\theta_2\sin\theta_3, \tag{3}$$

where $\kappa = \dfrac{s_1}{\sin\theta_3} = \dfrac{s_2}{\sin\theta_1} = \dfrac{s_3}{\sin\theta_2}$. Equivalently,

$$\mathrm{Area}(T) = \frac{-\sin\theta_i\sin\theta_{i+1}}{2\sin(\theta_i + \theta_{i+1})}\, s_i^2, \tag{4}$$

where the indicies i is understood modulo 3.

The signature of the quadratic form $\mathcal{A}_\theta$ is determined in [1] as follows.

Theorem 1. *The signature of the quadratic form $\mathcal{A}_\theta$ on $\mathcal{P}_\theta$ is*

$$\left(\frac{1}{\pi}\sum_{s=1}^{n}\nu_s - 1,\ \frac{1}{\pi}\sum_{s=1}^{n}(\pi - \nu_s) - 1\right)$$

where ν_s is a real number in $(0, \pi)$ such that $\nu_s \equiv \theta_s \mod \pi$.

Remark. For convex polygons, the exterior angles $\theta = (\theta_1, \dots, \theta_n)$ satisfy $0 < \theta_i < \pi$ and $\sum_{i=1}^n \theta_i = 2\pi$. Thus the signature in the convex case is $(1, n-3)$, which is studied in [9], [1], [5], [7], [3]. A nonconvex polygons bounding a region has negative exterior angles $-\pi < \theta_s < 0$ where corresponding ν_s is obtained by $\nu_s = \theta_s + \pi$. Thus if p is the number of negative exterior angles, the corresponding signature is $(p+1, n-p-3)$ (see [7]).

Theorem 1 is easily understood in the case of convex polygons. There is at least one triple of the sides of a convex polygon P whose extentions form a "big" triangle T_0 such that P is contained inside T_0 and whose sides touch the sides of T_0 only along those three sides. The complement of P in T_0 can be devided into $n-3$ "small" triangles T_i by suitably extending the side of P so that each T_i touches a unique side of P. Then the area of P is obtained by

$$\mathrm{Area}(P) = \mathrm{Area}(T_0) - \sum_{i=1}^{n-3}\mathrm{Area}(T_i) \tag{5}$$

By the sine rule, one can see the lenghs t_i of the sides of T_i's which touches P for $0 \leq i \leq n-3$ are linear combinations of $s_1, \ldots, s_n$ which turnes out to deinfe an isomorphism on the $(n-2)$-dimensional vector space $\mathcal{V}_\theta$. By (4), we see that

$$\text{Area}(P) = C_0 t_0^2 - \sum_{i=1}^{n-3} C_i t_i^2 \tag{6}$$

where the constants C_i $(0 \leq i \leq n-3)$ are expressed by the sines of exterior angles θ_i's. This leads the signature of the form $\mathcal{A}_\theta$ to be $(1, n-3)$. The signature of $\mathcal{A}_\theta$ in the nonconvex case is understood similarly. That is, by extending the sides of a polygon P, there appear triangles where the area of P is obtained by adding or subtracting in many possible ways the areas of some of these triangles, which contributes to count the positive or negatice vectors in $\mathcal{P}_\theta$ with respect to $\mathcal{A}_\theta$ (see [7]).

Let P_θ be the space of Euclidean n-gons with prescribed angles $\theta = (\theta_1, \ldots, \theta_n)$ up to similarities with positive area. Since each similarity class can be uniquely represented by a polygon with Area $= 1$, the space P_θ is identified with an open subset in the space $\mathcal{A}_\theta^{-1}(1)$ in $\mathcal{P}_\theta$ which endows a pseudo-Riemannian structure of dimension $n-3$. The closure $\bar{P}_\theta$ of P_θ in $\mathcal{V}_\theta$ is an $(n-3)$-dimensional polyhedon with a pseudo-Riemannian structure. Especially in the convex case, $\bar{P}_\theta$ is a hyperbolic polyhedron whose combinatorial and geometric structures are studied in [1], [5], [7], [3].

A trivial example. In the case of $n = 3$, for $\theta = (\theta_1, \theta_2, \theta_3)$ with $\theta_i > 0$, P_θ is a point in $\mathbb{R}^3$ with coordinates

$$\left(\sqrt{\frac{2\sin\theta_3}{\sin\theta_1 \sin\theta_2}}, \sqrt{\frac{2\sin\theta_1}{\sin\theta_2 \sin\theta_3}}, \sqrt{\frac{2\sin\theta_2}{\sin\theta_3 \sin\theta_1}})\right)$$

3. Spaces of Euclidean cone structures on the 2-sphere

A Euclidean cone metric on the 2-sphere $\mathbb{S}^2$ is a singular metric on $\mathbb{S}^2$ which is Euclidean except at finite points $p_1, \ldots, p_n$, $n \geq 3$ and the neighbourhood of each point p_k are modelled on the neighbourhood of a Euclidean cones with cone angle $\theta_k > 0$. The apex curvature at the cone points p_k is $2\pi - \theta_k$. We note that if the cone angle θ_k at p_k satisfies $0 < \theta_k < 2\pi$, the curvature at the cone point p_k is positive and if $\theta_k > 2\pi$, the curvature is negative. By Gauss-Bonnet theorem, the curvatures α_k's of a Euclidean cone metric on $\mathbb{S}^2$ satisfy the relation

$$\sum_{k=1}^{n} \alpha_k = 4\pi. \tag{7}$$

Let $(\alpha_1, \ldots, \alpha_n)$ be an n-tuple with $n \geq 3$ of real numbers satisfying (7). We denote by $C(\alpha_1, \ldots, \alpha_n)$ the space of Euclidean cone metrics on $\mathbb{S}^2$ with n labelled cone points of curvatures α_k, $1 \leq k \leq n$, up to orientation and label-preserving similarities. By Troyanov's theorem ([10]), $C(\alpha_1, \ldots, \alpha_n)$ is homeomorphic to the configuration space of n points on $\mathbb{C}P^1$ which is an $(n-3)$-dimensional manifold.

Let C be a Euclidean cone metric on $\mathbb{S}^2$ which represents an element in $C(\alpha_1, \ldots, \alpha_n)$. There is a function assigning each C its area Area(C). When

C has only positive curvatures on its cone points, that is $0 < \alpha_i < 2\pi$ for $1 \le i \le n$, it is shown in [9] that there is a complex $(n-2)$-dimensional local parametrisation of Euclidean cone metrics near C up to orientation and label preserving Euclidean isometries, with respect to which the area function is a Hermitian form $\mathcal{A}$ of type $(1, n-3)$ inducing a complex hyperbolic structure on $C(\alpha_1, \ldots, \alpha_n)$. When C has negative curvatures on some cone points, we see that there is also a complex $(n-2)$-dimensional local parametrisation, with respect to which the area function gives rise to a Hermitian form $\mathcal{A}$ of different type as follows.

Theorem 2. *The signature of the Hermitian form* $\mathcal{A}$ *is*

$$\left(\frac{1}{2\pi} \sum_{s=1}^{n} \mu_s - 1,\ \frac{1}{2\pi} \sum_{s=1}^{n} (2\pi - \mu_s) - 1 \right)$$

where $\mu_s = \lg(\exp(\sqrt{-1}\alpha_s))$ *is a real number in* $(0, 2\pi)$.

Example. In [8], we studied the pseudo-Hermitian form on the space $C((n-2)\pi, \underbrace{\pi, \ldots, \pi}_{n})$ of Euclidean cone structures on the 2-sphere where $n = 2m + 1$ is an odd interger. The signature of the area form $\mathcal{A}$ on the parameter space is (m, m). The same result appears in [6].

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