1. INTRODUCTION

The space of Euclidean polygons with prescribed exterior angles in the Euclidean plane up to similarities is a subspace of the moduli space $\mathcal{C}$ of Euclidean cone structures on the 2-sphere with prescribed cone angles. The latter space is known to be homeomorphic to the configuration space of points on $\mathbb{C}P^1$ ([10]), where the subspace consisting of configurations of all points on $\mathbb{R}P^1$ corresponds to the space of Euclidean polygons.

In [9], Thurston exhibits a complex hyperbolic structure on $\mathcal{C}$ by using the area form. It gives an alternative way to get complex hyperbolic orbifolds obtained by Deligne and Mostow in [2] where monodoromy of hypergeometric functions induces their lattice. The metric completion of the complex hyperbolic structure on $\mathcal{C}$ is identical with a partial compactification of the configuration space by adding $\alpha$-stable points modulo $PGL(2)$ in [2]. Here $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a sequence of $n$ real numbers with

\begin{equation}
0 < \alpha_i < 1 \quad \text{and} \quad \sum_{i=1}^{n} \alpha_i = 2,
\end{equation}

and a point $y = (y_1, \ldots, y_n)$ in $(\mathbb{C}P^1)^n$ is $\alpha$-stable if for all $x \in \mathbb{C}P^1$, $\sum_{y_i = x} \alpha_i < 1$. We note that the data (1) of $\alpha_i$'s exactly corresponds to the apex curvatures of cone points of Euclidean cone structures on the 2-sphere $\mathbb{S}^2$ by multiplying $2\pi$, and also the exterior angles of Euclidean polygons by $\pi$.

In this note, we shall address on the signature of Hermitian form given by the area function on the space of Euclidean cone metric on $\mathbb{S}^2$ with cone points of possibly negative curvatures, that is with some $\alpha$'s negative. A particular case is given in [4] and in general in [11]. We also restate the signature of the area form on the space of Euclidean polygons given in [1] in terms of exterior angles.

2. SPACES OF EUCLIDEAN POLYGONS

Let $n \geq 3$ and $P$ be a Euclidean polygon with $n$ cyclically ordered vertices $p_1, \ldots, p_n$ in $\mathbb{E}^2$. We call the $i$-th side of $P$ the vector $p_{i+1} - p_i$ and denote by $s_i$ its length. The exterior angle $\theta_i$ at the vertex $p_i$ is the oriented angle between the $(i-1)$-th side and $i$-th side. We say two polygons $P$ and $Q$ are congruent if there exists a congruent transformation of $\mathbb{E}^2$ which sends the set of vertices of $P$ to those of $Q$ with preserving their indices.
Let $\theta = (\theta_1, \ldots, \theta_n)$ be an $n$-tuple with $n \geq 3$ of real numbers satisfying $\sum_{i=1}^{n} \theta_i = 2\pi$. We assume $\theta_i \not\equiv 0 \mod \pi$. Let $\mathcal{P}_\theta$ denote the set of congruence classes of Euclidean $n$-gons with prescribed exterior angles $\theta = (\theta_1, \ldots, \theta_n)$. We note that the space $\mathcal{P}_\theta$ is parametrised by the side lengths $s_i$ satisfying the equation

$$
\sum_{k=1}^{n} s_k \exp \left( \sqrt{-1} \sum_{j=1}^{k} \theta_j \right) = 0.
$$

Let $\mathcal{V}_\theta$ be the codimension two subspace in $\mathbb{R}^n$ whose elements $(s_1, \ldots, s_n)$ satisfy the equation (2). Then $\mathcal{P}_\theta$ lies in the space $\mathcal{V}_\theta$ satisfying $s_i > 0$ for $1 \leq i \leq n$ and its closure $\bar{\mathcal{P}}_\theta$ forms a polyhedral cone in $\mathcal{V}_\theta$ satisfying $s_i \geq 0$.

Let $\text{Area}$ be the function on $\mathcal{P}_\theta$ assigning each $n$-gon $P$ its signed area $\text{Area}(P)$. Obviously $\text{Area}$ is quadratic on $\mathcal{P}_\theta$ and extends to a quadratic form $A_\theta$ on $\mathcal{V}_\theta$.

A trivial example. In the case of $n = 3$, the element $T = (s_1, s_2, s_3)$ in $\mathcal{P}_\theta$ is a triangle with prescribed exterior angles $\theta = (\theta_1, \theta_2, \theta_3)$. By the sine rule,

$$
\text{Area}(T) = \frac{\kappa^2}{2} \sin \theta_1 \sin \theta_2 \sin \theta_3,
$$

where $\kappa = \frac{s_1}{\sin \theta_3} = \frac{s_2}{\sin \theta_1} = \frac{s_3}{\sin \theta_2}$. Equivalently,

$$
\text{Area}(T) = -\frac{\sin \theta_i \sin \theta_{i+1}}{2 \sin(\theta_i + \theta_{i+1})} s_i^2,
$$

where the indices $i$ is understood modulo 3.

The signature of the quadratic form $A_\theta$ is determined in [1] as follows.

**Theorem 1.** The signature of the quadratic form $A_\theta$ on $\mathcal{P}_\theta$ is

$$
\left( \frac{1}{\pi} \sum_{s=1}^{n} \nu_s - 1, \frac{1}{\pi} \sum_{s=1}^{n} (\pi - \nu_s) - 1 \right)
$$

where $\nu_s$ is a real number in $(0, \pi)$ such that $\nu_s \equiv \theta_s \mod \pi$.

**Remark.** For convex polygons, the exterior angles $\theta = (\theta_1, \ldots, \theta_n)$ satisfy $0 < \theta_i < \pi$ and $\sum_{i=1}^{n} \theta_i = 2\pi$. Thus the signature in the convex case is $(1, n-3)$, which is studied in [9], [1], [5], [7], [3]. A nonconvex polygons bounding a region has negative exterior angles $-\pi < \theta_s < 0$ where corresponding $\nu_s$ is obtained by $\nu_s = \theta_s + \pi$. Thus if $p$ is the number of negative exterior angles, the corresponding signature is $(p+1, n-p-3)$ (see [7]).

Theorem 1 is easily understood in the case of convex polygons. There is at least one triple of the sides of a convex polygon $P$ whose extensions form a “big” triangle $T_0$ such that $P$ is contained inside $T_0$ and whose sides touch the sides of $T_0$ only along those three sides. The complement of $P$ in $T_0$ can be divided into $n-3$ “small” triangles $T_i$ by suitably extending the side of $P$ so that each $T_i$ touches a unique side of $P$. Then the area of $P$ is obtained by

$$
\text{Area}(P) = \text{Area}(T_0) - \sum_{i=1}^{n-3} \text{Area}(T_i)
$$
By the sine rule, one can see the lengths $t_i$ of the sides of $T_i$’s which touches $P$ for $0 \leq i \leq n - 3$ are linear combinations of $s_1, \ldots, s_n$ which turns out to define an isomorphism on the $(n - 2)$-dimensional vector space $V_\theta$. By (4), we see that

$$\text{Area}(P) = C_0 t_0^2 - \sum_{i=1}^{n-3} C_i t_i^2$$

where the constants $C_i$ ($0 \leq i \leq n - 3$) are expressed by the sines of exterior angles $\theta_i$’s. This leads the signature of the form $A_\theta$ to be $(1, n - 3)$. The signature of $A_\theta$ in the nonconvex case is understood similarly. That is, by extending the sides of a polygon $P$, there appear triangles where the area of $P$ is obtained by adding or subtracting in many possible ways the areas of some of these triangles, which contributes to count the positive or negative vectors in $P_\theta$ with respect to $A_\theta$ (see [7]).

Let $P_\theta$ be the space of Euclidean $n$-gons with prescribed angles $\theta = (\theta_1, \ldots, \theta_n)$ up to similarities with positive area. Since each similarity class can be uniquely represented by a polygon with Area $= 1$, the space $P_\theta$ is identified with an open subset in the space $A_\theta^{-1}(1)$ in $P_\theta$ which endows a pseudo-Riemannian structure of dimension $n - 3$. The closure $\bar{P}_\theta$ of $P_\theta$ in $V_\theta$ is an $(n - 3)$-dimensional polyhedron with a pseudo-Riemannian structure. Especially in the convex case, $\bar{P}_\theta$ is a hyperbolic polyhedron whose combinatorial and geometric structures are studied in [1], [5], [7], [3].

A trivial example. In the case of $n = 3$, for $\theta = (\theta_1, \theta_2, \theta_3)$ with $\theta_i > 0$, $P_\theta$ is a point in $\mathbb{R}^3$ with coordinates

$$\left(\sqrt{\frac{2 \sin \theta_3}{\sin \theta_1 \sin \theta_2}}, \sqrt{\frac{2 \sin \theta_1}{\sin \theta_2 \sin \theta_3}}, \sqrt{\frac{2 \sin \theta_2}{\sin \theta_3 \sin \theta_1}}\right)$$

3. Spaces of Euclidean cone structures on the 2-sphere

A Euclidean cone metric on the 2-sphere $S^2$ is a singular metric on $S^2$ which is Euclidean except at finite points $p_1, \ldots, p_n$, $n \geq 3$ and the neighbourhood of each point $p_k$ are modelled on the neighbourhood of a Euclidean cones with cone angle $\theta_k > 0$. The apex curvature at the cone points $p_k$ is $2\pi - \theta_k$. We note that if the cone angle $\theta_k$ at $p_k$ satisfies $0 < \theta_k < 2\pi$, the curvature at the cone point $p_k$ is positive and if $\theta_k > 2\pi$, the curvature is negative. By Gauss-Bonnet theorem, the curvatures $\alpha_k$’s of a Euclidean cone metric on $S^2$ satisfy the relation

$$\sum_{k=1}^{n} \alpha_k = 4\pi.$$  

Let $(\alpha_1, \ldots, \alpha_n)$ be an $n$-tuple with $n \geq 3$ of real numbers satisfying (7). We denote by $C(\alpha_1, \ldots, \alpha_n)$ the space of Euclidean cone metrics on $S^2$ with $n$ labelled cone points of curvatures $\alpha_k$, $1 \leq k \leq n$, up to orientation and label-preserving similarities. By Troyanov’s theorem ([10]), $C(\alpha_1, \ldots, \alpha_n)$ is homeomorphic to the configuration space of $n$ points on $\mathbb{C}P^1$ which is an $(n - 3)$-dimensional manifold.

Let $C$ be a Euclidean cone metric on $S^2$ which represents an element in $C(\alpha_1, \ldots, \alpha_n)$. There is a function assigning each $C$ its area $\text{Area}(C)$. When
$C$ has only positive curvatures on its cone points, that is $0 < \alpha_i < 2\pi$ for $1 \leq i \leq n$, it is shown in [9] that there is a complex $(n-2)$-dimensional local parametrisation of Euclidean cone metrics near $C$ up to orientation and label preserving Euclidean isometries, with respect to which the area function is a Hermitian form $\mathcal{A}$ of type $(1, n-3)$ inducing a complex hyperbolic structure on $C(\alpha_1, \ldots, \alpha_n)$. When $C$ has negative curvatures on some cone points, we see that there is also a complex $(n-2)$-dimensional local parametrisation, with respect to which the area function gives rise to a Hermitian form $\mathcal{A}$ of different type as follows.

**Theorem 2.** The signature of the Hermitian form $\mathcal{A}$ is

$$
\left( \frac{1}{2\pi} \sum_{s=1}^{n} \mu_s - 1, \frac{1}{2\pi} \sum_{s=1}^{n} (2\pi - \mu_s) - 1 \right)
$$

where $\mu_s = \log(\exp(\sqrt{-1} \alpha_s))$ is a real number in $(0, 2\pi)$.

**Example.** In [8], we studied the pseudo-Hermitian form on the space $C((n-2)\pi, \pi, \ldots, \pi)$ of Euclidean cone structures on the 2-sphere where $n = 2m + 1$ is an odd integer. The signature of the area form $\mathcal{A}$ on the parameter space is $(m, m)$. The same result appears in [6].

**References**


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