Josai Mathematical Monographs
vol 6 (2013), pp. 29–48

Linear differential equations on $\mathbb{P}^1$ and representations of quivers

Kazuki Hiroe*

Research Institute of Mathematical Sciences, Kyoto University,
Kyoto 606-8502 JAPAN.

Abstract

In this article we consider the additive Delinge-Simpson problem which is originally defined for systems of first order Fuchsian differential equations on the Riemann sphere. We will generalize this problem to systems of differential equations with at most unramified irregular singular points. A correspondence between the systems of differential equations and representations of quivers is given and applied to the additive Delinge-Simpson problem. This is a survey of the forthcoming paper [4] which contains the detailed proofs of the statements in this article.

1 Additive Deligne-Simpson problem

Let us recall the additive Deligne-Simpson problem (cf. [7]) for systems of linear Fuchsian differential equations. A system of first order linear differential equations is called *Fuchsian* if it is of the form

$$\frac{d}{dx}Y = \sum_{i=1}^{p} \frac{A_i}{x-a_i}Y, \quad A_i \in M(n,\mathbb{C}).$$

Namely, Fuchsian equations are systems of first order linear differential equations with coefficients in $\mathbb{C}(x)$ whose poles are at most simple poles. Here we call each A_i the *residue matrix* at the singular point a_i for $i = 1,\ldots,p$. Moreover $A_0 := -\sum_{i=1}^{p} A_i$ is called the *residue matrix* at ∞.

Let us fix $C_0,\ldots,C_p$, conjugacy classes of $M(n,\mathbb{C})$. Then the additive Delinge-Simpson problem for Fuchsian equations asks the following. *Does there exsit an irreducible Fuchsian equation*

$$\frac{d}{dx}Y = \sum_{i=1}^{p} \frac{A_i}{x-a_i}Y, \ (A_0 := -\sum_{i=1}^{p} A_i)$$

*E-mail:kazuki@kurims.kyoto-u.ac.jp

such that $A_i \in C_i$ *for all* $i = 0, \ldots, p$? Here we say that $\frac{d}{dx}Y = \sum_{i=1}^{p} \frac{A_i}{x-a_i}Y$ is *irreducible* if $A_0, \ldots, A_p$ have no nontrivial simultaneous invariant subspace of $\mathbb{C}^n$, i.e., if there exists $W \subsetneqq \mathbb{C}^n$ such that $A_iW \subset W$ for all $i = 0, \ldots, p$, then $W = \{0\}$. Namely the problem asks the existence of an irreducible Fuchsian equation with the prescribed local data, i.e., the conjugacy classes of residue matrices at singular points.

We can also reformulate this problem as a problem of matrices.

Definition 1.1 (the additive Deligne-Simpson problem for the conjugacy classes)**.** Let $C_0, \ldots, C_p$ be conjugacy classes of $M(n, \mathbb{C})$. Then let us consider the problem. *Does there exist a tuple of matrices* $(A_0, \ldots, A_p) \in C_0 \times \cdots \times C_p$ *such that*

1. $\sum_{i=0}^{p} A_i = 0$,
2. $(A_0, \ldots, A_p)$ *is irreducible, i.e., if there exists* $W \subsetneqq \mathbb{C}^n$ *such that* $A_iW \subset W$ *for all* $i = 0, \ldots, p$, *then* $W = \{0\}$?

We say the additive Deligne-Simpson problem for $C_0, \ldots, C_p$ is *solvable* if the above problem has a solution.

The study of this problem is developed by V. Kostov who gives an necessary and sufficient condition on the choices of $C_0, \ldots, C_p$, conjugacy classes of $M(n, \mathbb{C})$, for which the additive Deligne-Simpson problem are solvable under a generic condition. After his study, the complete necessary and sufficient condition is given by W. Crawley-Boevey.

As a generalization of this problem, it seems to be natural to consider the similar problem for non-Fuchsian equations (see for example [1], [8]). In particular we shall consider non-Fuchsian equations whose singular points are at most unramified irregular singular points. Our setting explained below can be seen as a natural generalization of the above Fuchsian case and its non-Fuchsian extensions studied in [1] and [8].

Before formulating the generalized problem precisely, let us recall some facts of local and formal theory of differential equations with irregular singular points. Let $M(n, \mathbb{C}((x)))$ be the ring of $n \times n$ matrices with the components in $\mathbb{C}((x)) := \{\sum_{i=r}^{\infty} c_i x^i \mid c_i \in \mathbb{C},\, r \in \mathbb{Z}\}$ which is the quotient field of the ring of formal power series $\mathbb{C}[[x]] := \{\sum_{i=0}^{\infty} c_i x^i \mid c_i \in \mathbb{C}\}$. The group of invertible elements of multiplication in $M(n, \mathbb{C}((x)))$ is denoted by $GL(n, \mathbb{C}((x)))$. Let us define the *valuation* $v(C)$ of $C = \sum_{i=r}^{\infty} c_i x^i \in M(n, \mathbb{C}((x))$ by $v(C) := \min\{i \mid c_i \neq 0\}$. For $A \in M(n, \mathbb{C}((x)))$ and $X \in GL(n, \mathbb{C}((x)))$ the *gauge transformation* of A by X is

$$X[A] := XAX^{-1} + \left(\frac{d}{dx}X\right)X^{-1}.$$

This definition reflects the fact that if the differential equation $\frac{d}{dx}y = Ay$ is satisfied by y, a $\mathbb{C}^n$-valued function, then $z = X^{-1}y$ satisfies the transformed equation $\frac{d}{dx}z = X[A]z$.

Definition 1.2 (Hukuhara-Turrittin-Levelt normal forms)**.** If an element $B \in M(n, \mathbb{C}((x)))$ is of the form

$$B = \mathrm{diag}(q_1(x^{-1})I_{n_1} + B_1 x^{-1}, \ldots, q_m(x^{-1})I_{n_m} + B_m x^{-1})$$

with $q_i(s) \in s^2\mathbb{C}[s]$ satisfying $q_i \neq q_j$ if $i \neq j$ and $B_i \in M(n_i, \mathbb{C})$, then B is called the *Hukuhara-Turrittin-Levelt normal form* or the *HTL normal form* shortly. Here I_m is the identity matrix of $M(m, \mathbb{C})$.

The theorem below is one of the most fundamental fact in the formal and local theory of the systems of linear differential equations.

Theorem 1.3 (Hukuhara, Turrittin, Levelt, see [10] for example)**.** *Let $A \in M(n, \mathbb{C}((x)))$. Then there exists a field extension $\mathbb{C}((t))$ of $\mathbb{C}((x))$ with $t^q = x$, $q \in \mathbb{Z}_{>0}$ and $X \in GL(n, \mathbb{C}((t)))$ such that A is reduced to the HTL normal form*

$$X[A] = \mathrm{diag}(q_1(t^{-1})I_{n_1} + B_1 t^{-1}, \ldots, q_m(t^{-1})I_{n_m} + B_m t^{-1}).$$

Here $q_i(s) \in s^2\mathbb{C}[s]$ satisfying $q_i \neq q_j$ if $i \neq j$ and $B_i \in M(n_i, \mathbb{C})$ with $\sum_{i=1}^m n_i = n$. Moreover if we fix the field extension $\mathbb{C}((t))$, then the HTL normal form is uniquely determined by A up to the action of $\prod_{i=1}^m GL(n_i, \mathbb{C})$ and the permutations of the indices $\{1, \ldots, m\}$.

As a counterpart of conjugacy classes C_i of residue matrices in the Fuchsian Deligne-Simpson problem, we shall introduce truncated orbits. Let us define $G_k := GL(n, \mathbb{C}[[x]]/x^k\mathbb{C}[[x]])$, $k \geq 1$, which can be identified with

$$\Big\{A_0 + A_1 x + \cdots + A_{k-1}x^{k-1} \mid A_0 \in GL(n, \mathbb{C}),\ A_i \in M(n, \mathbb{C}),\ i = 1, \ldots, k-1\Big\}.$$

Also define

$$\begin{aligned}\mathfrak{g}_k &:= M(n, \mathbb{C}[[x]]/x^k\mathbb{C}[[x]]) \\ &= \Big\{A_0 + A_1 x + \cdots + A_{k-1}x^{k-1} \mid A_i \in M(n, \mathbb{C}),\ i = 0, \ldots, k-1\Big\}.\end{aligned}$$

The dual vector space $\mathfrak{g}_k^* := \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}_k, \mathbb{C})$ is identified with

$$M(n, x^{-k}\mathbb{C}[[x]]/\mathbb{C}[[x]]) = \left\{\frac{A_k}{x^k} + \cdots + \frac{A_1}{x} \,\Big|\, A_i \in M(n, \mathbb{C})\right\}$$

by the nondegenerate bilinear form $\mathfrak{g}_k \times \mathfrak{g}_k^* \ni (A, B) \mapsto \mathrm{res}_{x=0}\mathrm{tr}(AB) \in \mathbb{C}$. Here $\mathrm{res}_{x=0}(\sum_{i=r}^{\infty} A_i x_i) := A_{-1}$.

Then an HTL normal form $B \in M(n, \mathbb{C}((x)))$ with $v(B) \geq -k$ can be seen as an element in $\mathfrak{g}_k^*$. Thus we can consider the G_k-orbit of B in $\mathfrak{g}_k^*$, $\mathcal{O}_B := \{gBg^{-1} \in \mathfrak{g}_k^* \mid g \in G_k\}$ called the *truncated orbit* of B. The integer k is called the *degree* of $\mathcal{O}_B$.

Now we can define a generalization of the Deligne-Simpson problem for the differential equation with unramified irregular singular points.

Definition 1.4 (generalized additive Deligne-Simpson problem)**.** Fix HTL normal forms $B^{(i)} \in \mathfrak{g}_{k_i}^* \subset M(n, \mathbb{C}((x)))$ for $i = 0, \ldots, p$. Then the *generalized additive Deligne-Simpson problem* asks the following.

Does there exist an irreducible system of linear differential equation

$$\frac{d}{dx}Y = \left(\sum_{i=1}^{p} \sum_{j=1}^{k_i} \frac{A_{i,j}}{(x - a_i)^j} + \sum_{j=2}^{k_0} A_{0,j} x^{j-2} \right) Y$$

satisfying that $A^{(i)}(x) \in \mathcal{O}_{B^{(i)}}$ *for* $i = 0, \ldots, p$*?*

Here $A^{(i)}(x) := \sum_{j=1}^{k_i} A_{i,j} x^{-j}$ for $i = 0, \ldots, p$ and $A_{0,1} := -\sum_{i=1}^{p} A_{i,1}$. We say

$$\frac{d}{dx}Y = \left(\sum_{i=1}^{p} \sum_{j=1}^{k_i} \frac{A_{i,j}}{(x - a_i)^j} + \sum_{j=2}^{k_0} A_{0,j} x^{j-2} \right) Y$$

is *irreducible* if the tuple $(A_{i,j})_{\substack{0 \le i \le p \\ 1 \le j \le k_i}}$ of coefficient matrices is irreducible.

This can be seen as a natural generalization of the additive Deligne-Simpson problem for Fuchsian equations because the original problem for Fuchsian equations corresponds to the case $k_0 = \cdots = k_p = 1$ in the above setting.

As the Fuchsian case we can reformulate the generalized additive Deligne-Simpson problem as follows.

Definition 1.5. Let $\mathcal{O}_{B^{(0)}}, \ldots, \mathcal{O}_{B^{(p)}}$ be truncated orbits of HTL normal forms $B^{(i)} \in \mathfrak{g}_{k_i}^*$, $i = 0, \ldots, p$. Then the *generalized additive Deligne-Simpson problem for the truncated orbits* $\mathcal{O}_{B^{(0)}}, \ldots, \mathcal{O}_{B^{(p)}}$ asks the following. *Does there exist an irreducible element in*

$$\left\{ \left(\sum_{j=1}^{k_0} A_j^{(0)} x^{-j}, \ldots, \sum_{j=1}^{k_p} A_j^{(p)} x^{-j} \right) \in \mathcal{O}_{B^{(0)}} \times \cdots \times \mathcal{O}_{B^{(p)}} \;\middle|\; \sum_{i=0}^{p} A_1^{(i)} = 0 \right\} ?$$

Here $\left(\sum_{j=1}^{k_0} A_j^{(0)} x^{-j}, \ldots, \sum_{j=1}^{k_p} A_j^{(p)} x^{-j} \right)$ is called *irreducible* if the tuple of matrices $(A_j^{(i)})_{\substack{0 \le i \le p \\ 1 \le j \le m_i}}$ is irreducible. We say that the generalized additive Deligne-Simpson problem for $\mathcal{O}_{B^{(0)}}, \ldots, \mathcal{O}_{B^{(p)}}$ is *solvable* if the above problem has a solution.

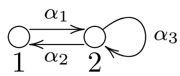
In the case $k_1 = \cdots = k_p = 1$ and $k_0 \le 3$, P. Boalch obtains the necessary and sufficient condition for the existence of a solution of the generalized additive Deligne- Simpson problem [1].

2 A review of Crawley-Boevey's theorem of representations of quivers

The study of additive Deligne-Simpson problem for Fuchsian equations is developed by V. Kostov. After Kostov's study, W. Crawley-Boevey gave the complete answer of the additive Deligne-Simpson problem for Fuchsian equations by using his theory of representations of deformed preprojective algebras. Let us give a quick review of the statement of one of Crawley-Boevey's theorems for the representation theory of deformed preprojective algebras (see [2] for the detail).

Definition 2.1 (quivers)**.** A *quiver* $Q = (Q_0, Q_1, s, t)$ is the quadruple consisting of Q_0, the set of *vertices*, Q_1, the set of *arrows* connecting vertices in Q_0, and two maps $s, t : Q_1 \to Q_0$ which associate to each arrow $\alpha \in Q_1$ its *source* $s(\alpha) \in Q_0$ and its *target* $t(\alpha) \in Q_0$ respectively.

For example let us consider the quiver Q.



Then $Q_0 = \{1, 2\}$, $Q_1 = \{\alpha_1, \alpha_2, \alpha_3\}$ and $s(\alpha_1) = 1$, $t(\alpha_1) = 2$, $s(\alpha_2) = 2$, $t(\alpha_2) = 1$, $s(\alpha_3) = t(\alpha_3) = 2$.

Although the notion of quivers appears in many topics of mathematics, here we are interested in the representations of quivers.

Definition 2.2 (representations of quivers)**.** Let Q be a finite quiver, i.e., Q_0 and Q_1 are finite sets. A *representation* M of Q is defined by the following data:

1. To each vertex a in Q_0, a finite dimensional $\mathbb{C}$- vector space $M_a = \mathbb{C}^{m_a}$ is associated.

2. To each arrow $\rho\colon a \to b$ in Q_1, a $\mathbb{C}$-linear map $\psi_\rho\colon M_a \to M_b$, equivalently $\psi_\rho \in M(m_b \times m_a, \mathbb{C})$, is associated.

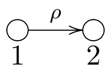
Let us call $\mathbf{dim}\, M := (m_a)_{a\in Q_0}$ the *dimension vector* of M. We denote the representation by $M = (M_a, \psi_\alpha)_{a\in Q_0, \alpha\in Q_1}$.

For each dimension vector $\alpha \in (\mathbb{Z}_{\geq 0})^{Q_0}$, we associate integers

$$q(\alpha) := \sum_{a\in Q_0} \alpha_a^2 - \sum_{\rho\in Q_1} \alpha_{s(\rho)}\alpha_{t(\rho)},$$
$$p(\alpha) := 1 - q(\alpha),$$

which will play important roles in the latter argument.

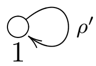
We denote the set of all representations of Q with the dimension vector $\alpha \in (\mathbb{Z}_{\geq 0})^{Q_0}$ by $\mathrm{Rep}\,(Q, \alpha)$. For example let us consider the quiver $Q^{(1)}$,



and the dimension vector $\alpha = (\alpha_1, \alpha_2)$. Then

$$\mathrm{Rep}(Q^{(1)}, \alpha) = M(\alpha_2 \times \alpha_1, \mathbb{C}).$$

Consider another quiver $Q^{(2)}$,



and the dimension vector $\beta = (\beta_1)$. Then

$$\mathrm{Rep}\,(Q^{(2)}, \beta) = M(\beta_1, \mathbb{C}).$$

Let us note that $\mathrm{Rep}\,(Q, \alpha)$ with the dimension vector $\alpha = (\alpha_a)_{a \in Q_0} \in (\mathbb{Z}_{\geq 0})^{Q_0}$ has an action of $\prod_{a \in Q_0} GL(\alpha_a, \mathbb{C})$ as below. For $M \in \mathrm{Rep}\,(Q, \alpha)$ and $g = (g_a) \in \prod_{a \in Q_0} GL(\alpha_a, \mathbb{C})$, the representation $g \cdot M \in \mathrm{Rep}\,(Q, \alpha)$ consists of the vector spaces M'_a, $(a \in Q_0)$ and $\psi'_\rho \in M(\alpha_{t(\rho)} \times \alpha_{s(\rho)}, \mathbb{C})$, $(\rho \in Q_1)$ as follows:

1. For each $a \in Q_0$, $M'_a := \mathbb{C}^{\alpha_a}$.
2. For each $\rho\colon a \to b \in Q_1$, $\psi'_\rho := g_b \psi_\rho g_a^{-1}$.

Let us consider the equivalent classes of the above examples. The equivalent classes $\mathrm{Rep}\,(Q^{(1)}, \alpha)/\prod_{a \in Q_0^{(1)}} GL(\alpha_a, \mathbb{C})$ can be identified with the finite set $\{1, 2, \ldots, \min\{\alpha_1, \alpha_2\}\}$, i.e., the set of ranks of elements in $M(\alpha_2 \times \alpha_1, \mathbb{C})$. On the other hand, the equivalent classes $\mathrm{Rep}\,(Q^{(2)}, \beta)/\prod_{a \in Q_0^{(1)}} GL(\beta_a, \mathbb{C})$ is classified by Jordan normal forms of $M(\beta_1, \mathbb{C})$.

Let $M = (M_a, \psi_\rho^M)_{a \in Q_0, \rho \in Q_1}$ and $N = (N_a, \psi_\rho^N)_{a \in Q_0, \rho \in Q_1}$ be representations of a quiver Q. Then N is called the *subrepresentation* of M if we have the following:

1. There exists the direct sum decomposition $M_a = N_a \oplus N'_a$ for each $a \in Q_0$.
2. For each $\rho\colon a \to b \in Q_1$, the equality $\psi_\rho^M|_{N_a} = \psi_\rho^N$ holds.

In this case we denote $N \subset M$. Moreover if we have

3. for each $\rho\colon a \to b \in Q_1$, we have $\psi_\rho^M|_{N'_a} \subset N'_b$,

we say M has the *direct sum decomposition* $M = N \oplus N'$ where $N' = (N'_a, \psi^M_\rho|_{N'_a})_{a\in Q_0,\rho\in Q_1}$.

The representation M is called *irreducible* if M has only subrepresentations M and $\{0\}$. Here $\{0\}$ is the representation of Q which consists of zero vector spaces and zero linear maps. On the other hand if any direct sum decomposition $M = N \oplus N'$ satisfies either $N = \{0\}$ or $N' = \{0\}$, then M is called *indecomposable*.

In [2] Crawley-Boevey considers the representations of the *doubles* of quivers. Let us recall the double of a quiver Q.

Definition 2.3 (double of a quiver)**.** Let $Q = (Q_0, Q_1)$ be a finite quiver. Then the *double* $\overline{Q}$ of Q is the quiver obtained by adjoining the reverse arrow $\rho^*\colon b \to a$ for each arrow $\rho\colon a \to b$. Namely $\overline{Q} = (\overline{Q}_0 := Q_0, \overline{Q}_1 := Q_1 \cup Q_1^*)$ where $Q_1^* := \{\rho^*\colon t(\rho) \to s(\rho) \mid \rho \in Q_1\}$.

Then the *moment map* $\mu_\alpha\colon \mathrm{Rep}(\overline{Q}, \alpha) \to \prod_{a\in Q_0} M(\alpha_a, \mathbb{C})$ is defined by

$$\mu_\alpha(x)_a := \sum_{\substack{\rho\in Q_1\\ t(\rho)=a}} \psi^x_\rho \psi^x_{\rho^*} - \sum_{\substack{\rho\in Q_1\\ s(\rho)=a}} \psi^x_{\rho^*}\psi^x_\rho, \quad a \in Q_0,$$

where $x = (x_a, \psi^x_\rho)_{a\in Q_0,\rho\in Q_1\cup Q_1^*} \in \mathrm{Rep}\,(\overline{Q}, \alpha)$.

Definition 2.4. Let Q be a finite quiver and $\overline{Q}$ the double of Q. Let us fix a dimension vector $\alpha \in (\mathbb{Z}_{\geq 0})^{Q_0}$ and a tuple of complex numbers $\lambda = (\lambda_a) \in \mathbb{C}^{Q_0}$. Then define the subspace of $\mathrm{Rep}\,(\overline{Q}, \alpha)$ by

$$\mathrm{Rep}\,(\overline{Q}, \alpha)_\lambda := \{M \in \mathrm{Rep}\,(\overline{Q}, \alpha) \mid \mu_\alpha(M)_a = \lambda_a I_{\alpha_a} \text{ for all } a \in Q_0\}.$$

In [2] Crawley-Boevey studies irreducible representations in $\mathrm{Rep}\,(\overline{Q}, \alpha)_\lambda$ and obtains the necessary and sufficient condition for the existence of the irreducible representations in terms of Kac-Moody root systems. Thus before seeing the existence theorem of irreducible representations, let us recall the definition of the root system of a quiver Q (cf. [6]).

Let Q be a finite quiver. The *Euler form* is

$$\langle \alpha, \beta\rangle := \sum_{a\in Q_0} \alpha_a\beta_a - \sum_{\rho\in Q_1} \alpha_{s(\rho)}\beta_{t(\rho)}$$

for $\alpha, \beta \in \mathbb{Z}^{Q_0}$ and *symmetric bilinear form* is

$$(\alpha, \beta) := \langle \alpha, \beta\rangle + \langle \beta, \alpha\rangle.$$

The each element $\epsilon(a) \in \mathbb{Z}^{Q_0}$, $a \in Q_0$ is called the *fundamental root* if $\epsilon(a)_a = 1$, $\epsilon(a)_b = 0$, $(b \in Q_0\backslash\{a\})$ and moreover there is no loop at the vertex a. Denote by Π the set of fundamental roots. For a fundamental root ϵ define the *fundamental reflection* r_ϵ by

$$r_\epsilon(\alpha) := \alpha - (\alpha, \epsilon)\epsilon \text{ for } \alpha \in \mathbb{Z}^{Q_0}.$$

The group $W \subset \operatorname{Aut} \mathbb{Z}^{Q_0}$ generated by all fundamental reflections is called *Weyl group* of the quiver Q. Note that the bilinear form $(\ ,\ )$ is W-invariant. Define the set of *real roots* by

$$\Delta^{\mathrm{re}} := \bigcup_{w \in W} w(\Pi).$$

For an element $\alpha = (\alpha_a)_{a \in Q_0} \in \mathbb{Z}^{Q_0}$ the *support* of α is the subquiver consists of the set of vertices a for which $\alpha_a \neq 0$ and all arrows joining these vertices. Define the *fundamental set* $F \subset \mathbb{Z}^{Q_0}$ by

$$F := \left\{\alpha \in (\mathbb{Z}_{\geq 0})^{Q_0} \backslash \{0\} \mid (\alpha, \epsilon) \leq 0 \text{ for all } \epsilon \in \Pi, \text{ support of } \alpha \text{ is connected}\right\}.$$

Then define the set of *imaginary roots* by

$$\Delta^{\mathrm{im}} := \bigcup_{w \in W} w(F \cup -F).$$

Then the *root system* is defined by

$$\Delta := \Delta^{\mathrm{re}} \cup \Delta^{\mathrm{im}}.$$

An element $\alpha \in \Delta \cap (\mathbb{Z}_{\geq 0})^{Q_0}$ is called *positive root* and denote by Δ^+ the set of positive roots.

Now let us recall Crawley-Boevey's theorem.

Theorem 2.5 (Crawley-Boevey [2]). *Let Q be a finite quiver and $\overline{Q}$ the double of Q. Let us fix a dimension vector $\alpha \in (\mathbb{Z}_{\geq 0})^{Q_0}$ and $\lambda = (\lambda_a) \in \mathbb{C}^{Q_0}$. Then there exists an irreducible representation in* $\operatorname{Rep}(\overline{Q}, \alpha)_\lambda$ *if and only if the following are satisfied,*

1. *$\alpha \in \Delta^+$ and $\lambda \cdot \alpha := \sum_{a \in Q_0} \lambda_a \alpha_a = 0$,*
2. *if there exists a decomposition $\alpha = \beta_1 + \beta_2 + \cdots$, with $\beta_i \in \Delta^+$ and $\lambda \cdot \beta_i = 0$, then $p(\alpha) > p(\beta_1) + p(\beta_2) + \cdots$.*

3 The additive Deligne-Simpson problem for Fuchsian equations

In [3] Crawley-Boevey gives the complete answer of the additive Deligne-Simpson problem for Fuchsian equations. He gives a one-to-one correspondence between irreducible Fuchsian equations with prescribed conjugacy classes of residue matrices and irreducible representations in $\operatorname{Rep}(\overline{Q}, \alpha)_\lambda$ with suitable Q, α and λ. Then Theorem 2.5 can be applied to give the answer of the additive Deligne-Simpson problem.

3.1 Conjugacy classes and representations of quivers

Let C be a conjugacy class of $M(n,\mathbb{C})$. Then there exist complex numbers $\xi_1,\ldots,\xi_d$ such that

$$\prod_{i=1}^{d}(A-\xi_i I_n)=0 \tag{1}$$

for all $A\in C$. For example consider the minimal polynomial. Moreover note that $m_i=\operatorname{rank}\prod_{j=1}^{i}(A-\xi_j I_n)$, $i=1,\ldots,d$ are independent of the choice of $A\in C$. Conversely, if $B\in M(n,\mathbb{C})$ satisfies $\operatorname{rank}\prod_{j=1}^{i}(B-\xi_j I_n)=m_i$ for all $i=1,\ldots,d$, then $B\in C$. Here we formally put $m_d=0$. This observation leads us to the following correspondence between the elements in C and some representations of a quiver.

Proposition 3.1 (Crawley-Boevey [3]). *Let us fix a conjugacy class of $M(n,\mathbb{C})$, C, and choose $\xi_1,\ldots,\xi_d\in\mathbb{C}$ as above. Put $m_k:=\operatorname{rank}\prod_{i=1}^{k}(A-\xi_i I_n)$, $k=1,\ldots,d-1$, for $A\in C$ and $m_0:=n$. Define the quiver Q as below.*

Put $\mathbf{m}:=(m_i)_{i\in Q_0}\in(\mathbb{Z}_{\geq 0})^{Q_0}$. Then

$$\Phi\colon\{A\in C\}\to \left\{M=(M_a,\psi_\rho)_{a\in Q_0,\rho\in Q_1\cup Q_1^*}\in\operatorname{Rep}(\overline{Q},\mathbf{m})\;\middle|\; \begin{array}{c}\mu_{\mathbf{m}}(M)_i=(\xi_{i+1}-\xi_i)I_{m_i}\text{ for all } i=1,\ldots,d-1,\\ \psi_\rho\text{: injective}, \psi_{\rho^*}\text{: surjective for all }\rho\in Q_1,\rho^*\in Q_1^*\end{array}\right\}\Big/\prod_{i=1}^{d-1}GL(m_i,\mathbb{C})$$

defined by $\Phi(A)=(M(A)_a,\psi(A)_\rho)_{a\in Q_0,\rho\in Q_1\cup Q_1^*}$ is bijection. Here

$$M(A)_0:=\mathbb{C}^n,\qquad M(A)_k:=\operatorname{Im}\prod_{i=1}^{k}(A-\xi_i I_n)\text{ for all }k=1,\ldots,d-1,$$

$$\psi_{\rho_i}:M(A)_{i+1}\hookrightarrow M(A)_i\text{ : inclusion},\qquad \psi_{\rho_i^*}:=(A-\xi_{i+1})|_{M(A)_i}.$$

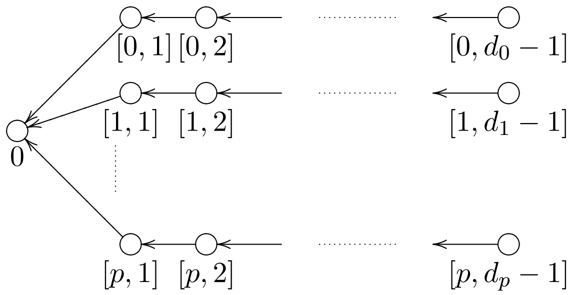
Moreover the inverse map is given by $(M_a,\psi_\rho)_{a\in Q_0,\rho\in Q_1\cup Q_1^*}\mapsto\psi_{\rho_1}\psi_{\rho_1^*}+\xi_1$.

The proposition gives a one-to-one correspondence between a conjugacy class and a subspace of the representations of a quiver. However to apply this correspondence to the additive Deligne-Simpson problem, we need a number of conjugacy classes simultaneously.

Theorem 3.2 (Crawley-Boevey [2]). *Let $C_0, \ldots, C_p$ be conjugacy classes of $M(n, \mathbb{C})$. For $i = 0, \ldots, p$, choose $\xi_{[i,1]}, \ldots, \xi_{[i,d_i]} \in \mathbb{C}$ so that*

$$\prod_{j=1}^{d_i} (A^{(i)} - \xi_{[i,j]} I_n) = 0$$

for all $A^{(i)} \in C_i$. Put $m_0 := n$ and $m_{[i,j]} := \operatorname{rank} \prod_{k=1}^{j} (A^{(i)} - \xi_{[i,k]} I_n)$ for $j = 1, \ldots, d_i - 1$. Consider the following quiver Q.



Define $\alpha = (\alpha_a)_{a \in Q_0} \in (\mathbb{Z}_{\geq 0})^{Q_0}$ by $\alpha_0 := m_0$ and $\alpha_{[i,j]} := m_{[i,j]}$ for $i = 0, \ldots, p$, $j = 1, \ldots, d_i - 1$. Also define $\lambda = (\lambda_a)_{a \in Q_0} \in \mathbb{C}^{Q_0}$ by $\lambda_0 := -\sum_{i=0}^{p} \xi_1$ and $\lambda_{[i,j]} := \xi_{[i,j+1]} - \xi_{[i,j]}$ for $i = 0, \ldots, p$, $j = 1, \ldots, d_i - 1$.

Then there exists a one-to-one correspondence

$$\left\{ (A_0, \ldots, A_p) \in C_0 \times \cdots \times C_p \,\middle|\, \substack{\sum_{i=0}^{p} A_i = 0, \\ (A_0, \ldots, A_p) \text{ is irreducible}} \right\} / GL(n, \mathbb{C}) \to$$
$$\left\{ M \in \operatorname{Rep}(\overline{Q}, \alpha)_\lambda \mid M \text{ is irreducible} \right\} / \prod_{a \in Q_0} GL(\alpha_a, \mathbb{C}).$$

This theorem tells us that the existence of a solution of the additive Deligne-Simpson problem follows from that of irreducible representations in $\operatorname{Rep}(\overline{Q}, \alpha)_\lambda$. Thus we can apply Theorem 2.5 to the additive Deligne-Simpson problem.

Theorem 3.3 (Crawley-Boevey [3]). *Let $C_0, \ldots, C_p$ be conjugacy classes of $M(n, \mathbb{C})$. Let us take a quiver Q and $\alpha \in (\mathbb{Z}_{\geq 0})^{Q_0}$ and $\lambda \in \mathbb{C}^{Q_0}$ as Theorem 3.2. Then the additive Deligne-Simpson problem for $C_0, \ldots, C_p$ is solvable if and only if the following are satisfied,*

1. *$\alpha \in \Delta^+$ and $\lambda \cdot \alpha := \sum_{a \in Q_0} \lambda_a \alpha_a = 0$,*
2. *if there exists a decomposition $\alpha = \beta_1 + \beta_2 + \cdots$, with $\beta_i \in \Delta^+$ and $\lambda \cdot \beta_i = 0$, then $p(\alpha) > p(\beta_1) + p(\beta_2) + \cdots$.*

4 Differential equations with poles of order 2 and representations of quivers

Now let us discuss a generalization of the additive Deligne-Simpson problem for non-Fuchsian equations. Before discussing in the general setting, we consider the case $k_0 = \cdots = k_p = 2$ in Definition 1.5.

Let $B \in \mathfrak{g}_2^*$ be a HTL normal form written by

$$B = \mathrm{diag}\left(\alpha_1 I_{n_1} x^{-2} + B_1 x^{-1}, \ldots, \alpha_m I_{n_m} x^{-2} + B_m x^{-1}\right).$$

Here $B_i \in M(n_i, \mathbb{C})$ and $\alpha_i \in \mathbb{C}$, $i = 0, \ldots, p$ satisfying $\alpha_i \neq \alpha_j$ if $i \neq j$.

First recall the structure of the truncated orbit $\mathcal{O}_B$. Let us put $B_{\mathrm{irr}} := \mathrm{diag}\,(\alpha_1 I_{n_1}, \ldots, \alpha_m I_{n_m})$ and denote by $V_i \subset \mathbb{C}^n$ the eigenspace of B_{irr} for each eigenvalue α_i, $i = 1, \ldots, m$. For any $X \in M(n, \mathbb{C})$ $X_{i,j}$ denote the $\mathrm{Hom}_{\mathbb{C}}(V_j, V_i)$-component of X with respect to the decomposition $M(n, \mathbb{C}) = \mathrm{End}_{\mathbb{C}}(\bigoplus_{i=1}^m V_i) = \bigoplus_{1 \leq i,j \leq m} \mathrm{Hom}_{\mathbb{C}}(V_i, V_j)$.

The following lemma is well-known.

Lemma 4.1. *Let $B \in \mathfrak{g}_2^*$ be the HTL normal form as above. Then $\mathcal{O}_B$ consists of $A(x) = \sum_{i=1}^2 A_i x^{-i} \in \mathfrak{g}_2^*$ satisfying the following. There exists $G \in GL(n, \mathbb{C})$ such that*

1. $GA_2G^{-1} = B_{irr}$,
2. $(GA_1G^{-1})_{i,i} = B_i$, $i = 1, \ldots, m$.

Next we consider a number of truncated orbits simultaneously and relate them to representations of a quiver. Let $B^{(0)}, \ldots, B^{(p)} \in \mathfrak{g}_2^*$ be HTL normal forms written by

$$B^{(i)} = \mathrm{diag}\left(\alpha_1^{(i)} I_{n_1^{(i)}} x^{-2} + B_1^{(i)} x^{-1}, \ldots, \alpha_{m_i}^{(i)} I_{n_{m_i}^{(i)}} x^{-2} + B_{m_i}^{(i)} x^{-1}\right).$$

Let $V_j^{(i)} \subset \mathbb{C}^n$ be the eigenspace of $B_{\mathrm{irr}}^{(i)}$ for each eigenvalue $\alpha_j^{(i)}$, $i = 0, \ldots, p$, $j = 1, \ldots, m_i$.

Let $X_{(i,j),(i'j')}$ be $\mathrm{Hom}_{\mathbb{C}}(V_{j'}^{(i')}, V_j^{(i)})$-component of $X \in M(n, \mathbb{C})$ with respect to $M(n, \mathbb{C}) = \bigoplus_{1 \leq j \leq m_i} \bigoplus_{1 \leq j' \leq m_{i'}} \mathrm{Hom}_{\mathbb{C}}(V_j^{(i')}, V_j^{(i)})$. We may write $X = \left(X_{(i,j),(i'j')}\right)_{\substack{1 \leq j \leq m_i, \\ 1 \leq j' \leq m_{i'}}}$.

Let us consider the quiver Q defined as follows. The set of vertices is

$$Q_0 := \{[i,j] \mid i = 0, \ldots, p,\ j = 1, \ldots, m_i\}.$$

The set of arrows is

$$Q_1 := \left\{\rho_{[i,j']}^{[0,j]} \colon [0,j] \to [i,j'] \,\middle|\, j = 1, \ldots, m_0,\ i = 1, \ldots, p,\ j' = 1, \ldots, m_i\right\}.$$

Take the dimension vector $\alpha = (\alpha_a)_{a \in Q_0} \in \mathbb{Z}^{Q_0}$ so that $\alpha_{[i,j]} := \dim_{\mathbb{C}} V_j^{(i)}$, $i = 0, \ldots, p$, $j = 1, \ldots, m_i$.

Proposition 4.2 ([4]). *We use the same notation as above. Then there exists a bijection*

$$\Phi\colon \left\{ \left(\sum_{i=1}^{2} A_i^{(0)} x^{-i}, \ldots, \sum_{i=1}^{2} A_i^{(p)} x^{-i} \right) \in \mathcal{O}_{B^{(0)}} \times \cdots \times \mathcal{O}_{B^{(p)}} \;\middle|\; \sum_{i=0}^{p} A_1^{(i)} = 0 \right\} / GL(n,\mathbb{C}) \longrightarrow$$

$$\left\{ (M_a, \psi_\rho) \in \operatorname{Rep}(\overline{Q}, \alpha) \;\middle|\; \det \left(\psi_{\rho^{[0,j]}_{[i,j']}} \right)_{\substack{1 \le j \le m_0 \\ 1 \le j' \le m_{j'}}} \neq 0,\ i = 1, \ldots, p, \right.$$

$$\left. \mu_\alpha(M)_{[i,j]} \in C_j^{(i)},\ [i,j] \in Q_0 \right\} / \prod_{i=0}^{p} \prod_{j=1}^{m_i} GL(\alpha_{[i,j]}, \mathbb{C}).$$

Here $C_j^{(i)}$ is the conjugacy class of each $B_j^{(i)}$, $i = 0, \ldots, p$, $j = 1, \ldots, m_i$.

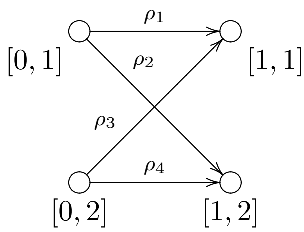
Let us explain the construction of Φ in the above proposition. Take an element $\mathbf{A} = \left(\sum_{i=1}^{2} A_i^{(0)} x^{-i}, \ldots, \sum_{i=1}^{2} A_i^{(p)} x^{-i} \right) \in \mathcal{O}_{B^{(0)}} \times \cdots \times \mathcal{O}_{B^{(p)}}$ with $\sum_{i=0}^{p} A_1^{(i)} = 0$ and choose $G_j \in GL(n,\mathbb{C})$ for each $\sum_{i=1}^{2} A_i^{(j)} x^{-i}$, $j = 0, \ldots, p$ as in Lemma 4.1. Under the conjugation by $GL(n,\mathbb{C})$, we may suppose $G_0 = I_n$. Then let us define $\Phi(\mathbf{A}) = (M_a, \psi_\rho)_{a \in Q_0, \rho \in Q_1 \cup Q_1^*}$ as follows,

1. $M_{[i,j]} := V_j^{(i)}$, $i = 0, \ldots, p$, $j = 1, \ldots, m_i$,
2. $\psi_{\rho^{[0,j]}_{[i,j']}} := (G^{(i)})_{(i,j'),(0,j)}$ and $\psi_{\left(\rho^{[0,j]}_{[i,j']}\right)^*} := \left(A_1^{(i)} (G^{(i)})^{-1} \right)_{(0,j),(i,j')}$, $j = 1, \ldots, m_0$, $i = 1, \ldots, p$, $j' = 1, \ldots, m_i$.

For example let us consider

$$B^{(0)} = \operatorname{diag}(\alpha_1 I_{m_1} x^{-2} + B_1^{(0)} x^{-1}, \alpha_2 I_{m_2} x^{-2} + B_2^{(0)} x^{-2}),$$
$$B^{(1)} = \operatorname{diag}(\beta_1 I_{n_1} x^{-2} + B_1^{(1)} x^{-1}, \beta_2 I_{n_2} x^{-2} + B_2^{(1)} x^{-2}),$$

and take $\mathbf{A} = \left(\sum_{i=1}^{2} A_i^{(0)} x^{-i}, \sum_{i=1}^{2} A_i^{(1)} x^{-i} \right) \in \mathcal{O}_{B^{(0)}} \times \mathcal{O}_{B^{(1)}}$ satisfying $A_1^{(0)} + A_1^{(1)} = 0$. Then the conjugation by $GL(n,\mathbb{C})$ allows to assume $A_2^{(0)} = B_{\mathrm{irr}}^{(0)}$, $(A_1^{(0)})_{i,i} = B_i^{(0)}$ for $i = 1, 2$ and there exists $G \in GL(n,\mathbb{C})$ such that $GA^{(1)}G^{-1} = B_{\mathrm{irr}}^{(1)}$ and $(GA_1^{(1)}G^{-1})_{i,i} = B_i^{(1)}$ for $i = 1, 2$. Then we can define the quiver Q,



,

and attach a representation $M = (M_a, \psi_\rho)_{a\in Q_0, \rho\in Q_1\cup Q_1^*}$ to $\mathbf{A}$ as follows:

$$M_{[0,i]} := \mathbb{C}^{m_i}, \quad M_{[1,i]} := \mathbb{C}^{n_i} \text{ for } i = 1, 2,$$
$$G = \left(\begin{array}{c|c} \psi_{\rho_1} & \psi_{\rho_3} \\ \hline \psi_{\rho_2} & \psi_{\rho_4} \end{array}\right), \quad A_1^{(1)}G^{-1} = \left(\begin{array}{c|c} \psi_{\rho_1^*} & \psi_{\rho_2^*} \\ \hline \psi_{\rho_3^*} & \psi_{\rho_4^*} \end{array}\right).$$

Then we have

$$\begin{aligned} GA_1^{(1)}G^{-1} &= \left(\begin{array}{c|c} \psi_{\rho_1}\psi_{\rho_1^*} + \psi_{\rho_3}\psi_{\rho_3^*} & * \\ \hline * & \psi_{\rho_2}\psi_{\rho_2^*} + \psi_{\rho_4}\psi_{\rho_4^*} \end{array}\right) \\ &= \left(\begin{array}{c|c} \mu_\alpha(M)_{[1,1]} & * \\ \hline * & \mu_\alpha(M)_{1,2} \end{array}\right). \end{aligned}$$

Similarly,

$$\begin{aligned} A_1^{(0)} &= -A_1^{(1)} = -A_1^{(1)}G^{-1}G \\ &= -\left(\begin{array}{c|c} \psi_{\rho_1^*}\psi_{\rho_1} + \psi_{\rho_2^*}\psi_{\rho_2} & * \\ \hline * & \psi_{\rho_3^*}\psi_{\rho_3} + \psi_{\rho_4^*}\psi_{\rho_4} \end{array}\right) \\ &= \left(\begin{array}{c|c} \mu_\alpha(M)_{[0,1]} & * \\ \hline * & \mu_\alpha(M)_{[0,2]} \end{array}\right). \end{aligned}$$

Thus we have $\mu_\alpha(M)_{[i,j]} = B_j^{(i)}$, $i = 0, 1$, $j = 1, 2$.

5 Truncated orbits and representations of quivers

In the previous section we restrict ourselves to the case $k_0 = \cdots = k_p = 2$. Let us consider truncated orbits of higher degrees and relate them to some representations of quivers.

Fix $k > 1$ and $B = \sum_{i=1}^{k} B^{[i]}x^{-i} \in \mathfrak{g}_k^*$ of HTL normal form written by

$$B = \operatorname{diag}\left(q_1(x^{-1})I_{n_1} + B_1x^{-1}, \ldots, q_m(x^{-1})I_{n_m} + B_mx^{-1}\right)$$

where $B_i \in M(n_i, \mathbb{C})$, $q_i(s) \in s^2\mathbb{C}[s]$, $i = 1, \ldots, m$ and $q_i \neq q_j$ if $i \neq j$.

Let $V_j^{[i]} \subset \mathbb{C}^n$, $i = 1, \ldots, k-1$, $j = 1, \ldots, m_{[i]}$ be simultaneous invariant spaces of $(B^{[i+1]}, \ldots, B^{[k-1]}, B^{[k]})$. Note that $m_{[1]} = m$.

Let $X_{i,j}$ denote the $\mathrm{Hom}_\mathbb{C}(V_j^{[1]}, V_i^{[1]})$- component of $X \in M(n, \mathbb{C}) = \bigoplus_{1\leq i,j\leq m_{[1]}} \mathrm{Hom}_\mathbb{C}(V_i^{[1]}, V_j^{[1]})$. For $g(x) = \sum_{i=r}^{\infty} g_ix^i \in M(n, \mathbb{C}((x)))$, define $g(x)_{j,j'} := \sum_{i=r}^{\infty}(g_i)_{j,j'}x^i$, $1 \leq j, j' \leq m_{[1]}$. In addition, with respect to the decomposition $M(n, \mathbb{C}) = \bigoplus_{1\leq i\leq m_{[1]}} \mathrm{Hom}_\mathbb{C}(V_i^{[1]}, \mathbb{C}^n)$, we denote $\mathrm{Hom}_\mathbb{C}(V_i^{[1]}, \mathbb{C}^n)$-component of $X \in M(n, \mathbb{C})$ by $X_{i,*}$ for $i = 1, \ldots, m_{[1]}$. Similarly $X_{*,i}$ denote the $\mathrm{Hom}_\mathbb{C}(\mathbb{C}^n, V_i^{[1]})$- component of $X \in M(n, \mathbb{C}) =$

$\bigoplus_{1\leq i\leq m_{[1]}} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^n, V_i^{[1]})$. We write $X = (X_{i,j})_{1\leq i,j\leq m_{[1]}} = (X_{*,i})_{1\leq i\leq m_{[1]}} = (X_{i,*})_{1\leq i\leq m_{[1]}}$.

Let $\pi_i : J_i := \{1,\ldots,m_{[i]}\} \to J_{i+1} := \{1,\ldots,m_{[i+1]}\}$ be the natural surjection such that $V_j^{[i]} \subset V_{\pi_i(j)}^{[i+1]}$. Put the total ordering $\{1 < 2 < \cdots < m_{[1]}\}$ on J_1 and also put the total ordering on J_i, $i = 2,\ldots,k-1$ so that

$$\text{if } j_1 < j_2, \text{ then } \pi_i(j_1) \leq \pi_i(j_2), \quad j_1, j_2 \in J_i.$$

For the pair $j \neq j' \in m_{[1]}$ we attach the number

$$d(j,j') = \max\{i \mid \pi_i(j) \neq \pi_i(j'),\, i = 0,\ldots,k-1\}. \tag{2}$$

Here we formally put $\pi_0 := id|_{J_1}$.

Let us define the subgroup of G_k,

$$G_k^0 := \left\{ \sum_{i=0}^{k-1} A_i x^i \in G_k \,\middle|\, A_0 = I_n \right\}$$

and its orbit $\mathcal{O}_B^0 := \{gBg^{-1} \in \mathfrak{g}_k^* \mid g \in G_k^0\}$.

According to the ordering on each J_i, $i = 1,\ldots,k-1$, define parabolic subalgebras of $M(n,\mathbb{C})$ as below,

$$\mathfrak{p}_i^+ := \bigoplus_{\substack{j_1,j_2\in J_i,\\ j_1\geq j_2}} \mathrm{Hom}_{\mathbb{C}}(V_{j_1}^{[i]}, V_{j_2}^{[i]}), \qquad \mathfrak{p}_i^- := \bigoplus_{\substack{j_1,j_2\in J_i,\\ j_1\leq j_2}} \mathrm{Hom}_{\mathbb{C}}(V_{j_1}^{[i]}, V_{j_2}^{[i]}),$$

and similarly nilpotent subalgebras

$$\mathfrak{u}_i^+ := \bigoplus_{\substack{j_1,j_2\in J_i,\\ j_1> j_2}} \mathrm{Hom}_{\mathbb{C}}(V_{j_1}^{[i]}, V_{j_2}^{[i]}), \qquad \mathfrak{u}_i^- := \bigoplus_{\substack{j_1,j_2\in J_i,\\ j_1< j_2}} \mathrm{Hom}_{\mathbb{C}}(V_{j_1}^{[i]}, V_{j_2}^{[i]}),$$

for $i = 1,\ldots,k-1$.

Also define the subsets of G_k^0,

$$P_k^{\pm} := \left\{ \sum_{i=0}^{k-1} P_i x^i \in G_k^0 \,\middle|\, P_i \in \mathfrak{p}_i^{\pm}, i = 1,\ldots,k-1 \right\},$$

$$U_k^{\pm} := \left\{ \sum_{i=0}^{k-1} U_i x^i \in G_k^0 \,\middle|\, U_i \in \mathfrak{u}_i^{\pm}, i = 1,\ldots,k-1 \right\},$$

and the subspace of $\mathfrak{g}_k^*$,

$$(\mathfrak{U}_k^{\mp})^* := \left\{ \sum_{i=1}^{k-1} U_i x^{-i-1} \,\middle|\, U_i \in \mathfrak{u}_i^{\pm},\, i = 1,\ldots,k-1 \right\}.$$

Then we can show the following decomposition.

Lemma 5.1. *For any $g \in G_k^0$, there uniquely exist $u_- \in U_k^-$ and $p_+ \in P_k^+$ such that $g = u_- \cdot p^+$.*

The above lemma shows that there exists a lower triangular matrix $u \in U_k^-$ and a upper triangular matrix $p \in P_k^+$ such that $A \in \mathcal{O}_B^0$ can be reduced to the upper triangular matrix $uAu^{-1} = p^{-1}Bp$. Furthermore we can show the following.

Proposition 5.2 ([4], [5]). *For any $A \in \mathcal{O}_B^0$ there uniquely exists $u \in U_{k-1}^-$ such that $\bar{B} := u^{-1}Au$ satisfies*

$$\bar{B} - B \in \mathfrak{U}_k^- \quad (mod\ x^{-1}\mathbb{C}[[x]]).$$

If we write $\bar{B} = \sum_{i=1}^k \bar{B}^{[i]}x^{-i}$, we can also show that $\bar{B}^{[k]} = B^{[k]}$ and $\bar{B}^{[1]} - B^{[1]} \in \mathfrak{u}_1^+ \oplus \mathfrak{u}_1^-$.

Let us take $A \in \mathcal{O}_B^0$, $u \in U_{k-1}^+$ and $\bar{B}$ as above proposition. Then $\tilde{B} = \sum_{i=1}^k \tilde{B}^{[i]}x^{-i} \in \mathfrak{g}_k^*$ is defined as follows:

$$\tilde{B}_{i,j} := 0 \text{ if } i > j,$$

$$\tilde{B}_{i,m_{[1]}} := \bar{B}_{i,m_{[1]}},$$

$$\tilde{B}_{i,m_{[1]}-j} :\equiv \bar{B}_{i,m_{[1]}-j} - \sum_{k=0}^{j-1} \tilde{B}_{i,m_{[1]}-k} u_{m_{[1]}-k,m_{[1]}-j} \ (\text{mod } \mathbb{C}[[x]]),$$

$$1 \le i \le m_{[1]},\ 1 \le j \le m_{[1]} - i.$$

Then the difference between the residue of $A \in \mathcal{O}_B^0$ and that of B can be computed by the above u and $\tilde{B}$ as follows.

Proposition 5.3. *Let us take $A \in \mathcal{O}_B^0$, $u \in U_{k-1}^+$ and $\tilde{B}$ as above. Then we have*

$$B_k - (\text{res}_{x=0}A)_{k,k} = \text{res}_{x=0}\left(-\sum_{i=1}^{k-1} b_{k,i}\tilde{B}_{i,k} + \sum_{i=k+1}^{m_{[1]}} \tilde{B}_{k,i}u_{i,k}\right).$$

Under these preparations, let us define the quiver Q as follows. The set of vertices is

$$Q_0 := \{0\} \cup \{1, \ldots, m_{[1]}\}.$$

The set of arrows is

$$Q_1 := \left\{\rho_{i,i'}^{[j]} : i \to i' \,\middle|\, 1 \le i < i' \le m_{[1]},\ j = 1, \ldots, d(i,i')\right\} \\ \cup \left\{\rho_i : 0 \to i \mid i = 1, \ldots, m_{[1]}\right\}.$$

Define the dimension vector $\alpha = (\alpha_a)_{a \in Q_0}$ by $\alpha_{[0]} := n$ and $\alpha_{[i]} := \dim_{\mathbb{C}} V_i^{[1]}$, $i = 1, \ldots, m_{[1]}$.

Proposition 5.4 ([4]). *There exists a bijection*

$$\Phi\colon \mathcal{O}_B \to \left\{ M=(M_a,\psi_\rho)\in \operatorname{Rep}(\overline{Q},\alpha) \;\middle|\; \begin{array}{l} \det(\psi_{\rho_i})_{1\le i\le m_{[1]}}\neq 0, \\ \mu_\alpha(M)_i\in C_i \text{ for } i=1,\dots,m_{[1]} \end{array}\right\} \Big/ \prod_{i=1}^{m_{[1]}} GL(\alpha_i,\mathbb{C}).$$

Here C_i are conjugacy classes of B_i for $i=1,\dots,m_{[1]}$.

Let us explain the construction of Φ. For $\bar{A}\in\mathcal{O}_B$ there exists $g\in GL(n,\mathbb{C})$ such that $A:=g\bar{A}g^{-1}\in\mathcal{O}^0_B$. Then by the above propositions, we can choose $u\in U^-_{k-1}$ and $\tilde{B}$ from A, and $\Phi(\bar{A})=(M_a,\psi_\rho)_{a\in Q_0,\rho\in Q_1\cup Q_1^*}$ is defined as follows:

$$\begin{aligned} M_0 &:= \mathbb{C}^n, & M_i &:= V_i^{[1]},\ i=1,\dots,m_{[1]},\\ \psi_{\rho^{[j]}_{i,i'}} &:= \tilde{B}^{[j+1]}_{i,i'}, & \psi_{(\rho^{[j]}_{i,i'})^*} &:= u^{[j]}_{i',i},\\ \psi_{\rho_i} &:= g_{i,*}, & \psi_{\rho_i^*} &:= \left(\operatorname{res}_{x=0}\bar{A}g^{-1}\right)_{*,i}. \end{aligned}$$

Here we set $\tilde{B}=\sum_{i=1}^k \tilde{B}^{[i]}x^{-i}$ and $u=\sum_{i=0}^{k-1}u^{[i]}x^i$. Then Proposition 5.3 tells us that $\mu_\alpha(M)_i=B_i\in C_i$ for $i=1,\dots,m_{[i]}$.

The compatibility between the symplectic structures of the coadjoint orbit $\mathcal{O}_B$ and that of the representation space of the quiver in Proposition 5.4 will be discussed in [5].

6 Generalized Delinge-Simpson problem and representations of a quiver

In the previous sections representations of quivers are associated with conjugacy classes of matrices (Proposition 3.1), tuples of truncated orbits of degree 2 (Proposition 4.2) and truncated orbits of higher degrees (Proposition 5.4). Glueing them together, now let us construct representations of quivers in order to apply to the generalized Deligne-Simpson problem.

Let $B^{(0)}=\sum_{i=1}^{k_0}B^{[0,i]}x^{-i}\in\mathfrak{g}^*_{k_0},\dots,B^{(p)}=\sum_{i=1}^{k_p}B^{[p,i]}x^{-i}\in\mathfrak{g}^*_{k_p}$ be HTL normal forms written by

$$B^{(i)}=\operatorname{diag}\left(q_1^{(i)}(x^{-1})I_{n_1^{(i)}}+B_1^{(i)}x^{-1},\dots,q_{m_i}^{(i)}(x^{-1})I_{n_{m_i}^{(i)}}+B_{m_i}^{(i)}x^{-1}\right)$$

for $i=0,\dots,p$ where $q_j^{(i)}(s)\in s^2\mathbb{C}[s]$ satisfying $q_j^{(i)}\neq q_{j'}^{(i)}$ if $j\neq j'$ and $B_j^{(i)}\in M(n_j^{(i)},\mathbb{C})$.

For $i=0,\dots,p$, $j=1,\dots,k_i-1$, let $V_k^{[i,j]}$, $k=1,\dots,m_{[i,j]}$, be the simultaneous eigenspaces of $(B^{[i,j+1]},\dots,B^{[i,k_i]})$. For each pair $j,j'\in\{1,\dots,m_{[i,1]}\}$, attach the integer $d_i(j,j')$ defined by the same way as in (2).

For each $B_j^{(i)}$, $i = 0, \ldots, p$ and $j = 1, \ldots, m_i$, let us choose complex numbers $\xi_1^{[i,j]}, \ldots, \xi_{e_{[i,j]}}^{[i,j]}$ so that

$$\prod_{k=1}^{e_{[i,j]}} (B_j^{(i)} - \xi_k^{[i,j]}) = 0.$$

Put $I_{\mathrm{irr}} := \{i \in \{0, \ldots, p\} \mid k_i > 1\} \cup \{0\}$ and $I_{\mathrm{reg}} := \{0, \ldots, p\} \backslash I_{\mathrm{irr}}$.

Now let us consider the following quiver Q. The set of vertices is

$$\begin{aligned} Q_0 := &\{[i,j] \mid i \in I_{\mathrm{irr}},\, j = 1, \ldots, m_i\} \\ &\cup \left\{[i,j,k] \mid i = 0, \ldots, p,\, j = 1, \ldots, m_i,\, k = 1, \ldots, e_{[i,j]} - 1\right\}. \end{aligned}$$

The set of arrows is

$$\begin{aligned} Q_1 := &\left\{\rho_{[i,j']}^{[0,j]} \colon [0,j] \to [i,j'] \,\middle|\, j = 1, \ldots, m_0,\, i \in I_{\mathrm{irr}} \backslash \{0\},\, j = 1, \ldots, m_i\right\} \\ &\cup \left\{\rho_{[i,j],[i,j']}^{[k]} \colon [i,j] \to [i,j'] \,\middle|\, i \in I_{\mathrm{irr}},\, 1 \leq j < j' \leq m_i,\, 1 \leq k \leq d(j,j')\right\} \\ &\cup \left\{\rho_1^{[i,j]} \colon [i,j,1] \to [i,j] \,\middle|\, i \in I_{\mathrm{irr}},\, j = 1, \ldots, m_i\right\} \\ &\cup \left\{\rho_{[0,j]}^{[i,1,1]} \colon [i,1,1] \to [0,j] \,\middle|\, i \in I_{\mathrm{reg}},\, j = 1, \ldots, m_0\right\} \\ &\cup \left\{\rho_k^{[i,j]} \colon [i,j,k] \to [i,j,k-1] \,\middle|\, i = 1, \ldots, p,\, j = 1, \ldots, m_i,\right. \\ &\qquad \left. k = 2, \ldots, e_{[i,j]} - 1\right\}. \end{aligned}$$

Let us define the dimension vector $\alpha = (\alpha_a)_{a \in Q_0}$ by $\alpha_{[i,j]} := n_j^{(i)}$, $\alpha_{[i,j,k]} := \dim_{\mathbb{C}} \left(\mathrm{rank} \prod_{l=1}^{k} (B_j^{(i)} - \xi_l^{[i,j]})\right)$. Also define $\lambda = (\lambda_a)_{a \in Q_0}$ by $\lambda_{[i,j]} := -\xi_1^{[i,j]}$ for $i \in I_{\mathrm{irr}} \backslash \{0\}$, $j = 1, \ldots, m_i$, $\lambda_{[0,j]} := -\xi_1^{[0,j]} - \sum_{i \in I_{\mathrm{reg}}} \xi_1^{[i,1]}$ for $j = 1, \ldots, m_0$, and $\lambda_{[i,j,k]} := \xi_{k+1}^{[i,j]} - \xi_k^{[i,j]}$ for $i = 0, \ldots, p$, $j = 1, \ldots, m_i$, $k = 1, \ldots, e_{[i,j]} - 1$.

Then combining Proposition 3.1, 4.2 and 5.4, we have the following bijection.

Theorem 6.1 ([4])**.** *Let $B^{(0)}, \ldots, B^{(p)}$ be HTL normal forms chosen as above. Then there exists a bijection*

$$\Phi \colon \left\{ \left(\sum_{j=1}^{k_0} A_j^{(0)} x^{-j}, \ldots, \sum_{j=1}^{k_p} A_j^{(p)} x^{-j} \right) \in \mathcal{O}_{B^{(0)}} \times \cdots \times \mathcal{O}_{B^{(p)}} \,\middle|\, \sum_{i=0}^{p} A_1^{(i)} = 0 \right\} / GL(n, \mathbb{C})$$

$$\to \Big\{ M = (M_a, \psi_{\rho \in Q_1 \cup Q_1*}) \in \mathrm{Rep}(\overline{Q}, \alpha)_\lambda \,\Big|$$

$$\det \left(\psi_{\rho^{[0,j]}_{[i,j']}} \right)_{\substack{1 \leq j \leq m_0 \\ 1 \leq j' \leq m_i}} \neq 0,\ i \in I_{irr} \backslash \{0\},$$

$$\left(\psi_{\rho^{[i,1,1]}_{[0,j]}} \right)_{1 \leq j \leq m_0} : M_{[i,1,1]} \to \bigoplus_{j=1}^{m_0} M_{[0,j]},\ \textit{injective},\ i \in I_{reg},$$

$$\left(\psi_{\left(\rho^{[i,1,1]}_{[0,j]}\right)^*} \right)_{1 \leq j \leq m_0} : \bigoplus_{j=1}^{m_0} M_{[0,j]} \to M_{[i,1,1]},\ \textit{surjective},\ i \in I_{reg},$$

$$\psi_{\rho^{[i,j]}_k},\ \textit{injective},\ \psi_{(\rho^{[i,j]}_k)*},\ \textit{surjective} \Big\} / \prod_{a \in Q_0} GL(\alpha_a, \mathbb{C}).$$

Unfortunately the above bijection Φ does not preserve irreducibility. Thus we introduce the following notion.

Definition 6.2 (quasi-irreducible)**.** If $X \in \mathrm{Rep}(\overline{Q}, \alpha)_\lambda$ has no nontrivial proper subrepresentation $Y \subsetneqq X$ in $\mathrm{Rep}(\overline{Q}, \alpha)_\lambda$ with $\mathbf{dim}\, Y = (\beta_a)_{a \in Q_0}$ satisfying

$$\sum_{j=1}^{m_0} \beta_{[0,j]} = \sum_{j=1}^{m_1} \beta_{[1,j]} = \cdots = \sum_{j=1}^{m_p} \beta_{[p,j]},$$

then X is called *quasi-irreducible.*

Then we have following correspondence between irreducible elements and quasi-irreducible representations.

Proposition 6.3. *There is the bijection*

$$\Phi\colon \left\{ \left(\sum_{j=1}^{k_0} A^{(0)}_j x^{-j}, \ldots, \sum_{j=1}^{k_p} A^{(p)}_j x^{-j} \right) \in \mathcal{O}_{B^{(0)}} \times \cdots \times \mathcal{O}_{B^{(p)}} \,\middle|\, \sum_{i=0}^{p} A^{(i)}_1 = 0,\ \textit{irreducible} \right\} / GL(n, \mathbb{C})$$

$$\to \Big\{ M = (M_a, \psi_{\rho \in Q_1 \cup Q_1*}) \in \mathrm{Rep}(\overline{Q}, \alpha)_\lambda \,\Big|\, \textit{quasi-irreducible},$$

$$\det \left(\psi_{\rho^{[0,j]}_{[i,j']}} \right)_{\substack{1 \leq j \leq m_0 \\ 1 \leq j' \leq m_i}} \neq 0,\ i \in I_{irr} \backslash \{0\} \Big\} / \prod_{a \in Q_0} GL(\alpha_a, \mathbb{C}).$$

Then as an application of the above correspondence, we obtain under a generic condition the necessary and sufficient condition of the existence of a solution of the generalized additive Deligne-Simpson problem.

Theorem 6.4 ([4]). *Let $B^{(0)}, \ldots, B^{(p)}$ be HTL normal forms. Let us take the quiver Q and the dimension vector $\alpha \in (\mathbb{Z}_{\geq 0})^{Q_0}$ and $\lambda \in \mathbb{C}^{Q_0}$ as in Theorem 6.1. Moreover assume that λ is generic. Then the generalized additive Delinge-Simpson problem for $\mathcal{O}_{B^{(0)}}, \ldots, \mathcal{O}_{B^{(p)}}$ is solvable if and only if the following are satisfied,*

1. $\alpha \in \Delta^+$ *and* $\lambda \cdot \alpha = 0$,
2. $q(\alpha) < 0$ *or* α *is indivisible.*

Here we say α is indivisible *if all components of α have no common divisors.*

Remark 6.5. If we assume the degree k_i of each $\mathcal{O}_{B^{(i)}}$ satisfies that $k_1 = \cdots = k_p = 1$ and $k_0 \leq 3$. P. Boalch obtains the complete answer of the generalized additive Deligne-Simpson problem [1] without generic condition of λ. Theorem 6.4 can be seen as a generalization of Oshima and Takemura (see Theorem 10.2 in [9]).

References

[1] P. Boalch, *Irregular connections and Kac-Moody root systems*, arXiv:0806.1050, 2008.

[2] W. Crawley-Boevey, *Geometry of the moment map for representations of quivers*, Compositio Math. **126**, no.3 (2001), 257–293.

[3] W. Crawley-Boevey, *On matrices in prescribed conjugacy classes with no common invariant subspace and sum zero*, Duke Math. J. **118**, no. 2 (2003), 339–352.

[4] K. Hiroe, *Systems of linear differential equations on $\mathbb{P}^1$ and representations of quivers*, in preparation.

[5] K. Hiroe, D. Yamakawa, in preparation.

[6] V. Kac, *Root systems, representations of quivers and invariant theory*, Lecture Notes in Mathematics Vol.996, Springer- Verlag, Berlin, 1983, pp. 74–108.

[7] V. Kostov, *The Deligne-Simpson problem-a survey*, J. Algebra **281**, no.1 (2004), 83–108.

[8] V. Kostov, *Additive Deligne-Simpson problem for non-Fuchsian systems*, Funkcialaj Ekvacioj **53** (2010), 395–410.

[9] T. Oshima, *Classification of Fuchsian systems and their connection problem*, arXiv:0811.2916, 2008, to appear in RIMS Kokyuroku Bessatsu.

[10] W. Wasow, *Asymptotic expansions for ordinary differential equations.* Pure and Applied Mathematics, Vol. XIV Interscience Publishers John Wiley & Sons, Inc 1965.