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AN ELEMENTARY APPROACH TO THE GAUSS HYPERGEOMETRIC FUNCTION

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Abstract. We give an introduction to the Gauss hypergeometric function, the hypergeometric equation and their properties in an elementary way. Moreover we explicitly and uniformly describe the connection coefficients, the reducibility of the equation and the monodromy group of the solutions.

1. Introduction

The Gauss hypergeometric function is the most fundamental and important special function and it has long been studied from various points of view. Many formulae for the function have been established and they are contained in the books on special functions such as [WW], [EMO], [WG], [SW] etc. In this paper we show and prove the fundamental formulae in an elementary way.

We first give local solutions of the Gauss hypergeometric equation for every parameter, recurrent relations among three consecutive functions and contiguous relations. Then we show the Gauss summation formula, the connection formula and the monodromy group which is expressed by an explicit base of the space of the solutions depending holomorphically on the parameters of the hypergeometric equation. Our results are valid without an exception of the value of the parameter.

The author recently shows in [O1] and [O2] that it is possible to analyze solutions of general Fuchsian linear ordinary differential equations and get the explicit formulae as in the case of the Gauss hypergeometric equations, in particular, in the case when the equation has a rigid spectral type.

Theorem 8 with Remark 9 may contain a new result but most results in this paper are known. The author hopes that this paper will be useful for the reader to understand the Gauss hypergeometric functions and moreover the analysis on general Fuchsian differential equations in [O2].

In this paper we will not use the theory of integrals nor gamma functions even for the connection formula and for the expression of the monodromy groups in contrast to [MS].

For example, the Liouville theorem is known to be proved by the Cauchy integral formula, whose generalization is the Fuchs relation on Fuchsian linear ordinary differential equations, is proved without the theory of integrals as follows.

Let $u(x) = \sum_{n=0}^{\infty} a_n x^n$ be a function on $\mathbb{C}$ defined by a power series whose radius of convergence is ∞. Suppose there exists a non-negative integer N such that $(1+|x|)^{-N}|u(x)|$ is bounded. Suppose moreover that $u(x)$ is not a polynomial. Replacing $u(x)$ by $\frac{1}{x^{N+1}}\bigl(u(x) - \sum_{n=0}^{N} a_n x^n\bigr)$, we may assume $\lim_{x\to\infty} |u(x)| = 0$. Then there exists $c \in \mathbb{C}$ satisfying $|u(x)| \le |u(c)| \ne 0$ for all $x \in \mathbb{C}$. Replacing $u(x)$ by $Cu(ax+c)$ with a certain complex numbers a and C, we may assume that there

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$$u(x) = 1 + x^m + \textstyle\sum_{n=1}^\infty b_n x^{m+n} \quad \text{and} \quad |u(x)| \le u(0) = 1 \qquad (\forall x \in \mathbb{C}).$$

But if $0 < \varepsilon \ll 1$, we have $\sum_{n=1}^\infty |b_n|\varepsilon^n \le \frac{1}{2}$ and $|u(\varepsilon)| \ge 1 + \frac{1}{2}\varepsilon^m$.

2. Gauss hypergeometric series and hypergeometric equation

For complex numbers α, β and γ, Euler studied that the Gauss *hypergeometric series*

$$F(\alpha,\beta,\gamma;x) = \sum_{n=0}^\infty \frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{x^n}{n!} = 1 + \frac{\alpha\beta}{\gamma}\frac{x}{1!} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}\frac{x^2}{2!} + \cdots \tag{2.1}$$

with

$$(a)_n := \prod_{\nu=0}^{n-1}(a+\nu) = a(a+1)\cdots(a+n-1) \qquad (a \in \mathbb{C}) \tag{2.2}$$

gives a solution of the Gauss *hypergeometric equation*

$$x(1-x)u'' + \bigl(\gamma - (\alpha+\beta+1)x\bigr)u' - \alpha\beta u = 0. \tag{2.3}$$

Here we note that

$$F(\alpha,\beta,\gamma;x) = F(\beta,\alpha,\gamma;x). \tag{2.4}$$

We will review this and obtain all the solutions of the equation around the origin. We introduce the notation

$$\partial := \tfrac{d}{dx}, \quad \vartheta := x\partial$$

and then the Gauss hypergeometric equation is

$$P_{\alpha,\beta,\gamma}u = 0 \tag{2.5}$$

with the linear ordinary differential operator

$$P_{\alpha,\beta,\gamma} := x(1-x)\partial^2 + \bigl(\gamma - (\alpha+\beta+1)x\bigr)\partial - \alpha\beta. \tag{2.6}$$

Since $\vartheta^2 = x^2\partial^2 + x\partial = x^2\partial^2 + \vartheta$, we have

$$\begin{aligned} xP_{\alpha,\beta,\gamma} &= (x^2\partial^2 + \gamma x\partial) - x\bigl(x^2\partial^2 + (\alpha+\beta+1)x\partial + \alpha\beta\bigr) \\ &= (\vartheta^2 - \vartheta + \gamma\vartheta) - x\bigl(\vartheta^2 + (\alpha+\beta)\vartheta + \alpha\beta\bigr) \\ &= \vartheta(\vartheta+\gamma-1) - x(\vartheta+\alpha)(\vartheta+\beta), \end{aligned} \tag{2.7}$$

the equation (2.3) is equivalent to

$$\vartheta(\vartheta+\gamma-1)u = x(\vartheta+\alpha)(\vartheta+\beta)u. \tag{2.8}$$

Putting $u = \sum_{n=0}^\infty c_n x^n$ and comparing the coefficients of x^n in the equation (2.8), we have

$$n(n+\gamma-1)c_n = (n-1+\alpha)(n-1+\beta)c_{n-1} \quad (c_{-1} = 0,\ n = 0,1,\ldots) \tag{2.9}$$

and therefore

$$\begin{aligned} c_n &= \frac{(\alpha+n-1)(\beta+n-1)}{(\gamma+n-1)n}c_{n-1} \\ &= \frac{(\alpha+n-1)(\alpha+n-2)(\beta+n-1)(\beta+n-2)}{(\gamma+n-1)(\gamma+n-2)n(n-1)}c_{n-2} = \cdots = \frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!}c_0, \end{aligned}$$

which shows that

$$u_{[\alpha,\beta,\gamma]}(x) := F(\alpha,\beta,\gamma;x) \tag{2.10}$$

is a solution of (2.3) if

$$\gamma \notin \{0,-1,-2,\ldots\}. \tag{2.11}$$

3. Local solutions

For a function $h(x)$ and a linear differential operator P we put

$$\operatorname{Ad}\bigl(h(x)\bigr)(P) := h(x)\circ P\circ h(x)^{-1}$$

and then

$$\operatorname{Ad}\bigl(h(x)\bigr)(\partial) = \partial - \frac{h'(x)}{h(x)},\quad \operatorname{Ad}(x^\lambda)(\vartheta) = \vartheta - \lambda \quad (\lambda\in\mathbb{C}).$$

Thus we have

$$\begin{aligned}\operatorname{Ad}\bigl(x^{\gamma-1}\bigr)(xP_{\alpha,\beta,\gamma}) &= (\vartheta-\gamma+1)\vartheta - x(\vartheta+\alpha-\gamma+1)(\vartheta+\beta-\gamma+1)\\ &= xP_{\alpha-\gamma+1,\beta-\gamma+1,2-\gamma}.\end{aligned}\tag{3.1}$$

Since $P_{\alpha,\beta,\gamma}u=0$ is equivalent to $\operatorname{Ad}\bigl(x^{\gamma-1}\bigr)(xP_{\alpha,\beta,\gamma})x^{\gamma-1}u=0$, we have another solution

$$v_{[\alpha,\beta,\gamma]}(x) := x^{1-\gamma}F(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma;x)\tag{3.2}$$

if $2-\gamma\notin\{0,-1,-2,\ldots\}$, namely,

$$\gamma\notin\{2,3,4,\ldots\}.\tag{3.3}$$

We have linearly independent solutions $u_{[\alpha,\beta,\gamma]}$ and $v_{[\alpha,\beta,\gamma]}$ when $\gamma\notin\mathbb{Z}$.

Since $u_{[\alpha,\beta,1]} = v_{[\alpha,\beta,1]}$, the function

$$w^{(0)}_{[\alpha,\beta,\gamma]} := (\gamma-1)^{-1}(u_{[\alpha,\beta,\gamma]} - v_{[\alpha,\beta,\gamma]})\tag{3.4}$$

is holomorphic with respect to γ when $|\gamma-1|<1$. Then we have

$$w^{(0)}_{[\alpha,\beta,1]}(x) = \log x\cdot F(\alpha,\beta,1;x) + \sum_{k=1}^{\infty} a_k x^k\tag{3.5}$$

with some $a_k\in\mathbb{C}$ and

$$w^{(0)}_{[\alpha,\beta,1]}(x) = \frac{d}{dt}\Bigl(x^t\cdot F(\alpha+t,\beta+t,1+t;x) - F(\alpha,\beta,1-t;x)\Bigr)\Big|_{t=0}.\tag{3.6}$$

Note that $u_{[\alpha,\beta,\gamma]}(x)$ and $w^{(0)}_{[\alpha,\beta,\gamma]}(x)$ give independent solutions when $|\gamma-1|<1$.

Now we examine the case when $\gamma=-m$ with a non-negative integer m. In this case the function $\bigl((\gamma+m)u_{[\alpha,\beta,\gamma]}\bigr)|_{\gamma=-m}$ is a solution of $P_{\alpha,\beta,-m}u=0$ and therefore

$$\bigl((\gamma+m)F(\alpha,\beta,\gamma;x)\bigr)\Big|_{\gamma=-m} = \frac{(\alpha)_{m+1}(\beta)_{m+1}}{(-m)_m(m+1)!}x^{m+1}F(\alpha+m+1,\beta+m+1,m+2;x)$$

and $(-m)_m = (-1)^m m!$. Hence the solution

$$w^{(m+1)}_{[\alpha,\beta,\gamma]} = u_{[\alpha,\beta,\gamma]} - \frac{(\alpha)_{m+1}(\beta)_{m+1}}{(\gamma)_{m+1}(m+1)!}v_{[\alpha,\beta,\gamma]}\qquad (m\in\{0,1,2,\ldots\})\tag{3.7}$$

is holomorphic with respect to γ if $|\gamma+m|<1$ and

$$\begin{aligned}w^{(m+1)}_{[\alpha,\beta,\gamma]}\Big|_{\gamma=-m} &= \sum_{k=0}^{m}\frac{(\alpha)_k(\beta)_k}{(-m)_k}\frac{x^k}{k!} + \sum_{k=m+1}^{\infty} b_k x^k\\ &\quad + \frac{(\alpha)_{m+1}(\beta)_{m+1}}{(-m)_m(m+1)!}\cdot x^{m+1}\log x\cdot F(\alpha+m+1,\beta+m+1,m+2;x).\end{aligned}$$

Then $v_{[\alpha,\beta,\gamma]}$ and $w^{(m+1)}_{[\alpha,\beta,\gamma]}$ are independent solutions when $|\gamma+m|<1$.

By the analytic continuation along the path $[0, 2\pi] \ni t \mapsto e^{\sqrt{-1}t}z$, the solutions change as follows

$$\begin{aligned}
u_{[\alpha,\beta,\gamma]}(e^{2\pi\sqrt{-1}}z) &= u_{[\alpha,\beta,\gamma]}(z),\\
v_{[\alpha,\beta,\gamma]}(e^{2\pi\sqrt{-1}}z) &= e^{2\pi\sqrt{-1}(1-\gamma)}v_{[\alpha,\beta,\gamma]}(z),\\
w^{(m+1)}_{[\alpha,\beta,-m]}(e^{2\pi\sqrt{-1}}z) &= \begin{cases} w^{(m+1)}_{[\alpha,\beta,-m]}(z) + 2\pi\sqrt{-1}v_{[\alpha,\beta,-m]}(z) & (m=-1),\\ w^{(m+1)}_{[\alpha,\beta,-m]}(z) \\ \quad +2\pi\sqrt{-1}\frac{(\alpha)_{m+1}(\beta)_{m+1}}{(-m)_m(m+1)!}v_{[\alpha,\beta,-m]}(z) & (m=0,1,\ldots).\end{cases}
\end{aligned}$$

When $|\gamma - m - 2| < 1$ with $m \in \{0, 1, 2, \ldots\}$, we have independent solutions $u_{[\alpha,\beta,\gamma]}$ and $x^{1-\gamma}w^{(m+1)}_{[\alpha-\gamma+1,\beta-\gamma+1,2-\gamma]}$, which is obtained by the correspondence $u(x) \mapsto x^{\gamma-1}u(x)$. Thus we have the following pairs of independent solutions.

$$\begin{aligned}
&\bigl(u_{[\alpha,\beta,\gamma]}, v_{[\alpha,\beta,\gamma]}\bigr) && (\gamma \notin \mathbb{Z}),\\
&\bigl(v_{[\alpha,\beta,\gamma]}, w^{(1-m)}_{[\alpha,\beta,\gamma]}\bigr) && (|\gamma - m| < 1,\ m \in \{1, 0, -1, -2, \ldots\}),\\
&\bigl(u_{[\alpha,\beta,\gamma]}, x^{1-\gamma}w^{(m-1)}_{[\alpha-\gamma+1,\beta-\gamma+1,2-\gamma]}\bigr) && (|\gamma - m| < 1,\ m \in \{2, 3, 4, \ldots\}).
\end{aligned} \tag{3.8}$$

Remark 1. It is easy to see from (2.9) that there exists a *polynomial solution* $u(x)$ with $u(0) = 1$ if and only if

$$\begin{aligned}
&\{\alpha,\ \beta\} \cap \{0, -1, -2, -3, \ldots\} \neq \emptyset\\
\text{and}\ \ &\gamma \notin \{0, -1, \ldots, 1-m\}\ \ \text{or}\ \ m = 0\\
\text{with}\ \ &m := -\max\bigl\{\{\alpha,\ \beta\} \cap \{0, -1, -2, -3, \ldots\}\bigr\}.
\end{aligned} \tag{3.9}$$

Then the polynomial solution is called a *Jacobi polynomial*[1] and equals

$$\sum_{k=0}^{m} \frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{x^k}{k!}. \tag{3.10}$$

Here (3.9) is equivalent to the existence of $m \in \{0, 1, 2, \ldots\}$ such that

$$(\alpha + m)(\beta + m) = 0 \ \text{ and } \ (\alpha + k)(\beta + k)(\gamma + k) \neq 0 \quad (k = 0, \ldots, m-1). \tag{3.11}$$

4. Symmetry of hypergeometric equation

By the coordinate transformation $T_{0\leftrightarrow1}$: $x \mapsto 1 - x$, we have $T_{0\leftrightarrow1}(\partial) = -\partial$ and

$$\begin{aligned}
T_{0\leftrightarrow1}(P_{\alpha,\beta,\gamma}) &= x(1-x)\partial^2 + \bigl(\gamma - (\alpha+\beta+1) + (\alpha+\beta+1)x\bigr)(-\partial) - \alpha\beta\\
&= P_{\alpha,\beta,\alpha+\beta-\gamma+1}.
\end{aligned}$$

Then we have local solutions for $|1 - x| < 1$:

$$\begin{aligned}
u_{[\alpha,\beta,\alpha+\beta-\gamma+1]}(1-x) &= F(\alpha, \beta, \alpha+\beta-\gamma+1; 1-x),\\
v_{[\alpha,\beta,\alpha+\beta-\gamma+1]}(1-x) &= (1-x)^{\gamma-\alpha-\beta}F(\gamma-\alpha, \gamma-\beta, \gamma-\alpha-\beta+1; 1-x).
\end{aligned} \tag{4.1}$$

By the coordinate transformation $T_{0\leftrightarrow\infty} : x \mapsto \frac{1}{x}$, we have $T_{0\leftrightarrow\infty}(\vartheta) = -\vartheta$ and

$$\begin{aligned}
x\operatorname{Ad}\bigl(x^{-\alpha}\bigr) \circ T_{0\leftrightarrow\infty}(xP_{\alpha,\beta,\gamma}) &= \operatorname{Ad}\bigl(x^{-\alpha}\bigr)\bigl(-x\vartheta(-\vartheta+\gamma-1) - (-\vartheta+\alpha)(-\vartheta+\beta)\bigr)\\
&= x(\vartheta+\alpha)(\vartheta+\alpha-\gamma+1) - \vartheta(\vartheta+\alpha-\beta) = -xP_{\alpha,\alpha-\gamma+1,\alpha-\beta+1}.
\end{aligned}$$

Then we have local solutions for $|x| > 1$:

$$\begin{aligned}
(\tfrac{1}{x})^{\alpha}u_{[\alpha,\alpha-\gamma+1,\alpha-\beta+1]}(\tfrac{1}{x}) &= (\tfrac{1}{x})^{\alpha}F(\alpha, \alpha-\gamma+1, \alpha-\beta+1; \tfrac{1}{x}),\\
(\tfrac{1}{x})^{\alpha}v_{[\alpha,\alpha-\gamma+1,\alpha-\beta+1]}(\tfrac{1}{x}) &= (\tfrac{1}{x})^{\beta}F(\beta, \beta-\gamma+1, \beta-\alpha+1; \tfrac{1}{x}).
\end{aligned} \tag{4.2}$$

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[1] $F(-m, \beta, \gamma; x)$ is called a Jacobi polynomial and the Legendre polynomial, spherical polynomial and Chebyshev polynomial are special cases of this Jacobi polynomial (cf. [WG] etc.).

We have local solutions at the singular points $x = 1$ and $x = \infty$ by using $(u_{[\alpha',\beta',\gamma']}, v_{[\alpha',\beta',\gamma']})$ if $\gamma' \notin \mathbb{Z}$. When $\gamma' \in \mathbb{Z}$, we have independent solutions by the pairs of functions given in (3.8).

We have the *Riemann scheme*

$$P\begin{Bmatrix} x=0 & 1 & \infty & \\ 0 & 0 & \alpha & ;\ x \\ 1-\gamma & \gamma-\alpha-\beta & \beta & \end{Bmatrix} \tag{4.3}$$

which indicates the characteristic exponents at the singular points 0, 1 and ∞ of the equation $P_{\alpha,\beta,\gamma}u = 0$ and represents the space of solutions of $P_{\alpha,\beta,\gamma}u = 0$.

In general a differential equation has a *characteristic exponent* λ at $x = c$ if it has a solution u whose singularity at $x = c$ is as follows. Under the coordinate $y = x - c$ or $y = \frac{1}{x}$ according to $c \neq \infty$ or $c = \infty$, there exists a positive integer k such that

$$\lim_{y\to 0} y^{-\lambda} \log^{1-k} y \cdot u(y)$$

is a non-zero constant. The maximal integer k is the multiplicity of the characteristic exponent λ. In most cases the *multiplicity* is free and then $k = 1$. The characteristic exponent of $P_{\alpha,\beta,\gamma}u = 0$ at the origin is multiplicity free if and only if $\gamma \neq 1$.

Then $T_{0\leftrightarrow 1}$ and $T_{0\leftrightarrow\infty}$ give

$$\begin{aligned} P\begin{Bmatrix} x=0 & 1 & \infty & \\ 0 & 0 & \alpha & ;\ x \\ 1-\gamma & \gamma-\alpha-\beta & \beta & \end{Bmatrix} &= P\begin{Bmatrix} x=0 & 1 & \infty & \\ 0 & 0 & \alpha & ;\ 1-x \\ \gamma-\alpha-\beta & 1-\gamma & \beta & \end{Bmatrix} \\ &= P\begin{Bmatrix} x=0 & 1 & \infty & \\ \alpha & 0 & 0 & ;\ \frac{1}{x} \\ \beta & \gamma-\alpha-\beta & 1-\gamma & \end{Bmatrix} \\ &= (\tfrac{1}{x})^{\alpha} P\begin{Bmatrix} x=0 & 1 & \infty & \\ 0 & 0 & \alpha & ;\ \frac{1}{x} \\ \beta-\alpha & \gamma-\alpha-\beta & \alpha-\gamma+1 & \end{Bmatrix}. \end{aligned}$$

which corresponds to the above solutions. Compositions of transformations that we have considered give

$$\begin{aligned} &(1-x)^{\alpha+\beta-\gamma} P\begin{Bmatrix} x=0 & 1 & \infty & \\ 0 & 0 & \alpha & ;\ x \\ 1-\gamma & \gamma-\alpha-\beta & \beta & \end{Bmatrix} \\ &\quad = P\begin{Bmatrix} x=0 & 1 & \infty & \\ 0 & \alpha+\beta-\gamma & \gamma-\beta & ;\ x \\ 1-\gamma & 0 & \gamma-\alpha & \end{Bmatrix} \end{aligned}$$

and

$$\begin{aligned} P\begin{Bmatrix} x=0 & 1 & \infty & \\ 0 & 0 & \alpha' & ;\ \frac{x}{x-1} \\ 1-\gamma' & \gamma'-\alpha'-\beta' & \beta' & \end{Bmatrix} &= P\begin{Bmatrix} x=0 & 1 & \infty & \\ 0 & \alpha' & 0 & ;\ x \\ 1-\gamma' & \beta' & \gamma'-\alpha'-\beta' & \end{Bmatrix} \\ &= (1-x)^{\alpha'} P\begin{Bmatrix} x=0 & 1 & \infty & \\ 0 & 0 & \alpha' & ;\ x \\ 1-\gamma' & \beta'-\alpha' & \gamma'-\beta' & \end{Bmatrix}. \end{aligned}$$

Put $\gamma' = \gamma$, $\alpha' = \alpha$ and $\beta' = \gamma - \beta$. The local holomorphic function at the origin in the above which takes the value 1 at the origin gives Kummer's formula

$$F(\alpha,\beta,\gamma;x) = (1-x)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\gamma-\beta,\gamma;x) \tag{4.4}$$

$$= (1-x)^{-\alpha}F(\alpha,\gamma-\beta,\gamma;\tfrac{x}{x-1}) \tag{4.5}$$

$$= (1-x)^{-\beta}F(\beta,\gamma-\alpha,\gamma;\tfrac{x}{x-1}). \tag{4.6}$$

5. Reccurent relations

For integers ℓ, m and n the function $F(\alpha+\ell,\beta+m,\gamma+n;x)$ is called the consecutive function of $F(\alpha,\beta,\gamma;x)$ and it has been shown by Gauss that among three consecutive functions F_1, F_2 and F_3, there exists a recurrence relation of the form

$$A_1F_1 + A_2F_2 + A_3F_3 = 0, \tag{5.1}$$

where A_1, A_2 and A_3 are rational functions of x. In short we put $F = F(\alpha,\beta,\gamma;z)$ and $F(\alpha+\ell,\gamma+n) = F(\alpha+\ell,\beta,\gamma+n;x)$, $F(\alpha-1) = F(\alpha-1,\beta,\gamma;x)$ etc. Then there is a recurrence relation among F and any two functions of 6 closed neighbors $F(\alpha\pm1)$, $F(\beta\pm1)$, $F(\gamma\pm1)$, which we give in this section. There are $\binom{6}{2} = 15$ recurrence relations of this type and they generate the recurrence relations (5.1).

Since

$$\frac{(\alpha+1)_n}{n!} - \frac{(\alpha)_n}{n!} = \frac{(\alpha+1)_{n-1}(\alpha+n-\alpha)}{n!} = \frac{(\alpha+1)_{n-1}}{(n-1)!},$$

we have

$$\frac{(\alpha+1)_n(\beta)_n}{(\gamma)_n n!}x^n - \frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!}x^n = \frac{\beta}{\gamma}\frac{(\alpha+1)_{n-1}(\beta+1)_{n-1}}{(\gamma+1)_{n-1}(n-1)!}x^n$$

and therefore

$$\gamma\bigl(F(\alpha+1)-F\bigr) = \beta xF(\alpha+1,\beta+1,\gamma+1). \tag{5.2}$$

Moreover since

$$(\gamma-1)\frac{(\alpha)_n}{(\gamma-1)_n} - \alpha\frac{(\alpha+1)_n}{(\gamma)_n} - (\gamma-\alpha-1)\frac{(\alpha)_n}{(\gamma)_n}$$
$$= \frac{(\alpha)_n}{(\gamma)_n}\bigl((\gamma+n-1)-(\alpha+n)-(\gamma-\alpha-1)\bigr) = 0,$$

we have

$$(\gamma-1)F(\gamma-1) - \alpha F(\alpha+1) - (\gamma-\alpha-1)F = 0. \tag{5.3}$$

We obtain other recurrence relations from these two as follows.

By the symmetry between α and β we have

$$(\gamma-1)F(\gamma-1) - \beta F(\beta+1) - (\gamma-\beta-1)F = 0 \tag{5.4}$$

and by the difference of the above two relations we have

$$\alpha F(\alpha+1) - \beta F(\beta+1) - (\alpha-\beta)F = 0. \tag{5.5}$$

Moreover $(5.2)|_{\gamma\mapsto\gamma-1} + (5.3)$ is

$$(\gamma-1)F(\alpha+1,\gamma-1) - \alpha F(\alpha+1) - (\gamma-\alpha-1)F = \beta xF(\alpha+1,\beta+1)$$

and therefore

$$(\gamma-1)F(\gamma-1) - (\alpha-1)F - (\gamma-\alpha)F(\alpha-1) - \beta xF(\beta+1) = 0.$$

Substituting $(5.3) + x \times (5.5)$ from the above, we have

$$\alpha(1-x)F(\alpha+1) + \big(\gamma - 2\alpha + (\alpha-\beta)x\big)F - (\gamma-\alpha)F(\alpha-1) = 0. \tag{5.6}$$

The equation $(5.2)|_{\alpha\mapsto\alpha-1} - x \times (5.4)|_{\gamma\mapsto\gamma+1}$ shows

$$\gamma(1-x)F - \gamma F(\alpha-1) + (\gamma-\beta)xF(\gamma+1) = 0 \tag{5.7}$$

and $(\gamma-\alpha)\times(5.7) - \gamma\times(5.6)$ shows

$$\gamma\big(\alpha - (\gamma-\beta)x\big)F - \alpha\gamma(1-x)F(\alpha+1) + (\gamma-\alpha)(\gamma-\beta)xF(\gamma+1) = 0 \tag{5.8}$$

and $(5.8) - \gamma(1-x)\times(5.3)$ shows

$$\begin{aligned}\gamma\big(\gamma-1- (2\gamma-\alpha-\beta-1)x\big)F + (\gamma-\alpha)(\gamma-\beta)xF(\gamma+1)\\ - \gamma(\gamma-1)(1-x)F(\gamma-1) = 0.\end{aligned} \tag{5.9}$$

The relations (5.6), its transposition of α and β, (5.3), (5.7) and (5.5) generate other 14 relations among the nearest neighbors except for (5.9).

In fact $(5.6) - (1-x)\times(5.5)$ gives

$$\beta(1-x)F(\beta+1) + (\gamma-\alpha-\beta)F - (\gamma-\alpha)F(\alpha-1) = 0 \tag{5.10}$$

and $(5.10)|_{\alpha\leftrightarrow\beta} - (5.6)$ gives

$$(\alpha-\beta)(1-x)F + (\gamma-\alpha)F(\alpha-1) - (\gamma-\beta)F(\beta-1) = 0 \tag{5.11}$$

and $(1-x)\times(5.3) + (5.6)$ gives

$$(\gamma-1)(1-x)F(\gamma-1) - \big(\alpha-1-(\gamma-\beta-1)x\big)F - (\gamma-\alpha)F(\alpha-1) = 0. \tag{5.12}$$

Then $(5.3)^*$, (5.5), $(5.6)^*$, $(5.7)^*$, $(5.8)^*$, (5.9), $(5.10)^*$, (5.11) and $(5.12)^*$ are the 15 recurrence relations. Here the sign * represents two relations under the transposition of α and β.

6. Contiguous relations

Since

$$\frac{d}{dx}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{x^n}{n!} = \frac{\alpha\beta}{\gamma}\cdot\frac{(\alpha+1)_{n-1}(\beta+1)_{n-1}}{(\gamma+1)_{n-1}}\frac{x^{n-1}}{(n-1)!}$$

we have

$$\tfrac{d}{dx}F(\alpha,\beta,\gamma;x) = \tfrac{\alpha\beta}{\gamma}F(\alpha+1,\beta+1,\gamma+1;x). \tag{6.1}$$

Combining this with (5.2), we have

$$\alpha F(\alpha+1) = \alpha F + \tfrac{\alpha\beta x}{\gamma}F(\alpha+1,\beta+1,\gamma+1) = \big(x\tfrac{d}{dx}+\alpha\big)F$$

and then (5.3) shows

$$\big(x\tfrac{d}{dx}+\alpha\big)F = (\gamma-1)F(\gamma-1) + (\alpha+1-\gamma)F.$$

Thus we have

$$\big(x\tfrac{d}{dx}+\alpha\big)F = \alpha\cdot F(\alpha+1), \tag{6.2}$$

$$\big(x\tfrac{d}{dx}+\beta\big)F = \beta\cdot F(\beta+1), \tag{6.3}$$

$$\big(x\tfrac{d}{dx}+\gamma-1\big)F = (\gamma-1)\cdot F(\gamma-1). \tag{6.4}$$

Since

$$\begin{aligned}P_{\alpha,\beta,\gamma} - (1-x)\partial\big(x\partial+\alpha\big) &= \big(\gamma - (\alpha+\beta+1)x - (1-x) - \alpha(1-x)\big)\partial - \alpha\beta\\ &= (\gamma-\alpha-1-\beta x)\partial - \alpha\beta\\ &= \tfrac{1}{x}(\gamma-\alpha-1-\beta x)(x\partial+\alpha) - \tfrac{\alpha}{x}(\gamma-\alpha-1),\end{aligned}$$

we have

$$xP_{\alpha,\beta,\gamma} = \big(x(1-x)\partial + \gamma - \alpha - 1 - \beta x\big)(x\partial + \alpha) - \alpha(\gamma - \alpha - 1) \tag{6.5}$$

and

$$\big(x(1-x)\partial + \gamma - \alpha - 1 - \beta x\big)\alpha F(\alpha+1) = \alpha(\gamma-\alpha-1)F.$$

Hence

$$\big(x(1-x)\partial + \gamma - \alpha - \beta x\big)F = (\gamma - \alpha)\cdot F(\alpha-1), \tag{6.6}$$

$$\big(x(1-x)\partial + \gamma - \beta - \alpha x\big)F = (\gamma - \beta)\cdot F(\beta-1). \tag{6.7}$$

In the same way, since

$$\begin{aligned}
P_{\alpha,\beta,\gamma} &- (1-x)\partial\big(x\partial + \gamma - 1\big)\\
&= \big(\gamma - (\alpha+\beta+1)x - (1-x) - (\gamma-1)(1-x)\big)\partial - \alpha\beta\\
&= (\gamma - \alpha - \beta - 1)x\partial - \alpha\beta\\
&= (\gamma - \alpha - \beta - 1)\big(x\partial + \gamma - 1\big) - (\gamma-\alpha-1)(\gamma-\beta-1),
\end{aligned}$$

we have

$$P_{\alpha,\beta,\gamma} = \big((1-x)\partial + \gamma - \alpha - \beta - 1\big)\big(x\partial + \gamma - 1\big) - (\gamma-\alpha-1)(\gamma-\beta-1) \tag{6.8}$$

and

$$(\gamma-1)\big((1-x)\partial + \gamma - \alpha - \beta - 1\big)F(\gamma-1) = (\gamma-\alpha-1)(\gamma-\beta-1)F.$$

Hence

$$\big((1-x)\tfrac{d}{dx} + \gamma - \alpha - \beta\big)F = \frac{(\gamma-\alpha)(\gamma-\beta)}{\gamma}\cdot F(\gamma+1). \tag{6.9}$$

Note that

$$P = P_1P_2 - c \ \Rightarrow\ P_2P = (P_2P_1 - c)P_2 \ \text{ and } \ PP_1 = P_1(P_2P_1 - c)$$

for linear differential operators P, P_1 and P_2 and $c \in \mathbb{C}$. We apply this to the equations (6.5) and (6.8). Then $P_2P_1 - c$ equals

$$\begin{aligned}
(x\partial + \alpha)&\big(x(1-x)\partial + \gamma - \alpha - 1 - \beta x\big) - \alpha(\gamma-\alpha-1)\\
&= x^2(1-x)\partial^2 + x(1 - 2x + \alpha(1-x) + \gamma - \alpha - 1 - \beta x)\partial - \beta x - \alpha\beta x\\
&= x\big(x(1-x)\partial^2 + (\gamma - (\alpha+\beta+2)x)\partial - \beta - \alpha\beta\big)\\
&= xP_{\alpha+1,\beta,\gamma}
\end{aligned}$$

and

$$\begin{aligned}
(x\partial + \gamma - 1)&\big((1-x)\partial + \gamma - \alpha - \beta - 1\big) - (\gamma-\alpha-1)(\gamma-\beta-1)\big)\\
&= x(1-x)\partial^2 + (-x + (\gamma-\alpha-\beta-1)x + (\gamma-1)(1-x))\partial - \alpha\beta\\
&= x(1-x)\partial^2 + (\gamma - 1 - (\alpha+\beta+1)x)\partial - \alpha\beta\\
&= P_{\alpha,\beta,\gamma-1},
\end{aligned}$$

respectively, and we have

(6.10)

$$\begin{aligned}
(x\partial + \alpha)xP_{\alpha,\beta,\gamma} &= xP_{\alpha+1,\beta,\gamma}(x\partial+\alpha),\\
xP_{\alpha,\beta,\gamma}\big(x(1-x)\partial + \gamma - \alpha - 1 - \beta x\big) &= \big(x(1-x)\partial + \gamma - \alpha - 1 - \beta x\big)xP_{\alpha+1,\beta,\gamma},\\
(x\partial + \gamma - 1)P_{\alpha,\beta,\gamma} &= P_{\alpha,\beta,\gamma-1}(x\partial + \gamma - 1),\\
P_{\alpha,\beta,\gamma}\big((1-x)\partial + \gamma - \alpha - \beta - 1\big) &= \big((1-x)\partial + \gamma - \alpha - \beta - 1\big)P_{\alpha,\beta,\gamma-1}.
\end{aligned}$$

Remark 2. Suppose $u(x)$ is a solution of $P_{\alpha,\beta,\gamma}u = 0$. Then (6.10) shows that

$$v(x) = (x\partial + \alpha)u(x)$$

is a solution of $P_{\alpha+1,\beta,\gamma}v = 0$. In fact, we have two linear maps

$$P\begin{Bmatrix} x=0 & 1 & \infty & \\ 0 & 0 & \alpha & ;\,x \\ 1-\gamma & \gamma-\alpha-\beta & \beta & \end{Bmatrix} \tag{6.11}$$

$$\underset{x(1-x)\frac{d}{dx}+\gamma-\alpha-1-\beta x}{\overset{x\frac{d}{dx}+\alpha}{\rightleftarrows}} \quad P\begin{Bmatrix} x=0 & 1 & \infty & \\ 0 & 0 & \alpha+1 & ;\,x \\ 1-\gamma & \gamma-\alpha-\beta-1 & \beta & \end{Bmatrix}.$$

If $\alpha(\gamma-\alpha-1) \neq 0$, (6.5) shows

$$u(x) = \tfrac{1}{\alpha(\gamma-\alpha-1)}\big(x(1-x)\partial + \gamma - \alpha - 1 - \beta x\big)v(x)$$

and hence these linear maps give the isomorphisms between the space of solutions of $P_{\alpha,\beta,\gamma}u = 0$ and that of $P_{\alpha+1,\beta,\gamma}v = 0$.

Suppose $\alpha(\gamma-\alpha-1) = 0$. Then $\big(x(1-x)\partial + \gamma - \alpha - 1 - \beta x\big)(x\partial + \alpha)u(x) = 0$. Since the kernels of these maps in (6.11) are of dimension 0 or 1, they are non-zero maps. Hence the dimensions of the kernels should be 1. For example, $x^{-\alpha}$ belongs to the left Riemann scheme in (6.11).

The same argument as above is valid for the linear maps

$$P\begin{Bmatrix} x=0 & 1 & \infty & \\ 0 & 0 & \alpha & ;\,x \\ 1-\gamma & \gamma-\alpha-\beta & \beta & \end{Bmatrix} \tag{6.12}$$

$$\underset{(1-x)\frac{d}{dx}+\gamma-\alpha-\beta-1}{\overset{x\frac{d}{dx}+\gamma-1}{\rightleftarrows}} \quad P\begin{Bmatrix} x=0 & 1 & \infty & \\ 0 & 0 & \alpha & ;\,x \\ 2-\gamma & \gamma-\alpha-\beta-1 & \beta & \end{Bmatrix}.$$

They are bijective if and only if $(\gamma-\alpha-1)(\gamma-\beta-1) \neq 0$.

7. Gauss summation formula

When $\mathrm{Re}\,(\gamma-\alpha-\beta) > 0$ and $\gamma \notin \{0,-1,-2,\ldots\}$, we have the *Gauss summation formula*

$$F(\alpha,\beta,\gamma;1) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!} = C(\gamma-\alpha,\gamma-\beta;\gamma,\gamma-\alpha-\beta). \tag{7.1}$$

Here we put

$$C(\alpha_1,\alpha_2;\beta_1,\beta_2) = C\big(\begin{smallmatrix}\alpha_1 & \alpha_2\\ \beta_1 & \beta_2\end{smallmatrix}\big) := \prod_{n=0}^{\infty} \frac{(\alpha_1+n)(\alpha_2+n)}{(\beta_1+n)(\beta_2+n)} = \frac{\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \tag{7.2}$$

with $\alpha_1+\alpha_2 = \beta_1+\beta_2$.

Since the solution of $P_{\alpha,\beta,\gamma}u = 0$ on the open interval $(0,1)$ is spanned by[2] $u_{[\alpha,\beta,\alpha+\beta-\gamma+1]}(1-x)$ and $v_{[\alpha,\beta,\alpha+\beta-\gamma+1]}(1-x)$ when

$$\gamma-\alpha-\beta \notin \mathbb{Z}, \tag{7.3}$$

[2] Let $p \in \mathbb{C}\setminus\{0,1\}$. Putting $x = p$ after applying ∂^n to (2.3), we see that $u^{(n+2)}(p)$ is determined by $u^{(0)}(p),\ldots,u^{(n+1)}(p)$ for $n = 0,1,\ldots$ and therefore $u^{(n+2)}(p)$ is determined by $u(p)$ and $u'(p)$, which implies that the dimension of local analytic solutions at p is at most 2. It is easy to show that any local solution can be analytically continued along any path which does not go through the singular point of the equation.

there exist constants $C_{\alpha,\beta,\gamma}$ and $C'_{\alpha,\beta,\gamma}$ such that

$$\begin{aligned}F(\alpha,\beta,\gamma;x) &= C_{\alpha,\beta,\gamma}\cdot F(\alpha,\beta,\alpha+\beta-\gamma+1;1-x)\\ &\quad + C'_{\alpha,\beta,\gamma}\cdot(1-x)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta+1;1-x).\end{aligned}\tag{7.4}$$

Since the hypergeometric series $F(\alpha,\beta,\gamma;x)$ is holomorphic with respect to the parameters α,β and γ under the condition

$$\gamma\notin\{0,-1,-2,\ldots\},\tag{7.5}$$

$C_{\alpha,\beta,\gamma}$ and $C'_{\alpha,\beta,\gamma}$ are holomorphic with respect to the parameters under the conditions (7.3) and (7.5).

Hence under these conditions, the equation (7.4) implies

$$\lim_{x\to 1-0}F(\alpha,\beta,\gamma;x)=C_{\alpha,\beta,\gamma}\tag{7.6}$$

if

$$\operatorname{Re}(\gamma-\alpha-\beta)>0.\tag{7.7}$$

We prepare the following lemma to prove the absolute convergence of (7.1).

Lemma 3. i) For any positive number b we have

$$\prod_{n=1}^{\infty}\Bigl(1+\frac{b^2}{n^2}\Bigr)<\infty.$$

ii) Let α, β and $\gamma\in\mathbb{C}$ satisfying $\gamma\notin\{0,-1,-2,\ldots\}$. Fix a non-negative integer N such that $\{\operatorname{Re}\alpha+N,\ \operatorname{Re}\beta+N,\ \operatorname{Re}\gamma+2N\}\subset(1,\infty)$. Then there exits a positive number $C>0$ satisfying

$$\left|\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\right|<C\frac{(\operatorname{Re}\alpha+N)_n(\operatorname{Re}\beta+N)_n}{(\operatorname{Re}\gamma+2N)_n}\quad(\forall n\in\{0,1,2,\ldots\}).$$

Proof. i) For a positive integers $N\ge m>b^2+1$ we have

$$\begin{aligned}\Bigl(\prod_{n=m}^{N}\Bigl(1+\frac{b^2}{n^2}\Bigr)\Bigr)^{-1}&\ge\prod_{n=m}^{N}\Bigl(1-\frac{b^2}{n^2}\Bigr)\ge 1-\sum_{n=m}^{N}\frac{b^2}{n^2}\ge 1-\sum_{n=m}^{N}\frac{b^2}{n(n-1)}\\ &=1-\frac{b^2}{m-1}+\frac{b^2}{N}>\frac{m-(b^2+1)}{m-1},\end{aligned}$$

which implies the claim i).

ii) We may assume $\alpha,\ \beta\notin\{0,-1,-2,\ldots\}$. The number

$$\left|\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\right|\cdot\left|\frac{(\alpha+1)_n(\beta+1)_n}{(\gamma+2)_n}\right|^{-1}=\left|\frac{\alpha\beta(\gamma+n)(\gamma+n+1)}{(\alpha+n)(\beta+n)\gamma(\gamma+1)}\right|$$

converges to $\bigl|\frac{\alpha\beta}{\gamma(\gamma+1)}\bigr|$ when $n\to\infty$, which implies that if $N>0$, then (α,β,γ,N) may be replaced by $(\alpha+1,\beta+1,\gamma+2,N-1)$ for the proof of ii). Hence we may assume that $\operatorname{Re}\alpha$, $\operatorname{Re}\beta$ and $\operatorname{Re}\gamma$ are larger than 1 and $N=0$. Putting $\alpha=a+bi$ with $a\ge 1$, we have

$$1\le\Bigl(\frac{|(\alpha)_n|}{(\operatorname{Re}\alpha)_n}\Bigr)^2=\prod_{k=0}^{n-1}\Bigl(\frac{(a+k)^2+b^2}{(a+k)^2}\Bigr)\le\prod_{k=1}^{n}\Bigl(1+\frac{b^2}{k^2}\Bigr)\le\prod_{k=1}^{\infty}\Bigl(1+\frac{b^2}{k^2}\Bigr)<\infty.$$

We have the similar estimate for β and hence the claim ii). □

Proposition 4. *If the conditions* (7.5) *and* (7.7) *are valid, the hypergeometric series* $F(\alpha,\beta,\gamma;x)$ *converges absolutely and uniformly to a continuous function on the closed unit disk* $\overline{D} := \{x \in \mathbb{C} \mid |x| \le 1\}$. *The function is also continuous for the parameters* α, β *and* γ *under the same conditions and*

$$\lim_{m\to\infty} F(\alpha,\beta,\gamma+m;x) = 1 \quad \textit{uniformly on } x \in \overline{D}.$$

Proof. If we use Starling's formula[3] for the Gamma function, we easily prove the first claim of the proposition but here we don't use it.

Lemma 3 assures that we may assume that α, β and γ are positive real numbers satisfying $\gamma > \alpha + \beta$. Fix $\epsilon \ge 0$ such that $0 < \gamma - \alpha - \beta - \epsilon \notin \mathbb{Z}$. Then the hypergeometric series $F(\alpha, \beta + \epsilon, \gamma; t)$ with $t \in (0, 1)$ is a sum of non-negative numbers and defines a monotonically increasing function on $(0, 1)$ and it satisfies

$$\sum_{n=0}^{\infty}\left|\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{x^n}{n!}\right| = F(\alpha,\beta,\gamma;|x|) \le F(\alpha,\beta+\epsilon,\gamma;|x|) \le C_{\alpha,\beta+\epsilon,\gamma} < \infty$$

for $x \in \overline{D}$ and therefore the proposition is clear. □

Suppose (7.5) and (7.7). If $\mathrm{Re}\bigl(\gamma - (\alpha+1) - \beta\bigr) > 0$, then F, $F(\gamma+1)$ and $F(\alpha+1)$ in (5.8) are continuous functions on $\overline{D}$ and we have[4]

$$-\gamma(\gamma-\alpha-\beta)F(\alpha,\beta,\gamma;1) + (\gamma-\alpha)(\gamma-\beta)F(\alpha,\beta,\gamma+1;1) = 0$$

by putting $x = 1$. This equality is valid by holomorphic continuation for γ under the conditions (7.5) and (7.7). Hence

$$F(\alpha,\beta,\gamma;1) = \frac{(\gamma-\alpha)(\gamma-\beta)}{\gamma(\gamma-\alpha-\beta)}F(\alpha,\beta,\gamma+1;1) = \frac{(\gamma-\alpha)_m(\gamma-\beta)_m}{(\gamma)_m(\gamma-\alpha-\beta)_m}F(\alpha,\beta,\gamma+m;1)$$

for $m = 1, 2, \ldots$. Putting $m \to \infty$ in the above, we have the Gauss summation formula[5].

8. A connection formula

Suppose (7.3) and (7.5). We have proved $C_{\alpha,\beta,\gamma} = C(\gamma-\alpha,\gamma-\beta;\gamma,\gamma-\alpha-\beta)$ in (7.4). Then (4.4) shows

$$\begin{aligned} F(\alpha,\beta,\gamma;x) &= (1-x)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\gamma-\beta,\gamma;x) \\ &= C_{\gamma-\alpha,\gamma-\beta,\gamma}(1-x)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta+1;1-x) \\ &\quad + C'_{\gamma-\alpha,\gamma-\beta,\gamma}F(\alpha,\beta,\alpha+\beta-\gamma+1;1-x). \end{aligned}$$

Comparing this equation with (7.4), we have $C'_{\alpha,\beta,\gamma} = C_{\gamma-\alpha,\gamma-\beta,\gamma} = C(\alpha,\beta;\gamma,\alpha+\beta-\gamma)$ and the *connection formula*

$$\begin{aligned} F(\alpha,\beta,\gamma;x) &= C\begin{pmatrix} \gamma-\alpha & \gamma-\beta \\ \gamma & \gamma-\alpha-\beta \end{pmatrix} \cdot F(\alpha,\beta,\alpha+\beta-\gamma+1;1-x) \\ &\quad + C\begin{pmatrix} \alpha & \beta \\ \gamma & \alpha+\beta-\gamma \end{pmatrix} \cdot (1-x)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta+1;1-x). \end{aligned} \tag{8.1}$$

This is valid when $\gamma \notin \{0, -1, -2, \ldots\}$ and $\gamma - \alpha - \beta \notin \mathbb{Z}$. If $F(\alpha',\beta',\gamma';x)$ with $\gamma' \in \mathbb{Z}$ appears in the above, we use $w^{(|1-\gamma'|)}_{[\alpha',\beta',\gamma']}$ to give one of the local solutions as in (3.8) and we get a connection formula from (8.1) with (3.4) or (3.7).

---

[3]Note that $(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$. Stirling's formula says $\Gamma(z) \sim \sqrt{2\pi}e^{-z}z^{z-\frac{1}{2}}$ when $|z| \to \infty$ and $\pi - |\mathrm{Arg}z|$ is larger than a fixed positive number. We can also prove the claim by the estimates $\frac{t}{2} \le \log(1+t) \le t$ for $0 \le t \le \frac{1}{2}$ and $\log(n+1) \le \sum_{m=1}^{n}\frac{1}{m} \le 1 + \log n$.

[4]Differentiating the first equalities in §8 by x, we also get this equality because the equality (6.1) shows $\frac{\alpha\beta}{\gamma}C_{\gamma-\alpha,\gamma-\beta,\gamma+1} = (\alpha+\beta-\gamma)C(\gamma-\alpha,\gamma-\beta,\gamma)$.

[5]Other proofs and generalizations of the formula are given in [O2].

By applying $\mathrm{Ad}\big(x^\lambda(1-x)^\mu\big)$ to $P_{\alpha,\beta,\gamma}$, the corresponding Riemann scheme of the equation $\tilde P\tilde u=0$ with $\tilde P := x(1-x)\,\mathrm{Ad}\big(x^\lambda(1-x)^\mu\big)P_{\alpha,\beta,\gamma}$ equals

$$P\begin{Bmatrix} x=0 & 1 & \infty & \\ \lambda_{0,1} & \lambda_{1,1} & \lambda_{\infty,1} & ;\ x \\ \lambda_{0,2} & \lambda_{1,2} & \lambda_{\infty,2} & \end{Bmatrix} = P\begin{Bmatrix} x=0 & 1 & \infty & \\ \lambda & \mu & \alpha-\lambda-\mu & ;\ x \\ 1-\gamma+\lambda & \gamma-\alpha-\beta+\mu & \beta-\lambda-\mu & \end{Bmatrix} \tag{9.1}$$

with

$$\begin{cases} \lambda_{0,1}=\lambda, & \lambda_{1,1}=\mu, & \lambda_{\infty,1}=\alpha-\lambda-\mu,\\ \lambda_{0,2}=1-\gamma+\lambda, & \lambda_{1,2}=\gamma-\alpha-\beta+\mu, & \lambda_{\infty,2}=\beta-\lambda-\mu, \end{cases} \tag{9.2}$$

$$\alpha=\lambda_{0,1}+\lambda_{1,1}+\lambda_{\infty,1},\quad \beta=\lambda_{0,1}+\lambda_{1,1}+\lambda_{\infty,2},\quad \gamma=\lambda_{0,1}-\lambda_{0,2}+1 \tag{9.3}$$

and $\tilde u(x)=x^{\lambda_{0,1}}(1-x)^{\lambda_{1,1}}u(x)$ is a solution of $\tilde P\tilde u=0$ for the solution $u(x)$ of $P_{\alpha,\beta,\gamma}u=0$. The Riemann scheme (9.1) represents the solutions of $\tilde P\tilde u=0$ and it is called the Riemann *P-function*. Here we have the *Fuchs relation*

$$\lambda_{0,1}+\lambda_{0,2}+\lambda_{1,1}+\lambda_{1,2}+\lambda_{\infty,1}+\lambda_{\infty,2}=1 \tag{9.4}$$

and

$$\begin{aligned}
\tilde P &= x(1-x)\Big(x(1-x)(\partial-\tfrac{\lambda}{x}+\tfrac{\mu}{1-x})^2+\big(\gamma-(\alpha+\beta+1)x\big)\big(\partial-\tfrac{\lambda}{x}+\tfrac{\mu}{1-x}\big)-\alpha\beta\Big)\\
&= x^2(1-x)^2\Big(\partial^2+(-\tfrac{2\lambda}{x}+\tfrac{2\mu}{1-x})\partial+\tfrac{\lambda}{x^2}+\tfrac{\mu}{(1-x)^2}+\tfrac{\lambda^2}{x^2}-\tfrac{2\lambda\mu}{x(1-x)}+\tfrac{\mu^2}{(1-x)^2}\Big)\\
&\quad+\big(\gamma-(\alpha+\beta+1)x\big)\big(x(1-x)\partial-\lambda(1-x)+\mu x\big)-\alpha\beta x(1-x)\\
&= x^2(1-x)^2\partial^2+x(1-x)\big(-2\lambda(1-x)+2\mu x+\gamma-(\alpha+\beta+1)x\big)\partial\\
&\quad+(\lambda^2+\lambda)(1-x)^2-2\lambda\mu x(1-x)+(\mu^2+\mu)x^2\\
&\quad+\big(\gamma-(\alpha+\beta+1)x\big)\big((\lambda+\mu)x-\lambda\big)+\alpha\beta(x^2-x)\\
&= x^2(1-x)^2\partial^2-x(1-x)\big((\alpha+\beta-2\lambda-2\mu+1)x+2\lambda-\gamma\big)\partial\\
&\quad+\big(\lambda^2+\lambda+2\lambda\mu+\mu^2+\mu-(\alpha+\beta+1)(\lambda+\mu)+\alpha\beta\big)x^2\\
&\quad+\big(-2\lambda^2-2\lambda-2\lambda\mu+\gamma(\lambda+\mu)+\lambda(\alpha+\beta+1)-\alpha\beta\big)x+\lambda^2+\lambda-\gamma\lambda\\
&= x^2(1-x)^2\partial^2-x(1-x)\big((\lambda_{\infty,1}+\lambda_{\infty,2}+1)x+\lambda_{0,1}+\lambda_{0,2}-1\big)\partial\\
&\quad+\lambda_{\infty,1}\lambda_{\infty,2}x^2+(\lambda_{1,1}\lambda_{1,2}-\lambda_{0,1}\lambda_{0,2}-\lambda_{\infty,1}\lambda_{\infty,2})x+\lambda_{0,1}\lambda_{0,2}\\
&= x^2(1-x)^2\Big(\partial^2-\tfrac{\lambda_{0,1}+\lambda_{0,2}-1}{x}\partial+\tfrac{\lambda_{1,1}+\lambda_{1,2}-1}{1-x}\partial+\tfrac{\lambda_{0,1}\lambda_{0,2}}{x^2}+\tfrac{\lambda_{1,1}\lambda_{1,2}}{(1-x)^2}+\\
&\quad\tfrac{\lambda_{0,1}\lambda_{0,2}+\lambda_{1,1}\lambda_{1,2}-\lambda_{\infty,1}\lambda_{\infty,2}}{x(1-x)}\Big).
\end{aligned}$$

Suppose $\lambda_{p,1}-\lambda_{p,2}\notin\mathbb{Z}$ for $p=0$, 1 and ∞. Let $u_p^{\lambda_{p,\nu}}(x)$ denote the normalized local solutions of $\tilde P\tilde u=0$ at $x=p$ such that

$$u_p^{\lambda_{p,\nu}}(x)=y^{\lambda_{p,\nu}}\phi_{p,\nu}(y),\quad y=\begin{cases} x & (p=0),\\ 1-x & (p=1),\\ \frac{1}{x} & (p=\infty)\end{cases} \tag{9.5}$$

and $\phi_{p,\nu}(t)$ are holomorphic if $|t| < 1$ and satisfies $\phi_{p,\nu}(0) = 1$. In fact, (4.1) and (4.2) imply

(9.6)
$$\begin{aligned}
\phi_{0,\nu}(x) &= (1-x)^{\lambda_{1,i}} F(\lambda_{0,\nu}+\lambda_{1,i}+\lambda_{\infty,1}, \lambda_{0,\nu}+\lambda_{1,i}+\lambda_{\infty,2}, \lambda_{0,\nu}-\lambda_{0,\bar\nu}+1; x),\\
\phi_{1,\nu}(x) &= x^{\lambda_{0,i}} F(\lambda_{0,i}+\lambda_{1,\nu}+\lambda_{\infty,1}, \lambda_{0,i}+\lambda_{1,\nu}+\lambda_{\infty,2}, \lambda_{1,\nu}-\lambda_{1,\bar\nu}+1; 1-x),\\
\phi_{\infty,\nu}(x) &= \left(\tfrac{x-1}{x}\right)^{\lambda_{1,i}} F(\lambda_{0,1}+\lambda_{1,i}+\lambda_{\infty,\nu}, \lambda_{0,2}+\lambda_{1,i}+\lambda_{\infty,\nu}, \lambda_{\infty,\nu}-\lambda_{\infty,\bar\nu}+1; \tfrac{1}{x}).
\end{aligned}$$

For indices i, j, $\nu \in \{1,2\}$, we put $\bar i = 3-i$, $\bar j = 3-j$ and $\bar\nu = 3-\nu$. Then for $i = 1$ and 2, we have the connection formula

(9.7)
$$\begin{aligned}
u_0^{\lambda_{0,i}}(x) &= \sum_{\nu=1}^{2} C\begin{pmatrix} \lambda_{0,i}+\lambda_{1,\bar\nu}+\lambda_{\infty,1} & \lambda_{0,i}+\lambda_{1,\bar\nu}+\lambda_{\infty,2} \\ \lambda_{0,i}-\lambda_{0,\bar i}+1 & \lambda_{1,\bar\nu}-\lambda_{1,\nu}\end{pmatrix} \cdot u_1^{\lambda_{1,\nu}}(x) \qquad \begin{cases}\bar i = 3-i,\\ \bar\nu = 3-\nu\end{cases}\\
&= \sum_{\nu=1}^{2} \frac{\Gamma(\lambda_{0,i}-\lambda_{0,\bar i}+1)\cdot\Gamma(\lambda_{1,\bar\nu}-\lambda_{1,\nu})}{\Gamma(\lambda_{0,i}+\lambda_{1,\bar\nu}+\lambda_{\infty,1})\cdot\Gamma(\lambda_{0,i}+\lambda_{1,\bar\nu}+\lambda_{\infty,2})} \cdot u_1^{\lambda_{1,\nu}}(x)
\end{aligned}$$

when $\lambda_{0,i} - \lambda_{0,\bar i} \notin \{-1,-2,\ldots\}$ and $\lambda_{1,\nu} - \lambda_{1,\bar\nu} \notin \mathbb{Z}$.

Since $C\begin{pmatrix} \lambda_{0,1}+\lambda_{1,2}+\lambda_{\infty,1} & \lambda_{0,1}+\lambda_{1,2}+\lambda_{\infty,2} \\ \lambda_{0,1}-\lambda_{0,2}+1 & \lambda_{1,2}-\lambda_{1,1}\end{pmatrix} = C\begin{pmatrix}\gamma-\alpha & \gamma-\beta \\ \gamma & \gamma-\alpha-\beta\end{pmatrix}$ and the connection coefficients in the right hand side of (9.7) are invariant under the transformation $u \mapsto \tilde u = x^\lambda(1-x)^\mu u$, (8.1) implies that the coefficient of $u_1^{\lambda_{1,1}}(x)$ is as given in (9.7). Hence (9.7) is valid because of the symmetries $\lambda_{1,1} \leftrightarrow \lambda_{1,2}$ and $\lambda_{0,1} \leftrightarrow \lambda_{0,2}$.

First note that

$$P\begin{Bmatrix} x=0 & 1 & \infty & \\ \lambda_{0,1} & \lambda_{1,1} & \lambda_{\infty,1} & ;\ x \\ \lambda_{0,2} & \lambda_{1,2} & \lambda_{\infty,2} & \end{Bmatrix} = P\begin{Bmatrix} x=0 & 1 & \infty & \\ \lambda_{0,2} & \lambda_{1,1} & \lambda_{\infty,1} & ;\ x \\ \lambda_{0,1} & \lambda_{1,2} & \lambda_{\infty,2} & \end{Bmatrix} \text{ etc.}$$

and $\tilde P$ is invariant under the transpositions $\lambda_{p,1} \leftrightarrow \lambda_{p,2}$ for $p = 0$, 1 and ∞. Moreover for $\lambda \in \mathbb{C}$ we have

$$x^\lambda P\begin{Bmatrix} x=0 & 1 & \infty & \\ \lambda_{0,1} & \lambda_{1,1} & \lambda_{\infty,1} & ;\ x \\ \lambda_{0,2} & \lambda_{1,2} & \lambda_{\infty,2} & \end{Bmatrix} = P\begin{Bmatrix} x=0 & 1 & \infty & \\ \lambda_{0,1}+\lambda & \lambda_{1,1} & \lambda_{\infty,1}-\lambda & ;\ x \\ \lambda_{0,2}+\lambda & \lambda_{1,2} & \lambda_{\infty,2}-\lambda & \end{Bmatrix}, \tag{9.8}$$

(9.9)
$$(1-x)^\lambda P\begin{Bmatrix} x=0 & 1 & \infty & \\ \lambda_{0,1} & \lambda_{1,1} & \lambda_{\infty,1} & ;\ x \\ \lambda_{0,2} & \lambda_{1,2} & \lambda_{\infty,2} & \end{Bmatrix} = P\begin{Bmatrix} x=0 & 1 & \infty & \\ \lambda_{0,1} & \lambda_{1,1}+\lambda & \lambda_{\infty,1}-\lambda & ;\ x \\ \lambda_{0,2} & \lambda_{1,2}+\lambda & \lambda_{\infty,2}-\lambda & \end{Bmatrix},$$

(9.10)
$$P\begin{Bmatrix} x=0 & 1 & \infty & \\ \lambda_{0,1} & \lambda_{1,1} & \lambda_{\infty,1} & ;\ 1-x \\ \lambda_{0,2} & \lambda_{1,2} & \lambda_{\infty,2} & \end{Bmatrix} = P\begin{Bmatrix} x=0 & 1 & \infty & \\ \lambda_{1,1} & \lambda_{0,1} & \lambda_{\infty,1} & ;\ x \\ \lambda_{1,2} & \lambda_{0,2} & \lambda_{\infty,2} & \end{Bmatrix},$$

$$P\begin{Bmatrix} x=0 & 1 & \infty & \\ \lambda_{0,1} & \lambda_{1,1} & \lambda_{\infty,1} & ;\ \frac{1}{x} \\ \lambda_{0,2} & \lambda_{1,2} & \lambda_{\infty,2} & \end{Bmatrix} = P\begin{Bmatrix} x=0 & 1 & \infty & \\ \lambda_{\infty,1} & \lambda_{1,1} & \lambda_{0,1} & ;\ x \\ \lambda_{\infty,2} & \lambda_{1,2} & \lambda_{0,2} & \end{Bmatrix}. \tag{9.11}$$

By the transformation $x \mapsto 1-x$ in (9.7), we have

(9.12)
$$\begin{aligned}
u_1^{\lambda_{1,i}}(x) &= \sum_{\nu=1}^{2} C\begin{pmatrix} \lambda_{1,i}+\lambda_{0,\bar\nu}+\lambda_{\infty,1} & \lambda_{1,i}+\lambda_{0,\bar\nu}+\lambda_{\infty,2} \\ \lambda_{1,i}-\lambda_{1,\bar i}+1 & \lambda_{0,\bar\nu}-\lambda_{0,\nu}\end{pmatrix} \cdot u_0^{\lambda_{0,\nu}}(x)\\
&= \sum_{\nu=1}^{2} \frac{\Gamma(\lambda_{1,i}-\lambda_{1,\bar i}+1)\cdot\Gamma(\lambda_{0,\bar\nu}-\lambda_{0,\nu})}{\Gamma(\lambda_{1,i}+\lambda_{0,\bar\nu}+\lambda_{\infty,1})\cdot\Gamma(\lambda_{1,i}+\lambda_{0,\bar\nu}+\lambda_{\infty,2})} \cdot u_0^{\lambda_{0,\nu}}(x).
\end{aligned}$$

Some combinations of (9.10) and (9.11) give similar identities of the Riemann schemes corresponding to the transformations $x \mapsto 1 - \frac{1}{x}$, $\frac{1}{1-x}$ and $\frac{x}{x-1}$.

In general, for the Riemann scheme

$$P\begin{Bmatrix} x = c_0 & c_1 & c_2 & \\ \lambda_{0,1} & \lambda_{1,1} & \lambda_{2,1} & ;\ x \\ \lambda_{0,2} & \lambda_{1,2} & \lambda_{2,2} & \end{Bmatrix} \text{ with } \{c_0, c_1, c_2\} = \{0, 1, \infty\}$$

satisfying the Fuchs relation $\sum \lambda_{j,\nu} = 1$, we fix local functions $u_{j,\nu}(x)$ belonging to the Riemann scheme corresponding to the characteristic exponents $\lambda_{j,\nu}$ of the points $x = c_j$, respectively. We normalize $u_{0,1}$, $u_{1,1}$ and $u_{1,2}$ as follows. Putting

$$\begin{cases} I = (0,1), & \varphi_0(x) = x, & \varphi_1(x) = 1 - x & \text{if } (c_0, c_1) = (0,1), \\ I = (0,1), & \varphi_0(x) = 1 - x, & \varphi_1(x) = x & \text{if } (c_0, c_1) = (1,0), \\ I = (1,\infty), & \varphi_0(x) = x - 1, & \varphi_1(x) = \frac{1}{x} & \text{if } (c_0, c_1) = (1,\infty), \\ I = (1,\infty), & \varphi_0(x) = \frac{1}{x}, & \varphi_1(x) = x - 1 & \text{if } (c_0, c_1) = (\infty,1), \\ I = (-\infty,0), & \varphi_0(x) = -x, & \varphi_1(x) = -\frac{1}{x} & \text{if } (c_0, c_1) = (0,\infty), \\ I = (-\infty,0), & \varphi_0(x) = -\frac{1}{x}, & \varphi_1(x) = -x & \text{if } (c_0, c_1) = (\infty,0), \end{cases} \tag{9.13}$$

we have

$$\lim_{I \ni x \to c_0} \varphi_0(x)^{-\lambda_{0,1}} u_{0,1}(x) = 1, \quad \lim_{I \ni x \to c_1} \varphi_1(x)^{-\lambda_{1,\nu}} u_{1,\nu}(x) = 1 \tag{9.14}$$

and $\varphi_0(x)^{-\lambda_{0,1}} u_{0,1}(x)$ is holomorphic at $x = c_0$ and $\varphi_1(x)^{-\lambda_{1,\nu}} u_{1,\nu}(x)$ are holomorphic at $x = c_1$ for $\nu = 1$ and 2. Note that $\varphi_0(x) > 0$ and $\varphi_1(x) > 0$ when $x \in I$. Then we have the connection formula

$$\begin{aligned} u_{0,1}(x) &= \sum_{\nu=1}^{2} C\Bigl(\begin{smallmatrix} \lambda_{0,1}+\lambda_{1,\bar\nu}+\lambda_{\infty,1} & \lambda_{0,1}+\lambda_{1,\bar\nu}+\lambda_{\infty,2} \\ \lambda_{0,1}-\lambda_{0,2}+1 & \lambda_{1,\bar\nu}-\lambda_{1,\nu} \end{smallmatrix}\Bigr) \cdot u_{1,\nu}(x) \\ &= \sum_{\nu=1}^{2} \frac{\Gamma(\lambda_{0,1} - \lambda_{0,2} + 1) \cdot \Gamma(\lambda_{1,\bar\nu} - \lambda_{1,\nu})}{\Gamma(\lambda_{0,1} + \lambda_{1,\bar\nu} + \lambda_{\infty,1}) \cdot \Gamma(\lambda_{0,1} + \lambda_{1,\bar\nu} + \lambda_{\infty,2})} \cdot u_{1,\nu}(x) \end{aligned} \tag{9.15}$$

for $x \in I$. Here we note that $\alpha_1 + \alpha_2 = \beta_1 + \beta_2$ in the definition of $C\bigl(\begin{smallmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{smallmatrix}\bigr)$.

In particular, we have

$$\begin{aligned} &F(\alpha, \beta, \gamma; x) \\ &\quad = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)} F(\alpha, \beta, \alpha + \beta - \gamma + 1; 1 - x) \\ &\qquad + (1 - x)^{\gamma - \alpha - \beta} \frac{\Gamma(\gamma)\Gamma(\alpha + \beta - \gamma)}{\Gamma(\alpha)\Gamma(\beta)} F(\gamma - \alpha, \gamma - \beta, \gamma - \alpha - \beta + 1; 1 - x) \\ &\quad = (-x)^{-\alpha} \frac{\Gamma(\gamma)\Gamma(\beta - \alpha)}{\Gamma(\beta)\Gamma(\gamma - \alpha)} F\Bigl(\alpha, \alpha - \gamma + 1, \alpha - \beta + 1; \frac{1}{x}\Bigr) \\ &\qquad + (-x)^{-\beta} \frac{\Gamma(\gamma)\Gamma(\alpha - \beta)}{\Gamma(\alpha)\Gamma(\gamma - \beta)} F\Bigl(\beta, \beta - \gamma + 1, \beta - \alpha + 1; \frac{1}{x}\Bigr). \end{aligned}$$

Here $F(a, b, c; z)$ is considered to be a holomorphic function on $\mathbb{C} \setminus [1, \infty)$.

In general, if the fractional linear transformation $x \mapsto y = \frac{ax+b}{cx+d}$ with $ad - bc \neq 0$ transforms 0, 1, ∞ to c_0, c_1 and c_2, respectively, then by this coordinate transformation we have the Riemann scheme

$$P\begin{Bmatrix} y = c_0 & c_1 & c_2 & \\ \lambda_{0,1} & \lambda_{1,1} & \lambda_{2,1} & ;\ y \\ \lambda_{0,2} & \lambda_{1,2} & \lambda_{2,2} & \end{Bmatrix} \qquad \Bigl(\sum_{j=0}^{2} \sum_{\nu=1}^{2} \lambda_{j,\nu} = 1\Bigr) \tag{9.16}$$

and the corresponding differential equation and its solutions. For simplicity, (9.16) is simply expressed by

$$P\begin{Bmatrix} c_0 & c_1 & c_2 \\ \lambda_{0,1} & \lambda_{1,1} & \lambda_{2,1} \\ \lambda_{0,2} & \lambda_{1,2} & \lambda_{2,2} \end{Bmatrix} \quad \text{or} \quad P\begin{Bmatrix} \lambda_{0,1} & \lambda_{1,1} & \lambda_{2,1} \\ \lambda_{0,2} & \lambda_{1,2} & \lambda_{2,2} \end{Bmatrix}.$$

If $|c_j| < \infty$ for $j = 0,\ 1,\ 2$, the equation $\hat{P}\hat{u} = 0$ of this Riemann scheme is given by

$$\hat{P} = \frac{d^2}{dx^2} - \Bigl(\sum_{j=0}^{2} \frac{\lambda_{j,1} + \lambda_{j,2} - 1}{x - c_j}\Bigr)\frac{d}{dx} + \sum_{j=0}^{2} \frac{\lambda_{j,1}\lambda_{j,2}\prod_{\nu\in\{0,1,2\}\setminus\{j\}}(c_j - c_\nu)}{(x - c_j)(x - c_0)(x - c_1)(x - c_2)}, \tag{9.17}$$

which is obtained by a direct calculation or by characteristic exponents at every singular point and the fact that ∞ is not a singular point. This equation was first given by Papperitz.

Applying $\mathrm{Ad}\bigl(x^{\lambda_{0,1}}(1-x)^{\lambda_{1,1}}\bigr)$ to (6.12), we have linear maps

$$P\begin{Bmatrix} x = 0 & 1 & \infty & \\ \lambda_{0,1} & \lambda_{1,1} & \lambda_{\infty,1} & ;\ x \\ \lambda_{0,2} & \lambda_{1,2} & \lambda_{\infty,2} & \end{Bmatrix} \tag{9.18}$$

$$\underset{(x-1)\frac{d}{dx}+\frac{\lambda_{0,1}}{x}-\lambda_{0,1}-\lambda_{1,1}-\lambda_{1,2}+1}{\overset{x\frac{d}{dx}+\frac{\lambda_{1,1}}{1-x}-\lambda_{0,1}-\lambda_{0,2}-\lambda_{1,1}}{\rightleftarrows}} P\begin{Bmatrix} x = 0 & 1 & \infty & \\ \lambda_{0,1} & \lambda_{1,1} & \lambda_{\infty,1} & ;\ x \\ \lambda_{0,2}+1 & \lambda_{1,2}-1 & \lambda_{\infty,2} & \end{Bmatrix},$$

which are bijective if and only if

$$(\lambda_{0,2} + \lambda_{1,1} + \lambda_{\infty,1})(\lambda_{0,2} + \lambda_{1,1} + \lambda_{\infty,2}) \neq 0. \tag{9.19}$$

Hence in general, there are non-zero linear maps

$$P\begin{Bmatrix} x = c_0 & c_1 & c_2 & \\ \lambda_{0,1} & \lambda_{1,1} & \lambda_{2,1} & ;\ x \\ \lambda_{0,2} & \lambda_{1,2} & \lambda_{2,2} & \end{Bmatrix} \underset{P_2}{\overset{P_1}{\rightleftarrows}} P\begin{Bmatrix} x = c_0 & c_1 & c_2 & \\ \lambda_{0,1} & \lambda_{1,1} & \lambda_{2,1} & ;\ x \\ \lambda_{0,2}+1 & \lambda_{1,2}-1 & \lambda_{2,2} & \end{Bmatrix} \tag{9.20}$$

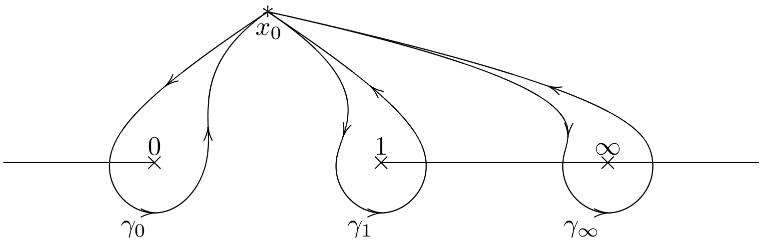
given by differential operators P_1 and P_2 and they are bijective if and only if

$$(\lambda_{0,2} + \lambda_{1,1} + \lambda_{2,1})(\lambda_{0,2} + \lambda_{1,1} + \lambda_{2,2}) \neq 0. \tag{9.21}$$

10. Monodromy and irreducibility

The equation $\tilde{P}\tilde{u} = 0$ has singularities at 0, 1 and ∞ but any local solution $u(x)$ in a small neighborhood of a point $x_0 \in X := (\mathbb{C} \cup \{\infty\}) \setminus \{0, 1\}$ is analytically continued along any path in X starting from x_0. It defines a (single-valued) holomorphic function on the simply connected domain $X' := \mathbb{C} \setminus \bigl((-\infty, 0] \cup [1, \infty)\bigr)$. In particular, $F(\alpha, \beta, \gamma; x)$ defines a holomorphic function on $\mathbb{C} \setminus [1, \infty)$.

Let (u_1, u_2) be a base of local solutions of $\tilde{P}\tilde{u} = 0$ at x_0. Let γ_p be closed paths starting from x_0 and circling around the point $x = p$ once in a counterclockwise direction for $p = 0$, 1 and ∞, respectively, as follows.



Let $\gamma_p u_j$ be the local solutions in a neighborhood of x_0 obtained by the analytic continuation of u_j along γ_p, respectively. Then there exist $M_p \in GL(2, \mathbb{C})$ satisfying

$(\gamma_p u_1, \gamma_p u_2) = (u_1, u_2)M_p$. Here $GL(2,\mathbb{C})$ is the group of invertible matrices of size 2 with entries in $\mathbb{C}$. The matrices M_p are called the *local generator matrices of monodromy* of the solution space of $\tilde{P}\tilde{u}$ and the subgroup of $GL(n,\mathbb{C})$ generated by M_0, M_1 and M_∞ is called the *monodromy group*. Here we note

$$M_\infty M_1 M_0 = I_2 \text{ (the identity matrix)} \tag{10.1}$$

and if we differently choose x_0 and (u_1, u_2), the set of local generator matrices of monodromy (M_0, M_1, M_p) changes into $(gM_0g^{-1}, gM_1g^{-1}, gM_\infty g^{-1})$ with a certain $g \in GL(2,\mathbb{C})$. If there exists a subspace V of $\mathbb{C}^2$ such that $\{0\} \subsetneqq V \subsetneqq \mathbb{C}^2$ and $M_pV \subset V$ for $p = 0, 1, \infty$, then we say that the monodromy of $\tilde{P}\tilde{u} = 0$ is *reducible*. If it is not reducible, it is called *irreducible*.

Remark 5. Suppose $\tilde{P}\tilde{u} = 0$ is reducible. Then there exists a non-zero local solution $v(x)$ in a neighborhood of x_0 which satisfies $\gamma_p v = C_p v$ with $C_p \in \mathbb{C} \setminus \{0\}$. Then the function $b(x) = \frac{v'(x)}{v(x)}$ satisfies $\gamma_p b = b$ and therefore $b(x) \in \mathbb{C}(x)$ and $v(x)$ is a solution of the differential equation $\partial v = b(x)v$. Here $\mathbb{C}(x)$ is the ring of rational functions with the variable x. Since $\tilde{P} - a_1(x)\partial\big(\partial - b(x)\big) = a_0(x)\partial + c(x)$ with $a_1(x) = x^2(1-x)^2$, $a_0(x)$, $c(x) \in \mathbb{C}(x)$, we have a division

$$\tilde{P} = \big(a_1(x)\partial + a_0(x)\big)\big(\partial - b(x)\big) + r(x)$$

with $r(x) \in \mathbb{C}(x)$. The condition $\tilde{P}v(x) = 0$ implies $r(x) = 0$ and therefore we have

$$\tilde{P} = \big(a_1(x)\partial + a_0(x)\big)\big(\partial - b(x)\big).$$

In general, let $W(x)$ denote the ring of ordinary differential operators with coefficients in rational functions. Then $P \in W(x)$ and the equation $Pu = 0$ are called reducible if there exist Q, $R \in W(x)$ such that $P = QR$ and the order of Q and that of R are both positive. If they are not reducible, they are called irreducible.

We note that $\tilde{P}\tilde{u} = 0$ is irreducible if and only if its monodromy is irreducible.

Lemma 6. *Let* $A_0 = \begin{pmatrix} \lambda_{0,1} & a_0 \\ 0 & \lambda_{0,2} \end{pmatrix}$ *and* $A_1 = \begin{pmatrix} \lambda_{1,1} & 0 \\ a_1 & \lambda_{1,2} \end{pmatrix}$ *in* $GL(2,\mathbb{C})$. *Then there exists a non-trivial proper simultaneous invariant subspace under the linear transformations of* $\mathbb{C}^2$ *defined by* A_0 *and* A_1 *if and only if*

$$a_0a_1\big(a_0a_1 + (\lambda_{0,1} - \lambda_{0,2})(\lambda_{1,1} - \lambda_{1,2})\big) = 0. \tag{10.2}$$

Proof. The lemma is clear when $a_0a_1 = 0$ and therefore we may assume $a_0a_1 \neq 0$. In this case if there exists a 1-dimensional invariant subspace, it is of the form $\mathbb{C}\binom{1}{c}$ with $c \neq 0$ and $A_0\binom{1}{c} = \lambda_{0,2}\binom{1}{c}$ and $A_1\binom{1}{c} = \lambda_{1,1}\binom{1}{c}$, which is equivalent to $\lambda_{0,1} + a_0c = \lambda_{0,2}$ and $a_1 + \lambda_{1,2}c = \lambda_{1,1}c$. This means $c = \frac{\lambda_{0,2}-\lambda_{0,1}}{a_0} = \frac{a_1}{\lambda_{1,1}-\lambda_{1,2}}$. □

Theorem 7. i) *When* $\operatorname{Re}\gamma \leq 1$, *the local monodromy of the Riemann Scheme* $P\begin{Bmatrix} x=0 & 1 & \infty & \\ 0 & 0 & \alpha & ;\, x \\ 1-\gamma & \gamma-\alpha-\beta & \beta & \end{Bmatrix}$ *at the origin is not semisimple if and only if* $1-\gamma \in \{0,1,2,\ldots\}$ *and* $-\alpha$, $-\beta \notin \{0,1,\ldots,-\gamma\}$. *When* $1-\gamma \in \{0,1,2,\ldots\}$, *the condition* $-\alpha$, $-\beta \notin \{0,1,\ldots,-\gamma\}$ *is equal to the non-existence of a polynomial solution of degree* $\leq -\gamma$ *with a non-zero value at the origin.*

ii) *The Riemann Scheme* $P\begin{Bmatrix} x=0 & 1 & \infty & \\ \lambda_{0,1} & \lambda_{1,1} & \lambda_{\infty,1} & ;\, x \\ \lambda_{0,2} & \lambda_{1,2} & \lambda_{\infty,2} & \end{Bmatrix}$ *with the Fuchs relation* $\sum_{p,\nu}\lambda_{p,\nu} = 1$ *has a non-semisimple local monodromy at the origin if and only if*

$$\begin{aligned} &\lambda_{0,1} - \lambda_{0,2} \in \mathbb{Z} \quad \textit{and} \\ &-\lambda_{0,k} - \lambda_{1,1} - \lambda_{\infty,\nu} \notin \{0,1,\ldots,|\lambda_{0,1}-\lambda_{0,2}|-1\} \quad \textit{for} \quad \nu = 1, 2. \end{aligned} \tag{10.3}$$

Here $k = 1$ *if* $\lambda_{0,2} - \lambda_{0,1} \geq 0$ *and* $k = 2$ *otherwise.*

Proof. i) Recall the description of local functions belonging to the Riemann scheme in §3. Then the local monodromy around the origin is not semisimple if $\gamma = 1$ and it is semisimple if $\gamma \notin \mathbb{Z}$. Putting $m = -\gamma$, we may assume $m \in \{0, 1, 2, \ldots\}$ to prove the claim. Then the semisimplicity of the monodromy is equivalent to the condition that $w^{(m+1)}_{[\alpha,\beta,-m]}$ in §3 has no logarithmic term, which equals the condition $(\alpha)_{m+1}(\beta)_{m+1} = 0$. Then Remark 1 implies that it also equals the existence of a polynomial $u(x)$ with $u(0) \neq 0$ in the Riemann scheme.

The claim in ii) follows from i) by the transformation of functions $u(x) \mapsto x^{-\lambda_{0,1}}(1 - x)^{-\lambda_{1,1}}u(x)$ or $x^{-\lambda_{0,2}}(1 - x)^{-\lambda_{1,1}}u(x)$ according to $\operatorname{Re}(\lambda_{0,2} - \lambda_{0,1})$ is non-negative or negative, respectively. □

Theorem 8. *For the Riemann scheme*

$$P\begin{Bmatrix} x = 0 & 1 & \infty & \\ \lambda_{0,1} & \lambda_{1,1} & \lambda_{\infty,1} & ;\ x \\ \lambda_{0,2} & \lambda_{1,2} & \lambda_{\infty,2} & \end{Bmatrix} \quad \text{with the Fuchs relation} \quad \sum_{p,\,\nu} \lambda_{p,\nu} = 1 \tag{10.4}$$

let M_0, M_1 *and* M_∞ *be the monodromy matrices around the points* 0, 1, ∞, *respectively, under a suitable base.*

i) (M_0, M_1, M_∞) *is irreducible if and only if*

$$\lambda_{0,1} + \lambda_{1,\nu} + \lambda_{\infty,\nu'} \notin \mathbb{Z} \quad (\forall \nu,\ \nu' \in \{1, 2\}). \tag{10.5}$$

ii) *Suppose*

$$\lambda_{0,2} + \lambda_{1,2} + \lambda_{\infty,\nu} \notin \{0, -1, -2, \ldots\} \quad (\nu = 1,\ 2). \tag{10.6}$$

We may assume

$$\lambda_{p,1} - \lambda_{p,2} \notin \{1, 2, 3, \ldots\} \quad (p = 0,\ 1) \tag{10.7}$$

by one or both of the permutations $\lambda_{0,1} \leftrightarrow \lambda_{0,2}$ *and* $\lambda_{1,1} \leftrightarrow \lambda_{1,2}$ *if necessary.*

Under these conditions the functions $u_0^{\lambda_{0,2}}(x)$ *and* $u_1^{\lambda_{1,2}}(x)$ *given by* (9.5) *are well-defined and linearly independent functions on* $(0, 1)$.

When

$$\lambda_{0,2} + \lambda_{1,1} + \lambda_{\infty,\nu} \notin \mathbb{Z} \quad (\nu = 1,\ 2), \tag{10.8}$$

there exists $g \in GL(2, \mathbb{C})$ *such that the monodromy matrices satisfy*

$$(gM_0g^{-1}, gM_1g^{-1}) = \left(\begin{pmatrix} e^{2\pi i\lambda_{0,2}} & a_0 \\ 0 & e^{2\pi i\lambda_{0,1}} \end{pmatrix}, \begin{pmatrix} e^{2\pi i\lambda_{1,1}} & 0 \\ a_1 & e^{2\pi i\lambda_{1,2}} \end{pmatrix}\right) \tag{10.9}$$

with

$$\begin{aligned} a_0 &= 2e^{-\pi i\lambda_{\infty,2}} \sin \pi(\lambda_{0,2} + \lambda_{1,1} + \lambda_{\infty,2}), \\ a_1 &= 2e^{-\pi i\lambda_{\infty,1}} \sin \pi(\lambda_{0,2} + \lambda_{1,1} + \lambda_{\infty,1}). \end{aligned} \tag{10.10}$$

When (10.8) *is not valid, we have* (10.9) *with a certain* $g \in GL(2, \mathbb{C})$ *and*

$$\begin{aligned} a_0 &= \begin{cases} 1 & \text{if } \lambda_{0,1} + \lambda_{1,2} + \lambda_{\infty,\nu} \notin \{0, -1, -2, \ldots\} \quad (\nu = 1,\ 2), \\ 0 & \text{otherwise}, \end{cases} \\ a_1 &= \begin{cases} 1 & \text{if } \lambda_{0,2} + \lambda_{1,1} + \lambda_{\infty,\nu} \notin \{0, -1, -2, \ldots\} \quad (\nu = 1,\ 2), \\ 0 & \text{otherwise}. \end{cases} \end{aligned} \tag{10.11}$$

Note that $a_0a_1 = 0$ *in this case.*

iii) *Under a change of indices* $\lambda_{p,\nu} \mapsto \lambda_{\sigma(p),\sigma_p(\nu)}$ *with suitable permutations* $(\sigma, \sigma_0, \sigma_1, \sigma_\infty) \in \mathfrak{S}_3 \times \mathfrak{S}_2^3$ *we have* (10.6) *and* (10.7). *Here* $\mathfrak{S}_3$ *and* $\mathfrak{S}_2$ *are identified with the permutation groups of* $\{0, 1, \infty\}$ *and* $\{1, 2\}$, *respectively.*

Proof. i), ii) Assume (10.7) in the proof. Then the functions $u_0^{\lambda_{0,2}}$ and $u_1^{\lambda_{1,2}}$ are well-defined and they satisfy $\gamma_0 u_0^{\lambda_{0,2}} = e^{2\pi i\lambda_{0,2}} u_0^{\lambda_{0,2}}$ and $\gamma_1 u_1^{\lambda_{1,2}} = e^{2\pi i\lambda_{1,2}} u_1^{\lambda_{1,2}}$.

Suppose that $u_0^{\lambda_{0,2}}$ and $u_1^{\lambda_{1,2}}$ are linearly dependent. Then

$$v_0 := x^{-\lambda_{0,2}}(1-x)^{-\lambda_{1,2}} u_0^{\lambda_{0,2}} \in P\begin{Bmatrix} x=0 & 1 & \infty \\ \lambda_{0,1}-\lambda_{0,2} & \lambda_{1,1}-\lambda_{1,2} & \lambda_{0,2}+\lambda_{1,2}+\lambda_{\infty,1} \\ 0 & 0 & \lambda_{0,2}+\lambda_{1,2}+\lambda_{\infty,2}\end{Bmatrix}$$

is an entire function, therefore a polynomial (cf. the last part of §1) and

$$\lambda_{0,2}+\lambda_{1,2}+\lambda_{\infty,k} \in \{0,-1,-2,\ldots\} \quad \text{for} \quad k=1 \text{ or } 2. \tag{10.12}$$

Suppose (10.12). Since $\lambda_{0,1}-\lambda_{0,2} \notin \{1,2,\ldots\}$ and

$$v_0(x) = F(\lambda_{0,2}+\lambda_{1,2}+\lambda_{\infty,1}, \lambda_{0,2}+\lambda_{1,2}+\lambda_{\infty,2}, 1-\lambda_{0,1}+\lambda_{0,2}; x),$$

Remark 1 shows that v_0 is a polynomial and hence (10.4) is reducible.

Suppose $u_0^{\lambda_{0,2}}$ and $u_1^{\lambda_{1,2}}$ are linearly independent. Under the base $(u_0^{\lambda_{0,2}}, u_1^{\lambda_{1,2}})$

$$M_0 = \begin{pmatrix} e^{2\pi i\lambda_{0,2}} & a_0 \\ & e^{2\pi i\lambda_{0,1}}\end{pmatrix},\ M_1 = \begin{pmatrix} e^{2\pi i\lambda_{1,1}} & \\ a_1 & e^{2\pi i\lambda_{1,2}}\end{pmatrix},\ M_\infty M_1 M_0 = I_2$$

and therefore

$$\begin{aligned}
\operatorname{trace} M_1M_0 &= e^{2\pi i(\lambda_{0,2}+\lambda_{1,1})} + a_0a_1 + e^{2\pi i(\lambda_{0,1}+\lambda_{1,2})} = e^{-2\pi i\lambda_{\infty,1}} + e^{-2\pi i\lambda_{\infty,2}},\\
a_0a_1 &= e^{-2\pi i\lambda_{\infty,1}} + e^{-2\pi i\lambda_{\infty,2}} - e^{2\pi i(\lambda_{0,2}+\lambda_{1,1})} - e^{2\pi i(\lambda_{0,1}+\lambda_{1,2})}\\
&= e^{-2\pi i\lambda_{\infty,2}}(e^{2\pi i(\lambda_{0,2}+\lambda_{1,1}+\lambda_{\infty,2})}-1)(e^{2\pi i(\lambda_{0,1}+\lambda_{1,2}+\lambda_{\infty,2})}-1)\\
&= e^{\pi i(\lambda_{0,1}+\lambda_{0,2}+\lambda_{1,1}+\lambda_{1,2})}\bigl(2i\sin\pi(\lambda_{0,2}+\lambda_{1,1}+\lambda_{\infty,2})\bigr)\\
&\quad\cdot\bigl(2i\sin\pi(\lambda_{0,1}+\lambda_{1,2}+\lambda_{\infty,2})\bigr)\\
&= 4e^{-\pi i(\lambda_{\infty,1}+\lambda_{\infty,2})}\sin\pi(\lambda_{0,2}+\lambda_{1,1}+\lambda_{\infty,2})\sin\pi(\lambda_{0,2}+\lambda_{1,1}+\lambda_{\infty,1}).
\end{aligned}$$

Hence the condition (10.8) implies $a_0a_1 \neq 0$ and then we have (10.9) with (10.10) by multiplying $u_0^{\lambda_{0,2}}$ by a suitable non-zero number.

Since

$$\begin{aligned}
&a_0a_1 + (e^{2\pi i\lambda_{0,2}} - e^{2\pi i\lambda_{0,1}})(e^{2\pi i\lambda_{1,1}} - e^{2\pi i\lambda_{1,2}})\\
&\quad= e^{-2\pi i\lambda_{\infty,1}} + e^{-2\pi i\lambda_{\infty,2}} - e^{2\pi i(\lambda_{0,1}+\lambda_{1,1})} - e^{2\pi i(\lambda_{0,2}+\lambda_{1,2})}\\
&\quad= e^{-2\pi i\lambda_{\infty,2}}(e^{2\pi i(\lambda_{0,1}+\lambda_{1,1}+\lambda_{\infty,2})}-1)(e^{2\pi i(\lambda_{0,2}+\lambda_{1,2}+\lambda_{\infty,2})}-1),
\end{aligned}$$

we have i) in view of Lemma 6.

Note that $a_1 = 0$ if and only if $\gamma_1 u_0^{\lambda_{0,2}} \in \mathbb{C}u_0^{\lambda_{0,2}}$. Since $u_0^{\lambda_{0,2}} \notin \mathbb{C}u_1^{\lambda_{1,2}}$ and $\lambda_{1,2}-\lambda_{1,1} \notin \{-1,-2,\ldots\}$, the condition $\gamma_1 u_0^{\lambda_{0,2}} \in \mathbb{C}u_0^{\lambda_{0,2}}$ is valid if and only if

$$\tilde v_0 := x^{-\lambda_{0,2}}(1-x)^{-\lambda_{1,1}} u_0^{\lambda_{0,2}} \in P\begin{Bmatrix} x=0 & 1 & \infty \\ \lambda_{0,1}-\lambda_{0,2} & \lambda_{1,2}-\lambda_{1,1} & \lambda_{0,2}+\lambda_{1,1}+\lambda_{\infty,1} \\ 0 & 0 & \lambda_{0,2}+\lambda_{1,1}+\lambda_{\infty,2}\end{Bmatrix}$$

is a polynomial. Remark 1 shows that this is also equivalent to

$$\lambda_{0,2}+\lambda_{1,1}+\lambda_{\infty,k} \in \{0,-1,-2,\ldots\} \quad \text{for} \quad k=1 \text{ or } 2$$

because $\lambda_{0,1}-\lambda_{0,2} \notin \{1,2,\ldots\}$. The condition $a_0 = 0$ is similarly examined.

iii) Suppose the claim iii) is not valid. If $\lambda_{0,2}+\lambda_{1,2}+\lambda_{\infty,1} \in \{0,-1,-2,\ldots\}$ and $\lambda_{\infty,2}-\lambda_{\infty,1} \notin \mathbb{Z}$, $\lambda_{0,1}+\lambda_{1,1}+\lambda_{\infty,\nu} \notin \{0,-1,-2,\ldots\}$ for $\nu = 1$, 2. Hence we may assume $\lambda_{p,2}-\lambda_{p,1} \in \{0,1,2,\ldots\}$ for $p = 0$, 1, ∞ and that the Riemann scheme is

$$P\begin{Bmatrix} \lambda_{0,1} & \lambda_{1,1} & \lambda_{\infty,1} \\ \lambda_{0,2} & \lambda_{1,2} & \lambda_{\infty,2}\end{Bmatrix} = P\begin{Bmatrix} c_0 & c_1 & -c_0-c_1-n_0-n_1-m \\ c_0+n_0 & c_1+n_1 & -c_0-c_1+m+1\end{Bmatrix}$$

with $n_0, n_1, m \in \{0,1,2,\ldots\}$. Then $\lambda_{0,\nu}+\lambda_{1,2}+\lambda_{\infty,2} \notin \{0,-1,\ldots\}$ for $\nu = 1$, 2. □

Remark 9. We calculate the monodromy matrices. We may assume (10.6) and (10.7). Then the monodromy matrices with respect to the base $(u_0^{\lambda_{0,2}}, u_1^{\lambda_{1,2}})$ depend holomorphically on the parameters $\lambda_{p,\nu}$. The function corresponding to the Riemann scheme $P\begin{Bmatrix} x=0 & 1 & \infty \\ \lambda_{0,1} & \lambda_{1,1} & \lambda_{\infty,1} \\ \lambda_{0,2} & \lambda_{1,2} & \lambda_{\infty,2} \end{Bmatrix}$ has the connection formula

$$u_0^{\lambda_{0,2}} = a\cdot u_1^{\lambda_{1,1}} + b\cdot u_1^{\lambda_{1,2}} \quad \text{with} \quad \begin{cases} a = C\binom{\lambda_{0,2}+\lambda_{1,2}+\lambda_{\infty,1}\ \lambda_{0,2}+\lambda_{1,2}+\lambda_{\infty,2}}{\lambda_{0,2}-\lambda_{0,1}+1\quad \lambda_{1,2}-\lambda_{1,1}} \\ b = C\binom{\lambda_{0,2}+\lambda_{1,1}+\lambda_{\infty,1}\ \lambda_{0,2}+\lambda_{1,1}+\lambda_{\infty,2}}{\lambda_{0,2}-\lambda_{0,1}+1\quad \lambda_{1,1}-\lambda_{1,2}} \end{cases}$$

as is given in (9.7) and therefore[6]

$$\begin{aligned}
(\gamma_1 u_0^{\lambda_{0,2}}, \gamma_1 u_1^{\lambda_{1,2}}) &= (\gamma_1 u_1^{\lambda_{1,1}}, \gamma_1 u_1^{\lambda_{1,2}}) \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \\
&= (u_1^{\lambda_{1,1}}, u_1^{\lambda_{1,2}}) \begin{pmatrix} e^{2\pi i\lambda_{1,1}} & \\ & e^{2\pi i\lambda_{1,2}} \end{pmatrix} \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \\
&= (u_0^{\lambda_{0,2}}, u_1^{\lambda_{1,2}}) \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}^{-1} \begin{pmatrix} e^{2\pi i\lambda_{1,1}} & \\ & e^{2\pi i\lambda_{1,2}} \end{pmatrix} \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \\
&= (u_0^{\lambda_{0,2}}, u_1^{\lambda_{1,2}}) \begin{pmatrix} a^{-1} & 0 \\ -a^{-1}b & 1 \end{pmatrix} \begin{pmatrix} ae^{2\pi i\lambda_{1,1}} & 0 \\ be^{2\pi i\lambda_{1,2}} & e^{2\pi i\lambda_{1,2}} \end{pmatrix} \\
&= (u_0^{\lambda_{0,2}}, u_1^{\lambda_{1,2}}) \begin{pmatrix} e^{2\pi i\lambda_{1,1}} & 0 \\ b(e^{2\pi i\lambda_{1,2}} - e^{2\pi i\lambda_{1,1}}) & e^{2\pi i\lambda_{1,2}} \end{pmatrix}, \\
e^{2\pi i\lambda_{1,1}} - e^{2\pi i\lambda_{1,2}} &= 2ie^{\pi i(\lambda_{1,1}+\lambda_{1,2})}\sin\pi(\lambda_{1,1}-\lambda_{1,2}) \\
&= 2\pi ie^{\pi i(\lambda_{1,1}+\lambda_{1,2})}(\lambda_{1,1}-\lambda_{1,2})\prod_{n=1}^{\infty}\Bigl(1-\frac{(\lambda_{1,1}-\lambda_{1,2})^2}{n^2}\Bigr) \\
&= 2\pi ie^{\pi i(\lambda_{1,1}+\lambda_{1,2})}\frac{1}{\Gamma(\lambda_{1,1}-\lambda_{1,2})\cdot\Gamma(1-\lambda_{1,1}+\lambda_{1,2})}, \\
b &= C\binom{\lambda_{0,2}+\lambda_{1,1}+\lambda_{\infty,1}\ \lambda_{0,2}+\lambda_{1,1}+\lambda_{\infty,2}}{\lambda_{0,2}-\lambda_{0,1}+1\quad \lambda_{1,1}-\lambda_{1,2}} \\
&= \frac{\Gamma(\lambda_{0,2}-\lambda_{0,1}+1)\cdot\Gamma(\lambda_{1,1}-\lambda_{1,2})}{\Gamma(\lambda_{0,2}+\lambda_{1,1}+\lambda_{\infty,1})\cdot\Gamma(\lambda_{0,2}+\lambda_{1,1}+\lambda_{\infty,2})}.
\end{aligned}$$

Hence putting $a_1 = b(e^{2\pi i\lambda_{1,1}} - e^{2\pi i\lambda_{1,2}})$, we have[7]

$$(\gamma_1 u_0^{\lambda_{0,2}}, \gamma_1 u_1^{\lambda_{1,2}}) = (u_0^{\lambda_{0,2}}, u_1^{\lambda_{1,2}}) \begin{pmatrix} e^{2\pi i\lambda_{1,1}} & \\ a_1 & e^{2\pi i\lambda_{1,2}} \end{pmatrix},$$

$$\begin{aligned}
a_1 = 2\pi ie^{\pi i(\lambda_{1,1}+\lambda_{1,2})} &\cdot \frac{\Gamma(\lambda_{0,2}-\lambda_{0,1}+1)}{\Gamma(\lambda_{1,2}-\lambda_{1,1}+1)} \\
&\cdot \frac{1}{\Gamma(\lambda_{0,2}+\lambda_{1,1}+\lambda_{\infty,1})\cdot\Gamma(\lambda_{0,2}+\lambda_{1,1}+\lambda_{\infty,2})}.
\end{aligned} \tag{10.13}$$

Note that under the conditions (10.6) and (10.7), the right hand side of the first line of (10.13) is holomorphic and never vanishes and the equality (10.13) is valid.

In the same way we have

$$(\gamma_0 u_0^{\lambda_{0,2}}, \gamma_0 u_1^{\lambda_{1,2}}) = (u_0^{\lambda_{0,2}}, u_1^{\lambda_{1,2}}) \begin{pmatrix} e^{2\pi i\lambda_{0,1}} & a_0 \\ & e^{2\pi i\lambda_{0,2}} \end{pmatrix},$$

---

[6]The Wronskian of $(u_0^{\lambda_{0,2}}, u_1^{\lambda_{1,2}})$ equals $(\lambda_{1,2}-\lambda_{1,1})ax^{\lambda_{0,1}+\lambda_{0,2}-1}(1-x)^{\lambda_{1,1}+\lambda_{1,2}-1}$.

[7]Under the base $(\tilde u_0^{\lambda_{0,2}}, \tilde u_1^{\lambda_{1,2}})$ with the functions $\tilde u_p^{\lambda_{p,2}} = \frac{u_p^{\lambda_{p,2}}}{\Gamma(\lambda_{p,2}-\lambda_{p,1}+1)}$, the monodromy matrices M_0 and M_1 are given by $a_p = \frac{2\pi ie^{\pi i(\lambda_{p,1}+\lambda_{p,2})}}{\Gamma(\lambda_{p,1}+\lambda_{1-p,2}+\lambda_{\infty,1})\cdot\Gamma(\lambda_{p,1}+\lambda_{1-p,2}+\lambda_{\infty,2})}$ for $p = 0, 1$. Note that the functions $\tilde u_p^{\lambda_{p,2}}$ holomorphically depend on the parameters $\lambda_{j,\nu}\in\mathbb{C}$ without a pole.

$$a_0 = 2\pi i e^{\pi i(\lambda_{0,1}+\lambda_{0,2})} \cdot \frac{\Gamma(\lambda_{1,2}-\lambda_{1,1}+1)}{\Gamma(\lambda_{0,2}-\lambda_{0,1}+1)} \cdot \frac{1}{\Gamma(\lambda_{0,1}+\lambda_{1,2}+\lambda_{\infty,1})\cdot\Gamma(\lambda_{0,1}+\lambda_{1,2}+\lambda_{\infty,2})}. \tag{10.14}$$

Here we use Gamma functions for the convenience to the reader but we may use functions of infinite products of linear functions in place of Gamma functions.

Remark 10. The differential operators P_1 and P_2 in (9.20) induce the identity maps of the monodromy groups of the Riemann schemes under the condition (9.21).

In particular, let $P\{\alpha,\beta,\gamma\}$ denote the Riemann scheme (4.3) which contains $F(\alpha,\beta,\gamma;x)$. Then we have the isomorphisms:

$$\begin{aligned} &P\{\alpha,\beta,\gamma\} \overset{\sim}{\leftrightarrow} P\{\alpha+1,\beta,\gamma\} \quad \text{if } \alpha(\gamma-\alpha-1)\neq 0,\\ &P\{\alpha,\beta,\gamma\} \overset{\sim}{\leftrightarrow} P\{\alpha,\beta+1,\gamma\} \quad \text{if } \beta(\gamma-\beta-1)\neq 0,\\ &P\{\alpha,\beta,\gamma\} \overset{\sim}{\leftrightarrow} P\{\alpha,\beta,\gamma-1\} \quad \text{if } (\gamma-\alpha-1)(\gamma-\beta-1)\neq 0. \end{aligned} \tag{10.15}$$

The isomorphisms are given by the differential operators appeared in (6.11) and (6.12). The above conditions are equivalent to say that under the shift of the parameters, any integer contained in $\{\alpha,\beta,\gamma-\alpha,\gamma-\beta\}$ does not change its property that it is positive or not.

Put $\{\alpha_1,\ldots,\alpha_N\} = \{\alpha,\beta,\gamma-\alpha,\gamma-\beta\}\cap\mathbb{Z}$. Then $N=0$ or 1 or 2 or 4. When $N=1$ or 2, we have $\alpha_j = 0$ or 1 for $1\le j\le N$ by suitable successive applications of the above isomorphisms.

Let α, β, $\gamma\in\mathbb{Z}$. Then it is easy to see that suitable successive applications of the above isomorphisms and the maps $\mathrm{Ad}(x^{\pm1})$, $\mathrm{Ad}\big((1-x)^{\pm1}\big)$, $T_{0\leftrightarrow1}$ and $T_{0\leftrightarrow\infty}$ transform the equation $P(\alpha,\beta,\gamma)u=0$ into $\partial^2u=0$ with $(\alpha,\beta,\gamma)=(0,-1,0)$ or $(\vartheta+1)\partial u=0$ with $(\alpha,\beta,\gamma)=(0,0,1)$ if $\#\{\alpha_\nu>0\mid \nu=1,\ldots,4\}$ is odd or even, respectively. Note that their solution spaces are $\mathbb{C}+\mathbb{C}x$ or $\mathbb{C}+\mathbb{C}\log x$, respectively.

11. Integral representation

Lastly we give an integral representation of Gauss hypergeometric function:

$$\int_{z_1}^{z_2}(t-z_1)^a(z_2-t)^b(z_3-t)^c dt = \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)} \cdot (z_2-z_1)^{a+b+1}(z_3-z_1)^c\cdot F\Big(a+1,-c,a+b+2;\frac{z_2-z_1}{z_3-z_1}\Big) \tag{11.1}$$

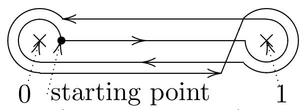
$$(a>-1,\ b>-1,\ |z_2-z_1|<|z_3-z_1|).$$

Putting $(z_1,z_2,z_3)=(0,1,\frac{1}{x})$ and $(a,b,c)=(\alpha-1,\gamma-\alpha-1,-\beta)$, we have an integral representation[8] of Gauss hypergeometric series, namely,

$$\int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-xt)^{-\beta}dt = \frac{\Gamma(\alpha)\Gamma(\gamma-\alpha)}{\Gamma(\gamma)}F(\alpha,\beta,\gamma;x). \tag{11.2}$$

By the complex integral through the Pochhammer contour $(1+,0+,0-,1-)$ along a double loop circuit, we have

$$\int^{(1+,0+,1-,0-)} z^{\alpha-1}(1-z)^{\gamma-\alpha-1}(1-xz)^{-\beta}dz = \frac{-4\pi^2e^{\pi i\gamma}}{\Gamma(1-\alpha)\Gamma(1+\alpha-\gamma)\Gamma(\gamma)}F(\alpha,\beta,\gamma;x). \tag{11.3}$$



$(1+,0+,0-,1-)$

Here we have no restriction on the values of the parameters α, β and γ. Put $\gamma'=\gamma-\alpha$. If α or γ' is a positive integer, the both sides in the above vanish.

[8]Putting $x=1$ in (11.2), we get the Gauss summation formula.

But replacing the integrand $\Phi(\alpha,\beta,\gamma';x,z) := z^{\alpha-1}(1-z)^{\gamma'-1}(1-xz)^{-\beta}$ by the function

$$\Phi^{(m)}(\alpha,\beta,\gamma';x,z) := \frac{\Phi(\alpha,\beta,\gamma';x,z)-\Phi(\alpha,\beta,m;x,z)}{\gamma'-m},$$

we get the integral representation of $F(\alpha,\beta,\gamma;x)$ even if $\gamma-\alpha=m$ is a positive integer because the function $\Phi^{(m)}(\alpha,\beta,\gamma',x,z)$ holomorphically depends on γ'. For example, when $m=1$, we have

$$\int^{(1+,0+,1-,0-)} z^{\alpha-1}\log(1-z)\cdot(1-xz)^{-\beta}dz = \frac{-4\pi^2e^{\pi i\alpha}}{\Gamma(1-\alpha)\Gamma(1+\alpha)}F(\alpha,\beta,\alpha+1;x).$$

A similar replacement of the integrand is valid when α is a positive integer. Namely, the function $\Gamma(1-\alpha)\Gamma(1+\alpha-\gamma)\int^{(1+,0+,1-,0-)} z^{\alpha-1}(1-z)^{\gamma-\alpha-1}\cdot(1-xz)^{-\beta}dz$ holomorphically depends on the parameters.

Putting $t=z_1+(z_2-z_1)s$ and $w=\frac{z_2-z_1}{z_3-z_1}$, the integral (11.1) equals

$$\begin{aligned}
&(z_2-z_1)^{a+b+1}(z_3-z_1)^c\int_0^1 s^a(1-s)^b(1-ws)^c ds\\
&=(z_2-z_1)^{a+b+1}(z_3-z_1)^c\int_0^1 s^a(1-s)^b\sum_{n=0}^\infty\frac{(-c)_n}{n!}(ws)^n\,ds\\
&=(z_2-z_1)^{a+b+1}(z_3-z_1)^c\sum_{n=0}^\infty\frac{w^n}{n!}\int_0^1 s^{a+n}(1-s)^b(-c)_n ds\\
&=(z_2-z_1)^{a+b+1}(z_3-z_1)^c\sum_{n=0}^\infty\frac{\Gamma(a+1+n)\Gamma(b+1)(-c)_n}{\Gamma(a+b+n+2)}w^n,
\end{aligned}$$

which implies (11.1). This function belongs to

$$P\begin{Bmatrix} z_3=z_1 & z_2 & \infty & \\ 0 & 0 & -c & ;\ z_3 \\ a+c+1 & b+c+1 & -a-b-c-1 & \end{Bmatrix}$$

as a function of z_3.

Remark 11. An analysis of the monodromy group associated with the Gauss hypergeometric function using integrals is given in [MS].

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