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Restriction of the system of differential equations satisfied by the matrix coefficient of the principal representation of $Sp(2,\mathbb{R})$

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1 Introduction

Oshima and Shimeno studied the system of ordinary differential equations which is obtained by restricting the A_{n-1}, BC_n-type Heckman-Opdam system to singular lines([9]). The result was applied to obtain the connection formula of Heckman-Opdam hypergeometric functions and the so-called Gauss summation formula.

In this paper, we will study the restriction of the system of differential equations satisfied by 2 dimensional spherical functions (matrix coefficients of the principal series representation of $Sp(2,\mathbb{R})$) of C_2-type.

2 The matrix coefficient of the principal series representations

In this section, we recall some facts about the principal series representations of $Sp(2,\mathbb{R})$ and their K-type. Notations are same as those of [4].

Let $G = Sp(2,\mathbb{R})$ be a split real semisimple Lie group of real rank 2 with a maximal compact subgroup K which is isomorphic to the unitary group $U(2)$. The restricted root system of G is of C_2-type such as $\{\pm 2e_1, \pm 2e_2, \pm e_1 \pm e_2\}$. We take a positive system $\{2e_1, 2e_2, e_1 \pm e_2\}$ and the corresponding minimal parabolic subgroup with the Langlands decomposition as $P = MAN$. Here, $M = \{\mathrm{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_1, \varepsilon_2) \mid \varepsilon_1, \varepsilon_2 \in \{\pm 1\}\}$, $A = \{\mathrm{diag}(a_1, a_2, a_1, a_2) \mid a_1, a_2 > 0\}$.

Let σ be an irreducible representation of M, $\mu = \mu_1 e_1 + \mu_2 e_2 \in \mathrm{Lie}(A)_{\mathbb{C}} \simeq \mathbb{C}^2$ and $\rho = 2e_1 + e_2 \in \mathrm{Lie}(A)_{\mathbb{C}}$ be the half sum of positive roots. The principal series representation of G is defined as $\pi_{\sigma,\mu} = \mathrm{Ind}_{MAN}^{G}(\sigma \boxtimes a^{\mu+\rho} \boxtimes \mathrm{id}_N)$ and let $V_{\sigma,\mu}$ be its representation space.

Let (τ_j, V_j) be K-types of $(\pi_{\sigma,\mu}, V_{\sigma,\mu})$ with multiplicity-free and ι_j be K-maps from V_j into $V_{\sigma,\mu}$ each of which is unique up to scalar multiple($j = 1, 2$). We denote maps of dual spaces(representation spaces of contragredient representation) V_j^* to $V_{\sigma,\mu}^*$ by ι_j^*.

We denote the vector-valued smooth function space

$$C^{\infty}_{\tau_1,\tau_2}(K\backslash G/K) = \{f : G \to V_1 \otimes V_2^* \mid f \text{ is smooth}, f(k_1 g k_2) = \tau_1(k_1) \otimes \tau_2^*(k_2)^{-1} f(g), \\ k_1, k_2 \in K, g \in G\}$$

We choose bases $\{v_k^j \mid k = 1, ..., \dim V_j\}$ of V_j and its dual bases $\{v_k^{j^*} \mid k = 1, ..., \dim V_j\}$ of V_j^*.

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Definition 2.1. We define the function in $C^\infty_{\tau_1,\tau_2}(K\backslash G/K)$

$$\phi(g)=\sum_{k,l}\langle \pi_{\sigma,\mu}(g)\iota_2(v_k^2),\iota_1^*(v_l^{1^*})\rangle v_l^1\otimes v_k^{2^*}$$

as (τ_1,τ_2)-spherical function.

Note that $\langle \pi_{\sigma,\mu}(g)\iota_2(v_k^2),\iota_1^*(v_l^{1^*})\rangle$ is a matrix coefficient of the principal series representation $\pi_{\sigma,\mu}$, and we denote this function by $\phi_{lk}(g)$.

By the Cartan decomposition $G=KAK$, ϕ is determined by the values on A. We use the coordinate (x_1,x_2) on $\mathrm{Lie}(A)$ as the coordinate of A, which is the dual basis of $\{e_1,e_2\}$ in $\mathrm{Lie}(A)^*$.

We take $\sigma=(\mathrm{id},\mathrm{sgn}),(\mathrm{sgn},\mathrm{id})$, where id is the trivial representation and sgn is the sign representation of $\{\pm1\}$. Then, $\pi_{\sigma,\mu}$ has 2 dimensional irreducible K-types as minimum dimensional components and we choose 2 dimensional K-type $\tau_1=(k,k-1),\tau_2=(l,l-1)$ with $k,l\in\mathbb{Z}$ (The detailed definition including the choice of the basis is found in [4]). We assume that $k\equiv l \mod 2$. From the symmetry with respect to the action of the Weyl group, we have the following lemma(Lemma 4.3 in [4]).

Lemma 2.2. *(1) Two of four components in ϕ are 0: i.e. $\phi_{00}(x_1,x_2)=\phi_{11}(x_1,x_2)=0$, hence*

$$\phi(x_1,x_2)=\phi_{01}(x_1,x_2)v_0^1\otimes v_1^{2^*}+\phi_{10}(x_1,x_2)v_1^1\otimes v_0^{2^*}.$$

(2) Moreover, we have the symmetry:

$$\phi_{10}(x_1,x_2)=-\phi_{01}(x_2,x_1).$$

Remark 2.3. If σ is either $(\mathrm{id},\mathrm{id})$ or $(\mathrm{sgn},\mathrm{sgn})$, the minimal dimensional K-type in $\pi_{\sigma,\mu}$ is 1. And if we take 1 dimensional K-type as τ_1,τ_2, the function ϕ is a scalar-valued function. The restriction for such ϕ, that is, the Heckman-Opdam hypergeometric function of type BC_2, was studied in [9].

We define functions ψ_{01},ψ_{10} as

$$\begin{aligned}\psi_{01}(x_1,x_2)&=(\cosh x_1\cosh x_2)^{\frac{l+k}{2}}(\sinh x_1\sinh x_2)^{\frac{l-k}{2}}\cosh^{-1}x_1\phi_{01}(x_1,x_2),\\ \psi_{10}(x_1,x_2)&=(\cosh x_1\cosh x_2)^{\frac{l+k}{2}}(\sinh x_1\sinh x_2)^{\frac{l-k}{2}}\cosh^{-1}x_2\phi_{10}(x_1,x_2).\end{aligned}$$

They also satisfy the symmetry : $\psi_{10}(x_1,x_2)=-\psi_{01}(x_2,x_1)$.

Theorem 2.4 (Equation (7.3) in [4]). *Both ψ_{01} and ψ_{10} satisfy the following system of differential equations.*

$$\begin{aligned}&\Bigg[\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\{2(k-1)\coth x_1+2(3-k-l)\coth 2x_1)\}\frac{\partial}{\partial x_1}\\
&\qquad+\{2k\coth x_2+2(1-k-l)\coth 2x_2)\}\frac{\partial}{\partial x_2}\\
&\qquad+\coth(x_1+x_2)\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}\right)+\coth(x_1-x_2)\left(\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_2}\right)\\
&\quad-\coth^2(x_1+x_2)-\coth^2(x_1-x_2)+2\coth 2x_1\coth(x_1+x_2)+2\coth 2x_1\coth(x_1-x_2)\\
&\qquad-\coth x_1\coth(x_1+x_2)-\coth x_1\coth(x_1-x_2)+2l^2-8l+5\Bigg]\psi_{01}\\
&-\{\cosh(x_1+x_2)\sinh^{-2}(x_1+x_2)+\cosh(x_1-x_2)\sinh^{-2}(x_1-x_2)\}\cosh^{-1}x_1\cosh x_2\psi_{10}\\
&\qquad\qquad=(\mu_1^2+\mu_2^2-5)\psi_{01},\end{aligned}$$

$$
\begin{aligned}
&\Big[\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \{2k\coth x_1 + 2(1-k-l)\coth 2x_1)\}\frac{\partial}{\partial x_1} \\
&\qquad + \{2(k-1)\coth x_2 + 2(3-k-l)\coth 2x_2)\}\frac{\partial}{\partial x_2} \\
&\qquad + \coth(x_1+x_2)\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right) + \coth(x_1-x_2)\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right) \\
&\quad - \coth^2(x_1+x_2) - \coth^2(x_1-x_2) + 2\coth 2x_2\coth(x_1+x_2) - 2\coth 2x_2\coth(x_1-x_2) \\
&\qquad - \coth x_2\coth(x_1+x_2) + \coth x_2\coth(x_1-x_2) + 2l^2 - 8l + 5\Big]\psi_{10} \\
&-\{\cosh(x_1+x_2)\sinh^{-2}(x_1+x_2) + \cosh(x_1-x_2)\sinh^{-2}(x_1-x_2)\}\cosh x_1\cosh^{-1} x_2\psi_{01} \\
&\qquad\qquad = (\mu_1^2 + \mu_2^2 - 5)\psi_{10}.
\end{aligned}
$$

The latter one is the consequence of the exchange of variables x_1 and x_2 in the former one.

We will write this system by the coordinates

$$y_1 = \exp(x_1 - x_2), \quad y_2 = \exp 2x_2,$$

which correspond to simple roots. Note that the point $(y_1, y_2) = (1, 1)$ corresponds to the identity in G and $\phi(y_1, y_2)$ is analytic around the point.

We regard ϕ and ψ as functions in variables y_1, y_2 hereafter.

Using this lemma, the above two equations are written in y_1, y_2 as follows.

Theorem 2.5. *$\psi_{01}(y_1, y_2), \psi_{10}(y_1, y_2)$ satisfy the following system of differential equations.*

$$
\begin{aligned}
&\left[\left(y_1\frac{\partial}{\partial y_1}\right)^2 + \left(-y_1\frac{\partial}{\partial y_1} + 2y_2\frac{\partial}{\partial y_2}\right)^2\right. \\
&+\left\{2(k-1)\frac{y_1^2y_2+1}{y_1^2y_2-1} + 2(3-k-l)\frac{y_1^4y_2^2+1}{y_1^4y_2^2-1} + \frac{y_1^2y_2^2+1}{y_1^2y_2^2-1} + \frac{y_1^2+1}{y_1^2-1}\right\}\left(y_1\frac{\partial}{\partial y_1}\right) \\
&+\left\{2k\frac{y_2+1}{y_2-1} + 2(1-k-l)\frac{y_2^2+1}{y_2^2-1} + \frac{y_1^2y_2^2+1}{y_1^2y_2^2-1} - \frac{y_1^2+1}{y_1^2-1}\right\}\left(-y_1\frac{\partial}{\partial y_1} + 2y_2\frac{\partial}{\partial y_2}\right) \\
&\left.-\left(\frac{y_1^2y_2^2+1}{y_1^2y_2^2-1}\right)^2 - \left(\frac{y_1^2+1}{y_1^2-1}\right)^2 + \left(2\frac{y_1^4y_2^2+1}{y_1^4y_2^2-1} - \frac{y_1^2y_2+1}{y_1^2y_2-1}\right)\left(\frac{y_1^2y_2^2+1}{y_1^2y_2^2-1} + \frac{y_1^2+1}{y_1^2-1}\right) + 2l^2 - 8l + 5\right]\psi_{01} \\
&-\left\{\frac{y_2(y_1^2y_2^2+1)}{(y_1^2y_2^2-1)^2} + \frac{y_1^2+1}{(y_1^2-1)^2}\right\}\frac{2y_1^2(y_2+1)}{y_1^2y_2+1}\psi_{10} \\
&= (\mu_1^2 + \mu_2^2 - 5)\psi_{01},
\end{aligned}
\tag{2.1}
$$

$$
\begin{aligned}
&\left[\left(-y_1\frac{\partial}{\partial y_1} + 2y_2\frac{\partial}{\partial y_2}\right)^2 + \left(y_1\frac{\partial}{\partial y_1}\right)^2\right. \\
&+\left\{2(k-1)\frac{y_2+1}{y_2-1} + 2(3-k-l)\frac{y_2^2+1}{y_2^2-1} + \frac{y_1^2y_2^2+1}{y_1^2y_2^2-1} - \frac{y_1^2+1}{y_1^2-1}\right\}\left(-y_1\frac{\partial}{\partial y_1} + 2y_2\frac{\partial}{\partial y_2}\right) \\
&+\left\{2k\frac{y_1^2y_2+1}{y_1^2y_2-1} + 2(1-k-l)\frac{y_1^4y_2^2+1}{y_1^4y_2^2-1} + \frac{y_1^2y_2^2+1}{y_1^2y_2^2-1} + \frac{y_1^2+1}{y_1^2-1}\right\}\left(y_1\frac{\partial}{\partial y_1}\right) \\
&\left.-\left(\frac{y_1^2y_2^2+1}{y_1^2y_2^2-1}\right)^2 - \left(\frac{y_1^2+1}{y_1^2-1}\right)^2 + \left(2\frac{y_2^2+1}{y_2^2-1} - \frac{y_2+1}{y_2-1}\right)\left(\frac{y_1^2y_2^2+1}{y_1^2y_2^2-1} - \frac{y_1^2+1}{y_1^2-1}\right) + 2l^2 - 8l + 5\right]\psi_{10} \\
&-\left\{\frac{y_2(y_1^2y_2^2+1)}{(y_1^2y_2^2-1)^2} + \frac{y_1^2+1}{(y_1^2-1)^2}\right\}\frac{2(y_1^2y_2+1)}{y_2+1}\psi_{01} \\
&= (\mu_1^2 + \mu_2^2 - 5)\psi_{10}
\end{aligned}
\tag{2.2}
$$

The latter is obtained by changing variables $(y_1, y_2) \to (1/y_1, y_1^2 y_2)$ in the former. This transformation corresponds the change variables $(x_1, x_2) \to (x_2, x_1)$.

Theorem 2.6 (Equation (7.4) in [4])**.** *ϕ satisfies the differential equation of 2×2 matrix type as $R(E^+_{l-1}) \circ R(E^-_l)\phi = -\{\mu_j^2 - (l-1)^2\}\phi$, where $j = 1$ when l is odd and $j = 2$ when l is even.*

$R(E^+_{l-1}), R(E^-_l)$ are written as follows.

$$\begin{aligned} R(E_l^-)\psi(x_1,x_2) \;=\; & (\cosh x_1 \cosh x_2)^{-\frac{l+k}{2}} (\sinh x_1 \sinh x_2)^{-\frac{l-k}{2}} \cosh x_1 \\ & \times \left[-\left\{ \frac{\partial}{\partial x_2} + \frac{1}{2}(\coth(x_1+x_2) - \coth(x_1-x_2)) \right\} \psi_{01} \right. \\ & \qquad \left. - \frac{1}{2}(\coth(x_1+x_2) - \coth(x_1-x_2))\psi_{10} \right] v_0^1 \otimes v_0^{3^*} \\ & + (\cosh x_1 \cosh x_2)^{-\frac{l+k}{2}} (\sinh x_1 \sinh x_2)^{-\frac{l-k}{2}} \cosh x_2 \\ & \times \left[\left\{ \frac{\partial}{\partial x_1} + \frac{1}{2}(\coth(x_1+x_2) + \coth(x_1-x_2)) \right\} \psi_{10} \right. \\ & \qquad \left. + \frac{1}{2}(\coth(x_1+x_2) + \coth(x_1-x_2))\psi_{10} \right] v_1^1 \otimes v_1^{3^*}, \end{aligned}$$

where $\tau_3 = (l-1, l-2)$ and $\{v_0^{3^*}, v_1^{3^*}\}$ is a basis of τ_3^*.

For $R(E_l^-)\psi(x_1,x_2) = (\cosh x_1 \cosh x_2)^{-\frac{l+k}{2}} (\sinh x_1 \sinh x_2)^{-\frac{l-k}{2}} \cosh x_1 \tilde{\psi}_{00}(x_1,x_2) v_0^1 \otimes v_0^{3^*}$ $+(\cosh x_1 \cosh x_2)^{-\frac{l+k}{2}} (\sinh x_1 \sinh x_2)^{-\frac{l-k}{2}} \cosh x_2 \tilde{\psi}_{11}(x_1,x_2) v_1^1 \otimes v_1^{3^*}$,

$$\begin{aligned} & R(E^+_{l-1}) \circ R(E^-_l)\psi \\ & = (\cosh x_1 \cosh x_2)^{-\frac{l+k}{2}} (\sinh x_1 \sinh x_2)^{-\frac{l-k}{2}} \cosh x_1 \\ & \times \left[\left\{ \frac{\partial}{\partial x_2} + 2k \coth x_2 + 2(1-k-l)\coth 2x_2 + \frac{1}{2}(\coth(x_1+x_2) - \coth(x_1-x_2)) \right\} \tilde{\psi}_{00} \right. \\ & \left. + \frac{1}{2}\left(\frac{1}{\sinh(x_1+x_2)} + \frac{1}{\sinh(x_1-x_2)} \right) \frac{\cosh x_2}{\cosh x_1} \tilde{\psi}_{11} \right] v_0^1 \otimes v_1^{2^*} \\ & + (\cosh x_1 \cosh x_2)^{-\frac{l+k}{2}} (\sinh x_1 \sinh x_2)^{-\frac{l-k}{2}} \cosh x_2 \\ & \times \left[-\left\{ \frac{\partial}{\partial x_1} + 2k \coth x_1 + 2(1-k-l)\coth 2x_1 + \frac{1}{2}(\coth(x_1+x_2) + \coth(x_1-x_2)) \right\} \tilde{\psi}_{11} \right. \\ & \left. - \frac{1}{2}\left(\frac{1}{\sinh(x_1+x_2)} - \frac{1}{\sinh(x_1-x_2)} \right) \frac{\cosh x_1}{\cosh x_2} \tilde{\psi}_{00} \right] v_1^1 \otimes v_0^{2^*} \end{aligned}$$

Then, we will describe the above differential equations with respect to the coordinate (y_1, y_2).

Theorem 2.7. *$\psi_{01}(y_1,y_2), \psi_{10}(y_1,y_2)$ satisfy the following system of differential equations.*

$$\begin{aligned} & \left\{ -y_1 \frac{\partial}{\partial y_1} + 2y_2 \frac{\partial}{\partial y_2} + 2k\frac{y_2+1}{y_2-1} + 2(1-k-l)\frac{y_2^2+1}{y_2^2-1} + \frac{1}{2}\left(\frac{y_1^2y_2^2+1}{y_1^2y_2^2-1} - \frac{y_1^2+1}{y_1^2-1} \right) \right\} \\ & \circ \left[-\left\{ -y_1\frac{\partial}{\partial y_1} + 2y_2\frac{\partial}{\partial y_2} + \frac{1}{2}\left(\frac{y_1^2y_2^2+1}{y_1^2y_2^2-1} - \frac{y_1^2+1}{y_1^2-1} \right) \right\} \psi_{01} - \frac{1}{2}\left(\frac{y_1^2y_2^2+1}{y_1^2y_2^2-1} - \frac{y_1^2+1}{y_1^2-1} \right) \psi_{10} \right] \\ & + \frac{y_1^2(y_2+1)}{y_1^2y_2+1}\left(\frac{y_2}{y_1^2y_2^2-1} + \frac{1}{y_1^2-1} \right) \circ \left[\left\{ y_1\frac{\partial}{\partial y_1} + \frac{1}{2}\left(\frac{y_1^2y_2^2+1}{y_1^2y_2^2-1} + \frac{y_1^2+1}{y_1^2-1} \right) \right\} \psi_{10} \right. \\ & \left. + \frac{1}{2}\left(\frac{y_1^2y_2^2+1}{y_1^2y_2^2-1} + \frac{y_1^2+1}{y_1^2-1} \right)\psi_{01} \right] = -\left\{ \mu_j^2 - (l-1)^2 \right\} \psi_{01} \qquad (2.3) \end{aligned}$$

$$
\begin{aligned}
&-\left\{y_1\frac{\partial}{\partial y_1}+2k\frac{y_1^2y_2+1}{y_1^2y_2-1}+2(1-k-l)\frac{y_1^4y_2^2+1}{y_1^4y_2^2-1}+\frac{1}{2}\left(\frac{y_1^2y_2^2+1}{y_1^2y_2^2-1}+\frac{y_1^2+1}{y_1^2-1}\right)\right\}\\
&\circ\left[\left\{y_1\frac{\partial}{\partial y_1}+\frac{1}{2}\left(\frac{y_1^2y_2^2+1}{y_1^2y_2^2-1}+\frac{y_1^2+1}{y_1^2-1}\right)\right\}\psi_{10}+\frac{1}{2}\left(\frac{y_1^2y_2^2+1}{y_1^2y_2^2-1}+\frac{y_1^2+1}{y_1^2-1}\right)\psi_{01}\right]\\
&-\frac{y_1^2y_2+1}{y_2+1}\left(\frac{y_2}{y_1^2y_2^2-1}-\frac{1}{y_1^2-1}\right)\circ\left[-\left\{-y_1\frac{\partial}{\partial y_1}+2y_2\frac{\partial}{\partial y_2}+\frac{1}{2}\left(\frac{y_1^2y_2^2+1}{y_1^2y_2^2-1}-\frac{y_1^2+1}{y_1^2-1}\right)\right\}\psi_{01}\right.\\
&\left.-\frac{1}{2}\left(\frac{y_1^2y_2^2+1}{y_1^2y_2^2-1}-\frac{y_1^2+1}{y_1^2-1}\right)\psi_{10}\right]=-\left\{\mu_j^2-(l-1)^2\right\}\psi_{10} \qquad (2.4)
\end{aligned}
$$

3 Leading components

Let $(\psi_{01}(y_1,y_2),\psi_{10}(y_1,y_2))=(\xi_1(y_1,y_2),\xi_2(y_1,y_2))$ be a local solution of the system of differential equations (2.1), (2.2), (2.3) and (2.4) which has the expansion around $(y_1,y_2)=(0,0)$ as

$$\begin{pmatrix}\xi_1(y_1,y_2)\\ \xi_2(y_1,y_2)\end{pmatrix}=y_1^\alpha y_2^\beta\sum_{p,q\in\mathbb{N}\cup\{0\}}\begin{pmatrix}c_{p,q}\\ d_{p,q}\end{pmatrix}y_1^py_2^q,$$

either $c_{00}\neq 0$ or $d_{00}\neq 0$. That is, (α,β) is the leading exponent of the local solution. Hereafter, we denote c_{00},d_{00} by c,d.

The indicial equations of the system are

$$
\begin{aligned}
&\left\{(\alpha-2\beta-l+1)^2+(\alpha+l-3)^2-\mu_1^2-\mu_2^2\right\}c=0, && (3.1)\\
&-2c+\left\{(\alpha-2\beta-l+1)^2+(\alpha+l-3)^2-\mu_1^2-\mu_2^2+4\alpha-4\beta-2\right\}d=0, && (3.2)\\
&\left\{(\alpha-2\beta-l+1)^2-\mu_j^2\right\}c=0, && (3.3)\\
&-(2\beta+2l-3)c+\{(\alpha+l-2)^2-\mu_j^2\}d=0. && (3.4)
\end{aligned}
$$

If $c\neq 0$, then equation (3.3) implies $\alpha-2\beta=l-1-\varepsilon_j\mu_j$, where ε_j has the same meaning as above. Then, from the equation (3.1), we have

$$\mu_j^2+(\alpha+l-3)^2-\mu_1^2-\mu_2^2=0.$$

By solving this equation, we obtain

$$(\alpha,\beta)=\left(-l+3+\varepsilon_{\bar{j}}\mu_{\bar{j}},-l+2+\frac{1}{2}\varepsilon_1\mu_1+\frac{1}{2}\varepsilon_2\mu_2\right),$$

where $\bar{j}$ and $\varepsilon_{\bar{j}}$ have same meanings as above.

For these α and β, equations (3.2) and (3.4) are

$$\begin{cases}-c+(1-\varepsilon_j\mu_j+\varepsilon_{\bar{j}}\mu_{\bar{j}})d=0,\\ (1+\varepsilon_j\mu_j+\varepsilon_{\bar{j}}\mu_{\bar{j}})\{-c+(1-\varepsilon_j\mu_j+\varepsilon_{\bar{j}}\mu_{\bar{j}})d\}=0.\end{cases}$$

Therefore, we have

$$\begin{pmatrix}c\\ d\end{pmatrix}=d\begin{pmatrix}1-\varepsilon_j\mu_j+\varepsilon_{\bar{j}}\mu_{\bar{j}}\\ 1\end{pmatrix}.$$

Especially, we have $1-\varepsilon_j\mu_j+\varepsilon_{\bar{j}}\mu_{\bar{j}}\neq 0$.

If $c=0$, then $d\neq 0$, equations (3.1), (3.3) are trivial and equations (3.2), (3.4) become

$$
\begin{aligned}
&\left\{(\alpha-2\beta-l+1)^2+(\alpha+l-3)^2-\mu_1^2-\mu_2^2+4\alpha-4\beta-2\right\}d=0, && (3.5)\\
&\{(\alpha+l-2)^2-\mu_j^2\}d=0. && (3.6)
\end{aligned}
$$

From the equation (3.6), we obtain

$$\alpha = -l + 2 + \varepsilon_j \mu_j,$$

where $\varepsilon_j \in \{\pm 1\}$. Then, we have

$$(-2\beta - 2l + 3 + \varepsilon_j \mu_j)^2 + (-1 + \varepsilon_j \mu_j)^2 - \mu_1^2 - \mu_2^2 - 4l + 8 + 4\varepsilon_j \mu_j - 4\beta - 2 = 0$$

from (3.5) and by solving this equation, we have

$$\beta = -l + 2 + \frac{1}{2}\varepsilon_1 \mu_1 + \frac{1}{2}\varepsilon_2 \mu_2,$$

and the leading exponent is

$$(\alpha, \beta) = \left(-l + 2 + \varepsilon_j \mu_j, -l + 2 + \frac{1}{2}\varepsilon_1 \mu_1 + \frac{1}{2}\varepsilon_2 \mu_2 \right).$$

Now, we have proved the following proposition.

Proposition 3.1. *There are two types of local solutions* $(\xi_1(y_1, y_2), \xi_2(y_1, y_2))$.

Type 1 *The leading exponent is*

$$(\alpha^{(1)}_{\varepsilon_1,\varepsilon_2}, \beta_{\varepsilon_1,\varepsilon_2}) = \left(-l + 3 + \varepsilon_{\bar{j}} \mu_{\bar{j}}, -l + 2 + \frac{1}{2}\varepsilon_1 \mu_1 + \frac{1}{2}\varepsilon_2 \mu_2 \right)$$

and

$$\begin{pmatrix} \xi_1(y_1, y_2) \\ \xi_2(y_1, y_2) \end{pmatrix} = y_1^{\alpha^{(1)}_{\varepsilon_1,\varepsilon_2}} y_2^{\beta_{\varepsilon_1,\varepsilon_2}} \left\{ d \begin{pmatrix} 1 - \varepsilon_j \mu_j + \varepsilon_{\bar{j}} \mu_{\bar{j}} \\ 1 \end{pmatrix} + \sum_{\substack{p,q \in \mathbb{N} \\ (p,q) \neq (0,0)}} \begin{pmatrix} c_{p,q} \\ d_{p,q} \end{pmatrix} y_1^p y_2^q \right\}.$$

Here, d is some constant, $(j, \bar{j}) = (1, 2)$ if l is odd and $(j, \bar{j}) = (2, 1)$ if l is even. $\varepsilon_j, \varepsilon_{\bar{j}}$ are in $\{\pm 1\}$ and independent from each other.

Type 2 *The leading exponent is*

$$(\alpha^{(2)}_{\varepsilon_1,\varepsilon_2}, \beta_{\varepsilon_1,\varepsilon_2}) = \left(-l + 2 + \varepsilon_j \mu_j, -l + 2 + \frac{1}{2}\varepsilon_1 \mu_1 + \frac{1}{2}\varepsilon_2 \mu_2 \right)$$

and

$$\begin{pmatrix} \xi_1(y_1, y_2) \\ \xi_2(y_1, y_2) \end{pmatrix} = y_1^{\alpha^{(2)}_{\varepsilon_1,\varepsilon_2}} y_2^{\beta_{\varepsilon_1,\varepsilon_2}} \left\{ d \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{\substack{p,q \in \mathbb{N} \\ (p,q) \neq (0,0)}} \begin{pmatrix} c_{p,q} \\ d_{p,q} \end{pmatrix} y_1^p y_2^q \right\}.$$

4 Restriction to $y_2 = 0$

In this section, we assume that l is odd so that $j = 1$ and $\bar{j} = 2$. To obtain the result corresponding to an even integer l, we have only to exchange ε_1, μ_1 and ε_2, μ_2.

Since $\psi_{01}(y_1, y_2), \psi_{10}(y_1, y_2)$ are analytic around $(y_1, y_2) = (1, 2)$, we can define analytic functions $f_{\varepsilon_1,\varepsilon_2}(y_1), g_{\varepsilon_1,\varepsilon_2}(y_1)$ around $0 < y_1 \leq 1$ by

$$\begin{pmatrix} \psi_{01}(y_1, y_2) \\ \psi_{10}(y_1, y_2) \end{pmatrix} = \sum_{\varepsilon_1,\varepsilon_2 \in \{\pm 1\}} y_2^{\beta_{\varepsilon_1,\varepsilon_2}} \left\{ \begin{pmatrix} f_{\varepsilon_1,\varepsilon_2}(y_1) \\ g_{\varepsilon_1,\varepsilon_2}(y_1) \end{pmatrix} + \begin{pmatrix} O(y_2) \\ O(y_2) \end{pmatrix} \right\}.$$

This is a linear combination of local solutions.

By inserting a local solution $y_2^{\beta_{\varepsilon_1,\varepsilon_2}}\left\{\begin{pmatrix} f_{\varepsilon_1,\varepsilon_2}(y_1) \\ g_{\varepsilon_1,\varepsilon_2}(y_1)\end{pmatrix} + \begin{pmatrix} O(y_2) \\ O(y_2)\end{pmatrix}\right\}$ into equations (2.1), (2.2), (2.3) and (2.4) and then multiplying $y_2^{-\beta_{\varepsilon_1,\varepsilon_2}}$ and taking the limit $y_2 \to 0$, we obtain the system of ordinary differential equations satisfied by $f_{\varepsilon_1,\varepsilon_2}(y_1), g_{\varepsilon_1,\varepsilon_2}(y_1)$.

Lemma 4.1. *The functions $f_{\varepsilon_1,\varepsilon_2}$ and $g_{\varepsilon_1,\varepsilon_2}(y_1)$ satisfy the following system of differential equations.*

$$\left\{\left(y_1\frac{d}{dy_1}\right)^2 + \left(-2\beta_{\varepsilon_1,\varepsilon_2} - 1 + \frac{y_1^2+1}{y_1^2-1}\right)y_1\frac{d}{dy_1} + 2\beta_{\varepsilon_1,\varepsilon_2}^2 + 2\beta_{\varepsilon_1,\varepsilon_2}\left(l-1-\frac{y_1^2}{y_1^2-1}\right)\right.$$
$$\left. + \frac{(l-1)^2+(l-3)^2-\mu_1^2-\mu_2^2}{2}\right\} f_{\varepsilon_1,\varepsilon_2}(y_1) = \frac{y_1^2(y_1^2+1)}{y_1^2-1}\{f_{\varepsilon_1,\varepsilon_2}(y_1)+g_{\varepsilon_1,\varepsilon_2}(y_1)\} \tag{2.1'}$$

$$\left\{\left(y_1\frac{d}{dy_1}\right)^2 + \left(-2\beta_{\varepsilon_1,\varepsilon_2} + 1 + \frac{y_1^2+1}{y_1^2-1}\right)y_1\frac{d}{dy_1} + 2\beta_{\varepsilon_1,\varepsilon_2}^2 + 2\beta_{\varepsilon_1,\varepsilon_2}\left(l-2-\frac{y_1^2}{y_1^2-1}\right)\right.$$
$$\left. + \frac{(l-1)^2+(l-3)^2-\mu_1^2-\mu_2^2}{2}\right\} g_{\varepsilon_1,\varepsilon_2}(y_1) = \frac{y_1^2+1}{(y_1^2-1)^2}\{f_{\varepsilon_1,\varepsilon_2}(y_1)+g_{\varepsilon_1,\varepsilon_2}(y_1)\} \tag{2.2'}$$

$$\left\{\left(y_1\frac{d}{dy_1}\right)^2 - 2\left(2\beta_{\varepsilon_1,\varepsilon_2}+l-1-\frac{y_1^2}{y_1^2-1}\right)y_1\frac{d}{dy_1} + (2\beta_{\varepsilon_1,\varepsilon_2}+l-1)^2 - \frac{2\beta_{\varepsilon_1,\varepsilon_2}y_1^2}{y_1^2-1} - \mu_1^2\right\} f_{\varepsilon_1,\varepsilon_2}(y_1)$$
$$= \frac{y_1^2}{(y_1^2-1)^2}\{(2\beta_{\varepsilon_1,\varepsilon_2}+2l-3)y_1^2 + (-2\beta_{\varepsilon_1,\varepsilon_2}-2l+5)\}\{f_{\varepsilon_1,\varepsilon_2}(y_1)+g_{\varepsilon_1,\varepsilon_2}(y_1)\} \tag{2.3'}$$

$$\left\{\left(y_1\frac{d}{dy_1}\right)^2 + 2\left(l-1+\frac{1}{y_1^2-1}\right)y_1\frac{d}{dy_1} - \frac{2\beta_{\varepsilon_1,\varepsilon_2}}{y_1^2-1} - \mu_1^2 + (l-1)^2\right\} g_{\varepsilon_1,\varepsilon_2}(y_1)$$
$$= \frac{1}{(y_1^2-1)^2}\{(-2\beta_{\varepsilon_1,\varepsilon_2}-2l+5)y_1^2 + (2\beta_{\varepsilon_1,\varepsilon_2}+2l-3)\}\{f_{\varepsilon_1,\varepsilon_2}(y_1)+g_{\varepsilon_1,\varepsilon_2}(y_1)\} \tag{2.4'}$$

To make the symbols simpler, we omit $\varepsilon_1, \varepsilon_2$ from $f_{\varepsilon_1,\varepsilon_2}(y_1), g_{\varepsilon_1,\varepsilon_2}(y_1)$ and $\beta_{\varepsilon_1,\varepsilon_2}$ temporarily.

Lemma 4.2. *Functions $f(y_1), g(y_1)$ satisfies the following differential equation respectively.*

$$\left[2(\beta+l-2)(y_1^2-1)\left(y_1\frac{d}{dy_1}\right)^2 - 2(\beta+l-2)\{(2\beta-1)y_1^2-2\beta-3\}y_1\frac{d}{dy_1}\right.$$
$$+\left\{2\beta^2+2\beta\left(l-1-\frac{y_1^2}{y_1^2-1}\right)+\frac{(l-1)^2+(l-3)^2-\mu_1^2-\mu_2^2}{2}\right\}\{(2\beta+2l-3)y_1^2-2\beta-2l+5\}$$
$$\left.-\left\{(2\beta+l-1)^2-\frac{2\beta y_1^2}{y_1^2-1}-\mu_1^2\right\}(y_1^2+1)\right]f(y_1)=0, \tag{4.1}$$

$$\left[-2(\beta+l-2)(y_1^2-1)\left(y_1\frac{d}{dy_1}\right)^2 + 2(\beta+l-2)\{(2\beta-3)y_1^2-2\beta-1\}y_1\frac{d}{dy_1}\right.$$
$$+\left\{2\beta^2+2\beta\left(l-2-\frac{y_1^2}{y_1^2-1}\right)+\frac{(l-1)^2+(l-3)^2-\mu_1^2-\mu_2^2}{2}\right\}\{(-2\beta-2l+5)y_1^2+2\beta+2l-3\}$$
$$\left.-\left\{-\frac{2\beta}{y_1^2-1}-\mu_1^2+(l-1)^2\right\}(y_1^2+1)\right]g(y_1)=0. \tag{4.2}$$

Proof. The first equation is obtained by $\{(2\beta+2l-3)y_1^2-2\beta-2l+5\}\times$ (2.1') $-(y_1^2+1)\times$ (2.3') and second one by $\{(-2\beta-2l+5)y_1^2+2\beta+2l-3\}\times$ (2.2') $-(y_1^2+1)\times$ (2.4'). □

Let $u=y_1^2$ and we define $\tilde{f}(u)$ and $\tilde{g}(u)$ by $\tilde{f}(u)=u^{l/2-2-\varepsilon_1\mu_1/2}f(y_1), \tilde{g}(u)=u^{l/2-1-\varepsilon_1\mu_1/2}g(y_1)$ respectively. We can derive differential equations satisfied by $\tilde{f}$ and $\tilde{g}$ from equations (4.1) and (4.2) as below.

$$\left[u(1-u)\frac{d^2}{du^2}+\left\{\frac{3}{2}+\frac{\varepsilon_1\mu_1}{2}-\frac{\varepsilon_2\mu_2}{2}-\left(\frac{7}{2}+\frac{\varepsilon_1\mu_1}{2}-\frac{\varepsilon_2\mu_2}{2}\right)u\right\}\frac{d}{du} -\frac{3}{4}(\varepsilon_1\mu_1-\varepsilon_2\mu_2+2)\right]\tilde{f}(u)=0 \tag{4.3}$$

$$\left[u(1-u)\frac{d^2}{du^2}+\left\{\frac{1}{2}+\frac{\mu_1}{2}-\frac{\mu_2}{2}-\left(\frac{5}{2}+\frac{\mu_1}{2}-\frac{\mu_2}{2}\right)u\right\}\frac{d}{du} -\frac{1}{2}(\mu_1-\mu_2+1)\right]\tilde{g}(u)=0 \tag{4.4}$$

Since $\tilde{f}(u)$ and $\tilde{g}(u)$ are analytic around $u=1$(i.e. $y_1=1$), we can express them by the Gaussian hypergeometric function ${}_2F_1$ as

$$\tilde{f}(u)=c_f{}_2F_1\left(\frac{3}{2},\frac{1}{2}\varepsilon_1\mu_1-\frac{1}{2}\varepsilon_2\mu_2+1;2;1-u\right),\quad \tilde{g}(u)=c_g{}_2F_1\left(\frac{1}{2},\frac{1}{2}\varepsilon_1\mu_1-\frac{1}{2}\varepsilon_2\mu_2+1;2;1-u\right),$$

where c_f, c_g are certain constants. From the formula

$$\begin{aligned}{}_2F_1(a,b;c;z)=&\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}{}_2F_1(a,b;a+b-c+1;1-z)\\&+\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b}{}_2F_1(c-a,c-b;c-a-b+1;1-z)\end{aligned}$$

and $f(y_1)=u^{-l/2+2+\varepsilon_1\mu_1/2}\tilde{f}(u)=y_1^{-l+4+\varepsilon_1\mu_1}\tilde{f}(y_1^2), g(y_1)=u^{-l/2+1+\varepsilon_1\mu_1/2}\tilde{g}(u)=y_1^{-l+2+\varepsilon_1\mu_1}\tilde{g}(y_1^2)$, we obtain the following theorem.

Theorem 4.3. *There exist constants c_f, c_g and*

$$\begin{aligned}f(y_1)=c_f&\left\{\frac{\Gamma\left(-\frac{1}{2}\varepsilon_1\mu_1+\frac{1}{2}\varepsilon_2\mu_2-\frac{1}{2}\right)}{\sqrt{\pi}\Gamma\left(-\frac{1}{2}\varepsilon_1\mu_1+\frac{1}{2}\varepsilon_2\mu_2+1\right)}y_1^{-l+4+\varepsilon_1\mu_1}{}_2F_1\left(\frac{3}{2},\frac{1}{2}\varepsilon_1\mu_1-\frac{1}{2}\varepsilon_2\mu_2+1;\frac{1}{2}\varepsilon_1\mu_1-\frac{1}{2}\varepsilon_2\mu_2+\frac{3}{2};y_1^2\right)\right.\\&\left.+\frac{2\Gamma\left(\frac{1}{2}\varepsilon_1\mu_1-\frac{1}{2}\varepsilon_2\mu_2+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{1}{2}\varepsilon_1\mu_1-\frac{1}{2}\varepsilon_2\mu_2+1\right)}y_1^{-l+3+\varepsilon_2\mu_2}{}_2F_1\left(\frac{1}{2},-\frac{1}{2}\varepsilon_1\mu_1+\frac{1}{2}\varepsilon_2\mu_2+1;-\frac{1}{2}\varepsilon_1\mu_1+\frac{1}{2}\varepsilon_2\mu_2+\frac{1}{2};y_1^2\right)\right\},\\ g(y_1)=c_g&\left\{\frac{2\Gamma\left(-\frac{1}{2}\varepsilon_1\mu_1+\frac{1}{2}\varepsilon_2\mu_2+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma\left(-\frac{1}{2}\varepsilon_1\mu_1+\frac{1}{2}\varepsilon_2\mu_2+1\right)}y_1^{-l+2+\varepsilon_1\mu_1}{}_2F_1\left(\frac{1}{2},\frac{1}{2}\varepsilon_1\mu_1-\frac{1}{2}\varepsilon_2\mu_2+1;\frac{1}{2}\varepsilon_1\mu_1-\frac{1}{2}\varepsilon_2\mu_2+\frac{1}{2};y_1^2\right)\right.\\&\left.+\frac{\Gamma\left(\frac{1}{2}\varepsilon_1\mu_1-\frac{1}{2}\varepsilon_2\mu_2-\frac{1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{1}{2}\varepsilon_1\mu_1-\frac{1}{2}\varepsilon_2\mu_2+1\right)}y_1^{-l+3+\varepsilon_2\mu_2}{}_2F_1\left(\frac{3}{2},-\frac{1}{2}\varepsilon_1\mu_1+\frac{1}{2}\varepsilon_2\mu_2+1;-\frac{1}{2}\varepsilon_1\mu_1+\frac{1}{2}\varepsilon_2\mu_2+\frac{3}{2};y_1^2\right)\right\}.\end{aligned}$$

Since $\Gamma(s+1)=s\Gamma(s)$ holds, we have

$$2\Gamma\left(\mp\frac{1}{2}\varepsilon_1\mu_1\pm\frac{1}{2}\varepsilon_2\mu_2+\frac{1}{2}\right)=(\mp\varepsilon_1\mu_2\pm\varepsilon_2\mu_2-1)\Gamma\left(\mp\frac{1}{2}\varepsilon_1\mu_1\pm\frac{1}{2}\varepsilon_2\mu_2-\frac{1}{2}\right).$$

From the definition of f and g, we have the following theorem.

Theorem 4.4. *For $\mu(\varepsilon)=\varepsilon_1\mu_1-\varepsilon_2\mu_2$, the asymtotic of the spherical function is*

$$\begin{pmatrix}\psi_{01}(y_1,y_2)\\ \psi_{10}(y_1,y_2)\end{pmatrix} = \sum_{\varepsilon_1,\varepsilon_2\in\{\pm1\}} \frac{y_2^{-l+2+\frac12\varepsilon_1\mu_1+\frac12\varepsilon_2\mu_2}}{\sqrt{\pi}}$$
$$\times\left\{\frac{\Gamma\left(-\frac12\mu(\varepsilon)-\frac12\right)}{\Gamma\left(-\frac12\mu(\varepsilon)+1\right)}y_1^{-l+2+\varepsilon_1\mu_1}\begin{pmatrix} c_{f_{\varepsilon_1,\varepsilon_2}}y_1^2{}_2F_1\left(\frac32,\frac12\mu(\varepsilon)+1;\frac12\mu(\varepsilon)+\frac32;y_1^2\right)\\ c_{g_{\varepsilon_1,\varepsilon_2}}(-\mu(\varepsilon)-1)_2F_1\left(\frac12,\frac12\mu(\varepsilon)+1;\frac12\mu(\varepsilon)+\frac12;y_1^2\right)\end{pmatrix}\right.$$
$$\left.+\frac{\Gamma\left(\frac12\mu(\varepsilon)-\frac12\right)}{\Gamma\left(\frac12\mu(\varepsilon)+1\right)}y_1^{-l+3+\varepsilon_2\mu_2}\begin{pmatrix} c_{f_{\varepsilon_1,\varepsilon_2}}(\mu(\varepsilon)-1)_2F_1\left(\frac12,-\frac12\mu(\varepsilon)+1;-\frac12\mu(\varepsilon)+\frac12;y_1^2\right)\\ c_{g_{\varepsilon_1,\varepsilon_2}}{}_2F_1\left(\frac32,-\frac12\mu(\varepsilon)+1;-\frac12\mu(\varepsilon)+\frac32;y_1^2\right)\end{pmatrix}+\begin{pmatrix}O(y_2)\\ O(y_2)\end{pmatrix}\right\}.$$

Subsequently, we will decide the ratio of constants c_f, c_d.

Theorem 4.5. $c_{f_{\varepsilon_1,\varepsilon_2}}=-c_{g_{\varepsilon_1,\varepsilon_2}}$

Proof. From Theorem 4.4, the ratio of coefficients of $y_1^{-l+3+\varepsilon_2\mu_2}$ is

$$c_{f_{\varepsilon_1,\varepsilon_2}}(\mu(\varepsilon)-1):c_{g_{\varepsilon_1,\varepsilon_2}}.$$

On the other hand, Proposition 3.1 says that this ratio should be

$$1-\mu(\varepsilon):1.$$

(Note that we assume l is odd and $j=1$.) Therefore, we obtain $c_{f_{\varepsilon_1,\varepsilon_2}}=-c_{g_{\varepsilon_1,\varepsilon_2}}$. □

So the spherical function is asymptotically

$$\begin{pmatrix}\psi_{01}(y_1,y_2)\\ \psi_{10}(y_1,y_2)\end{pmatrix} = \sum_{\varepsilon_1,\varepsilon_2\in\{\pm1\}} \frac{c_{f_{\varepsilon_1,\varepsilon_2}}}{\sqrt{\pi}}y_2^{-l+2+\frac12\varepsilon_1\mu_1+\frac12\varepsilon_2\mu_2}$$
$$\times\left\{\frac{\Gamma\left(-\frac12\mu(\varepsilon)-\frac12\right)}{\Gamma\left(-\frac12\mu(\varepsilon)+1\right)}y_1^{-l+2+\varepsilon_1\mu_1}\begin{pmatrix} y_1^2{}_2F_1\left(\frac32,\frac12\mu(\varepsilon)+1;\frac12\mu(\varepsilon)+\frac32;y_1^2\right)\\ (\mu(\varepsilon)+1)_2F_1\left(\frac12,\frac12\mu(\varepsilon)+1;\frac12\mu(\varepsilon)+\frac12;y_1^2\right)\end{pmatrix}\right.$$
$$\left.+\frac{\Gamma\left(\frac12\mu(\varepsilon)-\frac12\right)}{\Gamma\left(\frac12\mu(\varepsilon)+1\right)}y_1^{-l+3+\varepsilon_2\mu_2}\begin{pmatrix} (\mu(\varepsilon)-1)_2F_1\left(\frac12,-\frac12\mu(\varepsilon)+1;-\frac12\mu(\varepsilon)+\frac12;y_1^2\right)\\ -{}_2F_1\left(\frac32,-\frac12\mu(\varepsilon)+1;-\frac12\mu(\varepsilon)+\frac32;y_1^2\right)\end{pmatrix}+\begin{pmatrix}O(y_2)\\ O(y_2)\end{pmatrix}\right\}.$$

5 Restriction to $y_1=0$

As in Section 4, we consider a local solution

$$\begin{pmatrix}\psi_{01}(y_1,y_2)\\ \psi_{10}(y_1,y_2)\end{pmatrix}=\sum_{\varepsilon_1,\varepsilon_2\in\{\pm1\}}y_1^{\alpha^{(1)}_{\varepsilon_1,\varepsilon_2}}\left\{\begin{pmatrix}a^{(1)}_{\varepsilon_1,\varepsilon_2}(y_2)\\ b^{(1)}_{\varepsilon_1,\varepsilon_2}(y_2)\end{pmatrix}+\begin{pmatrix}O(y_1)\\ O(y_1)\end{pmatrix}\right\}+\sum_{\varepsilon_1,\varepsilon_2\in\{\pm1\}}y_1^{\alpha^{(2)}_{\varepsilon_1,\varepsilon_2}}\left\{\begin{pmatrix}a^{(2)}_{\varepsilon_1,\varepsilon_2}(y_2)\\ b^{(2)}_{\varepsilon_1,\varepsilon_2}(y_2)\end{pmatrix}+\begin{pmatrix}O(y_1)\\ O(y_1)\end{pmatrix}\right\}$$

of equations (2.1), (2.2), (2.3) and (2.4).

Inserting a local solution $y_1^{\alpha^{(1)}_{\varepsilon_1,\varepsilon_2}}\left\{\begin{pmatrix}a^{(1)}_{\varepsilon_1,\varepsilon_2}(y_2)\\ b^{(1)}_{\varepsilon_1,\varepsilon_2}(y_2)\end{pmatrix}+\begin{pmatrix}O(y_1)\\ O(y_1)\end{pmatrix}\right\}$ into (2.3) and then multiplying $y_1^{-\alpha^{(1)}_{\varepsilon_1,\varepsilon_2}}$ and taking the limit $y_1\to0$, we obtain the ordinary differential equation

$$\left[\left(y_2\frac{d}{dy_2}\right)^2+\left(-\alpha^{(1)}_{\varepsilon_1,\varepsilon_2}+k\frac{y_2+1}{y_2-1}+(1-k-l)\frac{y_2^2+1}{y_2^2-1}\right)y_2\frac{d}{dy_2}\right.$$
$$\left.+\frac{\alpha^2}{4}-\frac12\left(k\frac{y_2+1}{y_2-1}+(1-k-l)\frac{y_2^2+1}{y_2^2-1}\right)\alpha^{(1)}_{\varepsilon_1,\varepsilon_2}+\frac{(l-1)^2-\mu_j^2}{4}\right]a^{(1)}_{\varepsilon_1,\varepsilon_2}(y_2)=0.$$

We put $a_{\varepsilon_1,\varepsilon_2}(y_2) = y_2^{\beta_{\varepsilon_1,\varepsilon_2}} a^{(1)}_{\varepsilon_1,\varepsilon_2}(y_2)$. Then, the function $a_{\varepsilon_1,\varepsilon_2}$ satisfies the following differential equation.

$$\left[\frac{d^2}{dy_2^2} + \left(\frac{1+\varepsilon_j\mu_j}{y_2} + \frac{1+k-l}{y_2-1} + \frac{1-k-l}{y_2+1}\right)\frac{d}{dy_2} + \frac{(l-1-\varepsilon_j\mu_j)\{(l-1)y_2-k\}}{y_2(y_2-1)(y_2+1)}\right] a_{\varepsilon_1,\varepsilon_2}(y_2) = 0.$$

This is called the Heun's differential equation, whose Riemann scheme is

$$\mathcal{P}\left\{\begin{matrix} 0 & 1 & -1 & \infty \\ 0 & 0 & 0 & 1-l \\ -\varepsilon_j\mu_j & l-k & l+k & 1+\varepsilon_j\mu_j-l \end{matrix}\right\}$$

and the accessory parameter is $(l-1-\varepsilon_j\mu_j)k$.

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