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A CONSTRUCTION OF GENERATORS OF $Z(\mathfrak{so}_n)$

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Abstract. We construct generators of the center of the universal enveloping algebra of the complex orthogonal Lie algebra realized as the alternative matrices of size n. These elements are constructed in accordance with the Iwasawa decomposition of the real rank one indefinite orthogonal Lie algebra. We also discuss the Iwasawa decomposition of the Pfaffian.

1. Introduction and Main results

Let $\mathfrak{g} = \mathfrak{so}_n$ be the complex orthogonal Lie algebra realized as the alternative matrices of size n. Denote by $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$ and by $Z(\mathfrak{g})$ the center of $U(\mathfrak{g})$. It is well known for experts that a set of generators of $Z(\mathfrak{g})$ is given with determinant and Pfaffian ([3], [7], [4]).

The author is now interested in the structure of the space of Whittaker functions on real reductive Lie groups. In order to determine the composition series of the standard Whittaker $(\mathfrak{g}, K)$-modules ([9]), he needed to write the action of central elements on the space of Whittaker functions. For the case of indefinite unitary group $U(n-1,1)$, he succeeded in this task by using the determinant type generators of $Z(\mathfrak{gl}_n)$. He also tried the case of indefinite orthogonal group $SO_0(n-1,1)$, but, in his narrow idea, it seems that it is very difficult to write the differential equations characterizing Whittaker functions on $SO_0(n-1,1)$ by using the above determinant type generators of $Z(\mathfrak{so}_n)$. Under such backgrounds, he tried to write the action of $Z(\mathfrak{so}_n)$ in a different way. As a result, a new construction of the generators of $Z(\mathfrak{so}_n)$ is obtained. This is the main object of this paper.

In order to express the main result, we introduce some notation. In general, for a real Lie group L, the Lie algebra of it is denoted by $\mathfrak{l}_0$ and its complexification by $\mathfrak{l}$. This notation will be applied to groups denoted by other Roman letters in the same way without comment. The Kronecker delta is denoted by $\delta_{i,j}$. Let $E_{i,j} := (\delta_{i,k}\,\delta_{j,l})_{k,l=1}^n$ be the matrix units and define $A_{j,i} = E_{j,i} - E_{i,j}$. These are the standard generators of the space of alternative matrices. The diagonal $n \times n$ matrix $\sum_{i=1}^{n-1} E_{i,i} - E_{n,n}$ is denoted by $I_{n-1,1}$. The field of real (resp. complex) numbers is denoted by $\mathbb{R}$ (resp. $\mathbb{C}$). For a complex matrix $Z = (z_{i,j})_{i,j}$, define $\overline{Z} = (\overline{z_{i,j}})_{i,j}$, where $\overline{z}$ is the complex conjugate of a complex number z.

We realize the group $SO(n,\mathbb{C})$ as the subgroup of $SL(n,\mathbb{C})$ consisting of those elements which satisfy ${}^tg = g^{-1}$. Its Lie algebra $\mathfrak{g} = \mathfrak{so}_n$ is spanned by $A_{j,i}$, $1 \le i < j \le n$. The Lie group $G = SO_0(n-1,1)$ is the identity component of the real form of $SO(n,\mathbb{C})$ defined by the complex conjugation $g \mapsto I_{n-1,1}\overline{g}I_{n-1,1}$. Let $\theta(g) = I_{n-1,1}gI_{n-1,1}$ be a Cartan involution on $SO_0(n-1,1)$. Denote by K the maximal compact subgroup of G consisting of the fixed points of θ. Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$

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be the corresponding Cartan decomposition of $\mathfrak{g}_0 = \mathfrak{so}(n-1,1)$. More explicitly,

$$K \simeq SO(n-1),$$
$$\mathfrak{k}_0 = \mathbb{R}\text{-span}\{A_{j,i} \,|\, 1 \le i < j \le n-1\}, \quad \mathfrak{p}_0 = \mathbb{R}\text{-span}(\{\sqrt{-1}A_{n,i} \,|\, 1 \le i \le n-1\}.$$

As a maximal abelian subspace $\mathfrak{a}_0$ of $\mathfrak{p}_0$, we choose

$$\mathfrak{a}_0 = \mathbb{R}H, \qquad H := \sqrt{-1}A_{n,n-1}.$$

The subgroup $\exp \mathfrak{a}_0$ is denoted by A. Define a basis α of the complex dual space $\mathfrak{a}^*$ by $\alpha(H) = 1$. Then $\Sigma^+ = \{\alpha\}$ is a positive system of the root system $\Sigma(\mathfrak{g}_0, \mathfrak{a}_0)$. Dente by $\mathfrak{n}_0$ the nilpotent subalgebra corresponding to Σ^+. We choose

$$X_i := A_{n-1,i} + \sqrt{-1}A_{n,i}, \qquad i = 1, \ldots, n-2$$

as a basis of $\mathfrak{n}_0$. Define $N := \exp \mathfrak{n}_0$. Then we get Iwasawa decompositions $G = NAK$ and $\mathfrak{g}_0 = \mathfrak{n}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{k}_0$. As a consequence of Poincaré-Birkhoff-Witt theorem, $U(\mathfrak{g})$ is isomorphic to $U(\mathfrak{n}) \otimes U(\mathfrak{a}) \otimes U(\mathfrak{k})$.

As usual, denote by M the centralizer of A in K. This group is isomorphic to $SO(n-2)$. Define a Cartan subalgebra $\mathfrak{t}_\mathfrak{m}$ of $\mathfrak{m}$ by

$$\mathfrak{t}_\mathfrak{m} := \sum_{i=1}^{\lfloor (n-2)/2 \rfloor} \mathbb{C}T_i, \qquad T_i := \sqrt{-1}A_{n-2i,n-1-2i}.$$

We set $\mathfrak{t} = \mathfrak{t}_\mathfrak{m}$ if n is even, and $\mathfrak{t} = \mathfrak{t}_\mathfrak{m} + \mathbb{C}T_{\lfloor (n-1)/2 \rfloor}$ if n is odd. Here $T_{\lfloor (n-1)/2 \rfloor} := \sqrt{-1}A_{n-1,1}$. This $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{k}$. Define a basis $\{e_1, \ldots, e_{\lfloor (n-1)/2 \rfloor}\}$ of $\mathfrak{t}^*$ by $e_i(T_j) = \delta_{i,j}$. We regard $\{e_1, \ldots, e_{\lfloor (n-2)/2 \rfloor}\}$ as a basis of $(\mathfrak{t}_\mathfrak{m})^*$.

Choose a Borel subalgebra $\mathfrak{b}_\mathfrak{m} = \mathfrak{t}_\mathfrak{m} \oplus \mathfrak{u}$ of $\mathfrak{m}$. Set $\mathfrak{h} := \mathfrak{t}_\mathfrak{m} \oplus \mathfrak{a}$ and $\mathfrak{N} := \mathfrak{n} \oplus \mathfrak{u}$. The nilpotent subalgebras opposite to $\mathfrak{n}$, $\mathfrak{u}$ and $\mathfrak{N}$ are denoted by $\overline{\mathfrak{n}}$, $\overline{\mathfrak{u}}$ and $\overline{\mathfrak{N}}$, respectively. Denote by γ the Harish-Chandra map defined by the projection $U(\mathfrak{g}) \simeq U(\mathfrak{h}) \oplus (\mathfrak{N}U(\mathfrak{g}) + U(\mathfrak{g})\overline{\mathfrak{N}}) \to U(\mathfrak{h})$ composed by rho shift.

Theorem 1.1. *Suppose* $\mathfrak{g} = \mathfrak{so}_n$. *Let*

$$\Omega_{n-2} = \sum_{1 \le i < j \le n-2} (A_{j,i})^2$$

be a multiple of the Casimir element of $\mathfrak{so}_{n-2}$. *For a parameter* $u \in \mathbb{C}$, *define elements* $C_n(u) \in U(\mathfrak{g})$ *inductively by the following formulas:*

$$C_0(u) = C_1(u) = 1,$$

(1.1)

$$\begin{aligned} C_n(u) = &- \left\{ \left(H - \frac{n-2}{2} \right)^2 - u^2 + \sum_{i=1}^{n-2} X_i^2 \right\} C_{n-2}(u) \\ &+ \sum_{i=1}^{n-2} X_i \left(H - \frac{n-5}{2} \right) [A_{n-1,i}, C_{n-2}(u)] + 2 \sum_{i=1}^{n-2} X_i \, C_{n-2}(u) A_{n-1,i} \\ &- \frac{1}{2} \sum_{i=1}^{n-2} X_i \, [\Omega_{n-2}, [A_{n-1,i}, C_{n-2}(u)]] \\ &- \frac{1}{2} \sum_{i,j=1}^{n-2} X_i X_j \, [A_{n-1,i}, [A_{n-1,j}, C_{n-2}(u)]] \quad \textit{for} \quad n = 2, 3, \ldots \end{aligned}$$

Then $C_n(u)$ *is an element of* $Z(\mathfrak{g})$ *for any* $u \in \mathbb{C}$.

Moreover, the image $\gamma(C_n(u))$ of the Harish-Chandra map γ is

$$\gamma(C_n(u)) = (u^2 - H^2)(u^2 - T_1^2)\cdots(u^2 - T_{\lfloor (n-2)/2\rfloor}^2). \tag{1.2}$$

This paper is organized as follows. In §2, we explain the K-type shift operators and write it explicitly. §3 is the main part of this paper, in which Theorem 1.1 is proved. For the proof, we relate the element $C_n(u)$ and some composition of K-type shift operators. This relationship is explained in Lemma 3.2. The proof of this lemma is done in §4. For completeness, we discuss the Iwasawa decomposition of the Pfaffian in §5 and we relate it to a K-type shift operator.

2. Shift operators

In order to show that $C_n(u)$ is invariant under the adjoint action of K, we use the K-type shift operators on the space of smooth functions on G.

Let us review the definition of shift operators briefly. Denote by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the complexified Cartan decomposition. For a finite dimensional representation (τ, V) of K, define

$$C_\tau^\infty(K\backslash G) := \{f : G \xrightarrow{C^\infty} V \mid f(kg) = \tau(k)f(g),\ \ k \in K, g \in G\}.$$

This space is isomorphic to the intertwining space $\mathrm{Hom}_K(V^*, C^\infty(G)_{K\text{-finite}})$, where V^* is the contragredient representation of (τ, V).

Choose an orthonormal basis $\{W_i\}$ of $\mathfrak{p}_0$ with respect to an invariant bilinear form $\langle\ ,\ \rangle$ on $\mathfrak{g}$ which is negative (resp. positive) definite on $\mathfrak{k}_0$ (resp. $\mathfrak{p}_0$). Define a differential-difference operator ∇ by

$$\nabla\phi_\tau := \sum_i L(W_i)\phi_\tau \otimes W_i, \qquad \phi_\tau \in C_\tau^\infty(K\backslash G).$$

Here, $L(*)$ is the left regular representation. It is easy to see that ∇ does not depend on the choice of an orthonormal basis $\{W_i\}$ of $\mathfrak{p}_0$. As a consequence, the image of ∇ is an element of $C^\infty_{\tau\otimes\mathrm{Ad}}(K\backslash G)$;

$$\nabla\phi_\tau(kg) = (\tau \otimes \mathrm{Ad})(k)\,\nabla\phi_\tau(g), \quad k \in K, g \in G, \tag{2.1}$$

Here "Ad" is the adjoint representation of K on $\mathfrak{p}$.

Let $\lambda \in \mathfrak{t}^*$ be a dominant integral weight of K and let $(\tau_\lambda, V_\lambda)$ be the irreducible representation of K with highest weight λ. For notational convenience, set $e_{-\ell} = -e_\ell$ for $\ell = 1, 2, \ldots$ and $e_0 = 0$. In the case when $G = SO_0(n-1, 1)$ and λ is sufficiently regular, the irreducible decomposition of $V_\lambda \otimes \mathfrak{p}$ is

$$V_\lambda \otimes \mathfrak{p} \simeq \bigoplus_{1\le|\ell|\le\lfloor(n-1)/2\rfloor} V_{\lambda+e_\ell}\,(\oplus V_{\lambda+e_0} \text{ if } n \text{ is even})\,. \tag{2.2}$$

The projection operator from $V_\lambda \otimes \mathfrak{p}$ to $V_{\lambda+e_\ell}$ is denoted by pr_ℓ. Define a K-type shift operator P_ℓ by

$$P_\ell = \mathrm{pr}_\ell \circ \nabla : C^\infty_{\tau_\lambda}(K\backslash G) \to C^\infty_{\tau_{\lambda+e_\ell}}(K\backslash G).$$

The basis $\{\sqrt{-1}A_{n,i} \mid 1 \le i \le n-1\}$ of $\mathfrak{p}_0$ is orthonormal with respect to an appropriately normalized invariant bilinear form. Therefore, the operator ∇ is

$$\nabla\phi_\tau(g) = \sum_{i=1}^{n-1} L(\sqrt{-1}A_{n,i})\,\phi_\tau(g) \otimes \sqrt{-1}A_{n,i}.$$

For an irreducible representation τ of $K \simeq SO(n-1)$, the shift operators are explicitly calculated in [8]. To state the results, we introduce the Gelfand-Tsetlin basis of irreducible representations of $SO(n-1)$.

Definition 2.1. Let $\lambda = (\lambda_1, \dots, \lambda_{\lfloor (n-1)/2 \rfloor})$ be a dominant integral weight of $SO(n-1)$. A *(λ-)Gelfand-Tsetlin pattern* is a set of vectors $Q = (\boldsymbol{q}_1, \dots, \boldsymbol{q}_{n-2})$ such that

(1) $\boldsymbol{q}_i = (q_{i,1}, q_{i,2}, \dots, q_{i,\lfloor (i+1)/2 \rfloor})$.
(2) The numbers $q_{i,j}$ are all integers.
(3) $q_{2i+1,j} \geq q_{2i,j} \geq q_{2i+1,j+1}$, for any $j = 1, \dots, i-1$.
(4) $q_{2i+1,i} \geq q_{2i,i} \geq |q_{2i+1,i+1}|$.
(5) $q_{2i,j} \geq q_{2i-1,j} \geq q_{2i,j+1}$, for any $j = 1, \dots, i-1$.
(6) $q_{2i,i} \geq q_{2i-1,i} \geq -q_{2i,i}$.
(7) $q_{n-2,j} = \lambda_j$.

The set of all λ-Gelfand-Tsetlin patterns is denoted by $GT(\lambda)$.

Notation 2.2. For any set or number $*$ depending on $Q \in GT(\lambda)$, we denote it by $*(Q)$, if we need to specify Q. For example, $q_{i,j}(Q)$ is the $q_{i,j}$ part of $Q \in GT(\lambda)$.

Theorem 2.3 ([2])**.** *For a dominant integral weight λ of $SO(n-1)$, the set $GT(\lambda)$ of Gelfand-Tsetlin patterns is identified with a basis of $(\tau_\lambda, V_\lambda)$.*

The action of the elements in $\mathfrak{so}(n-1)$ is expressed as follows. For $j > 0$, let

$$l_{2i-1,j} := q_{2i-1,j} + i - j, \qquad l_{2i-1,-j} := -l_{2i-1,j},$$
$$l_{2i,j} := q_{2i,j} + i + 1 - j, \qquad l_{2i,-j} := -l_{2i,j} + 1,$$

and let $l_{2i,0} = 0$. Define $a_{p,q}(Q)$ by

$$a_{2i-1,j}(Q) = \operatorname{sgn} j \sqrt{-\frac{\prod_{1 \leq |k| \leq i-1}(l_{2i-1,j} + l_{2i-2,k}) \prod_{1 \leq |k| \leq i}(l_{2i-1,j} + l_{2i,k})}{4 \prod_{\substack{1 \leq |k| \leq i, \\ k \neq \pm j}} (l_{2i-1,j} + l_{2i-1,k})(l_{2i-1,j} + l_{2i-1,k} + 1)}},$$

for $j = \pm 1, \dots, \pm i$, and

$$a_{2i,j}(Q) = \epsilon_{2i,j}(Q) \sqrt{-\frac{\prod_{1 \leq |k| \leq i}(l_{2i,j} + l_{2i-1,k}) \prod_{1 \leq |k| \leq i+1}(l_{2i,j} + l_{2i+1,k})}{(4 l_{2i,j}^2 - 1) \prod_{\substack{0 \leq |k| \leq i \\ k \neq \pm j}} (l_{2i,j} + l_{2i,k})(l_{2i,j} - l_{2i,k})}},$$

for $j = 0, \pm 1, \dots, \pm i$, where $\epsilon_{2i,j}(Q)$ is $\operatorname{sgn} j$ if $j \neq 0$, and $\operatorname{sgn}(q_{2i-1,i}\, q_{2i+1,i+1})$ if $j = 0$. Let $\sigma_{a,b}$ be the shift operator, sending $\boldsymbol{q}_a$ to $\boldsymbol{q}_a + (0, \dots, \overset{|b|}{\operatorname{sgn}(b)}, 0, \dots, 0)$.

Under the above notation, the action of the Lie algebra is expressed as

$$\tau_\lambda(A_{2i+1,2i})Q = \sum_{1 \leq |j| \leq i} a_{2i-1,j}(Q)\, \sigma_{2i-1,j} Q,$$
$$\tau_\lambda(A_{2i+2,2i+1})Q = \sum_{0 \leq |j| \leq i} a_{2i,j}(Q)\, \sigma_{2i,j} Q.$$

Remark 2.4. This basis is compatible with the restriction to smaller orthogonal groups. More precisely, the restriction of τ_λ to $SO(n-2)$ is multiplicity free, and the vector $Q = (\boldsymbol{q}_1, \dots, \boldsymbol{q}_{n-2})$ is contained in the irreducible representation of $SO(n-2)$ whose highest weight is $\boldsymbol{q}_{n-2}$.

In order to write the projection operator pr_ℓ explicitly, we embed V_λ and $V_{\lambda+e_\ell}$ into an appropriately chosen irreducible representation of $SO(n)$. For example, when we consider the projection pr_1, we embed V_λ and $V_{\lambda+e_1}$ into the irreducible representation of $SO(n)$ whose highest weight is $\widetilde{\lambda} = (\lambda_1+1, \lambda_2, \dots)$. If we do so, then "$a_{n-2,\ell}(Q)\sigma_{n-2,\ell}Q$" in the following (for example in (2.3)) makes sense.

Just in the way as the proof of [5, Proposition 4.3], we get the following formulas.

Lemma 2.5. *For $Q \in GT(\lambda)$ and $\ell = 0, \pm 1, \dots, \pm\lfloor (n-1)/2 \rfloor$,*

$$\mathrm{pr}_\ell(Q \otimes \sqrt{-1}A_{n,n-1}) = a_{n-2,\ell}(Q)\sigma_{n-2,\ell}Q. \tag{2.3}$$

Remark 2.6. This lemma says that, if we embed V_λ and $V_{\lambda+e_\ell}$ into an irreducible representation $V_{\widetilde{\lambda}}$ of $SO(n)$, then we may identify $\mathrm{pr}_\ell(Q \otimes \sqrt{-1}A_{n,n-1})$ with the $V_{\lambda+e_\ell} \subset V_{\widetilde{\lambda}}|_{SO(n-1)}$ component of $\tau_{\widetilde{\lambda}}(A_{n,n-1})Q \in V_{\widetilde{\lambda}}$.

Let us write the operator P_ℓ explicitly. The action of $\mathfrak{k}$ on ϕ_{τ_λ} is given by

$$L(W)\phi_{\tau_\lambda}(a) = -\tau_\lambda(W)\phi_{\tau_\lambda}(a) \quad \text{for } W \in \mathfrak{k}. \tag{2.4}$$

Let ϖ_ℓ be operators from $GT(\lambda)$ to $GT(\lambda+e_\ell)$ defined by

$$\varpi_\ell Q := a_{n-2,\ell}(Q)\sigma_{n-2,\ell}Q, \qquad \ell = 0, \pm 1, \dots, \lfloor (n-1)/2 \rfloor. \tag{2.5}$$

Here, ϖ_0 is defined only when n is even. Then (2.3) is $\mathrm{pr}_\ell(Q \otimes \sqrt{-1}A_{n,n-1}) = \varpi_\ell Q$, and

$$\begin{aligned}
&\mathrm{pr}_\ell(Q \otimes \sqrt{-1}A_{n,i}) \\
&\quad = \mathrm{pr}_\ell(\tau_\lambda(A_{n-1,i})Q \otimes \sqrt{-1}A_{n,n-1}) - \mathrm{pr}_\ell\{(\tau_\lambda \otimes \mathrm{ad})(A_{n-1,i})(Q \otimes \sqrt{-1}A_{n,n-1})\} \\
&\quad = \varpi_\ell \tau_\lambda(A_{n-1,i})Q - \tau_{\lambda+e_\ell}(A_{n-1,i})\varpi_\ell Q.
\end{aligned}$$

For simplicity, we omit the symbols τ_λ and $\tau_{\lambda+e_\ell}$ hereafter. For example, we write $[\varpi_\ell, A_{n-1,i}]$ instead of $\varpi_\ell\tau_\lambda(A_{n-1,i}) - \tau_{\lambda+e_\ell}(A_{n-1,i})\varpi_\ell$, so the projection above is

$$\mathrm{pr}_\ell(Q \otimes \sqrt{-1}A_{n,i}) = [\varpi_\ell, A_{n-1,i}]Q.$$

In order to express $\phi_{\tau_\lambda}(g) \in C^\infty_{\tau_\lambda}(K\backslash G)$ explicitly, we use the Gelfand-Tsetlin basis. The coefficient function of Q is denoted by $c(Q;g)$. Namely, we write

$$\phi_{\tau_\lambda}(g) = \sum_{Q \in GT(\lambda)} c(Q;g)\, Q.$$

Lemma 2.7 ([8])**.** *For $\ell = 0, \pm 1, \dots, \pm\lfloor (n-1)/2 \rfloor$, the following formulas hold:*

$$\begin{aligned}
P_\ell \phi_{\tau_\lambda}(g) = \sum_{Q \in GT(\lambda)} \Big\{ &\big(L(H) + l_{n-2,\ell} - \lfloor \tfrac{n-2}{2} \rfloor\big)\, c(Q;g)\, \varpi_\ell Q \\
&+ \sum_{i=1}^{n-2} L(X_i)c(Q;g)[\varpi_\ell, A_{n-1,i}]Q \Big\}.
\end{aligned} \tag{2.6}$$

$$
\begin{aligned}
&P_{-\ell}P_\ell\phi_{\tau_\lambda}(g)\\
&=\sum_{Q\in GT(\lambda)}\Big\{(L(H)-l_{n-2,\ell}-\lfloor\frac{n-1}{2}\rfloor)(L(H)+l_{n-2,\ell}-\lfloor\frac{n-2}{2}\rfloor)\,c(Q;g)\,\varpi_{-\ell}\varpi_\ell Q\\
&\qquad+\sum_{i=1}^{n-2}L(X_i)(L(H)-l_{n-2,\ell}-\lfloor\frac{n-3}{2}\rfloor)c(Q;g)\varpi_{-\ell}[\varpi_\ell,A_{n-1,i}]Q\\
&\qquad+\sum_{i=1}^{n-2}L(X_i)(L(H)+l_{n-2,\ell}-\lfloor\frac{n-2}{2}\rfloor)c(Q;g)[\varpi_{-\ell},A_{n-1,i}]\varpi_\ell Q\\
&\qquad+\sum_{i,j=1}^{n-2}L(X_i)L(X_j)c(Q;g)[\varpi_{-\ell},A_{n-1,i}][\varpi_\ell,A_{n-1,j}]Q\Big\}.
\end{aligned}
\tag{2.7}
$$

Proof. When n is odd, (2.6) is obtained in [8, Proposition 5.1.4]. The calculation there is also valid if n is even. (2.7) can be obtained by composing two operators $P_{-\ell}$ and P_ℓ. For the proof of (2.7), we use the following identity: If $\ell\neq 0$,

$$
\begin{aligned}
&l_{n-2,-\ell}(\sigma_{n-2,\ell}Q)-\lfloor\frac{n-2}{2}\rfloor=l_{n-2,-\ell}(Q)-1-\lfloor\frac{n-2}{2}\rfloor\\
&\qquad=\begin{cases}-l_{n-2,\ell}(Q)-\frac{n-2}{2} & (n\text{ is even})\\ -l_{n-2,\ell}(Q)-1-\frac{n-3}{2} & (n\text{ is odd})\end{cases}=-l_{n-2,\ell}(Q)-\lfloor\frac{n-1}{2}\rfloor.
\end{aligned}
$$

The conclusion of this equality is valid when n is even and $\ell=0$. □

3. Proof of the main theorem

In this section, we prove Theorem 1.1. For the proof, we introduce notation.

Definition 3.1. Define

$$
u_\ell=\begin{cases}\lambda_\ell+n/2-\ell, & \text{if }\ell>0,\\ 1-(\lambda_{|\ell|}+n/2-|\ell|), & \text{if }\ell<0,\\ 0, & \text{when } n \text{ is even and } \ell=0.\end{cases}
$$

In other words, $u_\ell=l_{n-2,\ell}+1/2$ when n is odd and $u_\ell=l_{n-2,\ell}$ when n is even.

Next lemma is a key to show our main theorem.

Lemma 3.2. *Assume that $C_{n-2}(u)\in Z(\mathfrak{so}_{n-2})$ and that it satisfies* (1.2). *For $\ell=0,\pm1,\dots,\pm\lfloor(n-1)/2\rfloor$ and for every irreducible representation (τ_λ,V_λ) of $K\simeq SO(n-1)$, there exists a non-zero constant $d_{\lambda,\ell}$ determined by ℓ and the highest weight λ such that*

$$
P_{-\ell}P_\ell\phi_{\tau_\lambda}(g)=d_{\lambda,\ell}\,L(C_n(u_\ell))\phi_{\tau_\lambda}(g),\qquad \phi_{\tau_\lambda}\in C^\infty_{\tau_\lambda}(K\backslash G).
\tag{3.1}
$$

This lemma is proved by direct calculation. Since it is elementary but messy, we prove it in the next section. Here we complete the proof of Theorem 1.1.

Proposition 3.3. *Assume that $C_{n-2}(u)\in Z(\mathfrak{so}_{n-2})$ and that it satisfies* (1.2). *For any $k\in K\simeq SO(n-1)$, $\mathrm{Ad}(k)C_n(u)=C_n(u)$.*

Proof. Let $X(\mu,\nu)$, $\mu\in\widehat{M}$, $\nu\in\mathfrak{a}^*$ be the Harish-Chandra module of the principal series representation induced from $\mu\boxtimes e^{\nu+\rho}$. If $G=SO_0(n-1,1)$, then $X(\mu,\nu)$ is K-multiplicity free because of the Frobenius reciprocity $\mathrm{Hom}_K(V_\lambda^*,X(\mu,\nu))\simeq\mathrm{Hom}_M(V_\lambda^*,\mu)$ and the multiplicity freeness of $V_\lambda|_{SO(n-2)}$.

Suppose that a function $\phi_{\tau_\lambda} \in C^\infty_{\tau_\lambda}(K\backslash G)$ corresponds to an intertwining operator in $\mathrm{Hom}_K(V^*_{\tau_\lambda}, X(\mu,\nu))$. Since $X(\mu,\nu)$ is K-multiplicity free, the function $P_{-\ell}P_\ell\phi_{\tau_\lambda}$ is a constant multiple of ϕ_{τ_λ}. It follows that

$$L(k)P_{-\ell}P_\ell L(k^{-1})\phi_{\tau_\lambda} = P_{-\ell}P_\ell\phi_{\tau_\lambda}.$$

By Lemma 3.2, $P_{-\ell}P_\ell\phi_{\tau_\lambda} = d_{\lambda,\ell}\, L(C_n(u_\ell))\phi_{\tau_\lambda}$. Therefore,

$$\begin{aligned} L(\mathrm{Ad}(k)\, C_n(u_\ell))\, \phi_{\tau_\lambda} &= L(k)\, L(C_n(u_\ell))\, L(k^{-1})\, \phi_{\tau_\lambda} \\ &= (d_{\lambda,\ell})^{-1} L(k)\, P_{-\ell}P_\ell\, L(k^{-1})\, \phi_{\tau_\lambda} \\ &= (d_{\lambda,\ell})^{-1} P_{-\ell}P_\ell\, \phi_{\tau_\lambda} \\ &= L(C_n(u_\ell))\, \phi_{\tau_\lambda}. \end{aligned}$$

By the definition of $l_{n-2,\ell}$ and u_ℓ, the u_ℓ's satisfy $u_1 > u_2 > \cdots > u_{\lfloor (n-1)/2\rfloor} > 0$ if n is odd and $u_1 > u_2 > \cdots > u_{\lfloor (n-1)/2\rfloor} > u_0 = 0$ if n is even. It follows that $L(\mathrm{Ad}(k)C_n(u))\phi_{\tau_\lambda}$ and $L(C_n(u))\phi_{\tau_\lambda}$ are identical for $\lfloor n/2\rfloor$ points $u^2 = u_1^2, \ldots, u^2_{\lfloor n/2\rfloor}$. By the definition (1.1) of $C_n(u)$, it is a *monic* polynomial in u^2 of degree $\lfloor n/2\rfloor$. Therefore $L(\mathrm{Ad}(k)C_n(u))\phi_{\tau_\lambda} = L(C_n(u))\phi_{\tau_\lambda}$ for any $u \in \mathbb{C}$ and for every K-type $(\tau^*_\lambda, V^*_{\tau_\lambda})$ in $X(\mu,\nu)$. It follows that $\mathrm{Ad}(k)C_n(u) - C_n(u)$ annihilates every principal series. By the subrepresentation theorem, it annihilates every irreducible Harish-Chandra module. According to the Plancherel formula for G, the element $\mathrm{Ad}(k)C_n(u) - C_n(u)$ acts trivially on the space $C^\infty_c(G)(\subset L^2(G))$ of smooth functions of compact support, so it is the zero element in $U(\mathfrak{g})$. □

Proof of Theorem 1.1. We shall show $C_n(u) \in Z(\mathfrak{so}_n)$ for any $u \in \mathbb{C}$ and it satisfies (1.2) by induction on n. If $n = 0, 1$, then this is trivial since $C_0(u) = C_1(u) = 1$ by definition.

Assume that $C_{n-2}(u) \in Z(\mathfrak{so}_{n-2})$ and that it satisfies (1.2). Consider the (generalized) Harish-Chandra maps

$$\gamma_{\mathfrak{n}} : U(\mathfrak{g}) \simeq U(\mathfrak{m}\oplus\mathfrak{a}) \oplus (\mathfrak{n}U(\mathfrak{g}) + U(\mathfrak{g})\bar{\mathfrak{n}}) \to U(\mathfrak{m}\oplus\mathfrak{a}) \overset{\text{rho shift}}{\longrightarrow} U(\mathfrak{m}\oplus\mathfrak{a}), \tag{3.2}$$

$$\gamma_{\mathfrak{u}} : U(\mathfrak{m}\oplus\mathfrak{a}) \simeq U(\mathfrak{h}) \oplus (\mathfrak{u}U(\mathfrak{m}\oplus\mathfrak{a}) + U(\mathfrak{m}\oplus\mathfrak{a})\bar{\mathfrak{u}}) \to U(\mathfrak{h}) \overset{\text{rho shift}}{\longrightarrow} U(\mathfrak{h}). \tag{3.3}$$

For notation, see §1. Then by (1.1),

$$\gamma_{\mathfrak{n}}(C_n(u)) = (u^2 - H^2)C_{n-2}(u). \tag{3.4}$$

By the hypothesis of induction, $C_{n-2}(u)$ is a central element of $U(\mathfrak{so}_{n-2})$ and $\gamma_{\mathfrak{u}}(C_{n-2}(u))$ is $(u^2 - T_1^2)\cdots(u^2 - T^2_{\lfloor (n-2)/2\rfloor})$. Since the Harish-Chandra map γ given in §1 is the composition $\gamma = \gamma_{\mathfrak{u}} \circ \gamma_{\mathfrak{n}}$, (1.2) is shown.

The image $\gamma(C_n(u))$ given in (1.2) is invariant under the action of the Weyl group of $\mathfrak{g}$. Therefore, there exists an element $z \in Z(\mathfrak{g})$ such that $\gamma(z) = \gamma(C_n(u))$. By (3.4) and the hypothesis of induction, $\gamma_{\mathfrak{n}}(C_n(u) - z)$ is an element of $Z(\mathfrak{m}\oplus\mathfrak{a})$. Since the restriction of $\gamma_{\mathfrak{u}}$ to $Z(\mathfrak{m}\oplus\mathfrak{a})$ is injective, $\gamma_{\mathfrak{n}}(C_n(u) - z) = 0$.

Consider the projection

$$p : U(\mathfrak{g}) \simeq U(\mathfrak{n}) \otimes U(\mathfrak{a}) \otimes U(\mathfrak{k}) \simeq (U(\mathfrak{a}) \otimes U(\mathfrak{k})) \oplus \mathfrak{n}U(\mathfrak{g}) \to U(\mathfrak{a}) \otimes U(\mathfrak{k}).$$

Recall the definition (1.1) of $C_n(u)$. The right hand of (3.4) is the same as $C_n(u) \mod \mathfrak{n}U(\mathfrak{g})$ composed by rho shift. It follows that $\gamma_{\mathfrak{n}}(C_n(u)) = (\text{rho shift}) \circ p(C_n(u))$. By a result of Lepowsky's ([6]), the restriction of p to the subalgebra

$U(\mathfrak{g})^{\mathfrak{k}}$ of $\mathfrak{k}$-invariants in $U(\mathfrak{g})$ is injective. Proposition 3.3 says that $C_n(u)$ is an element of $U(\mathfrak{g})^{\mathfrak{k}}$. It follows from $\gamma_{\mathfrak{n}}(C_n(u)-z)=0$ that $C_n(u)-z\in U(\mathfrak{g})^{\mathfrak{k}}\cap \operatorname{Ker} p=\{0\}$. This shows $C_n(u)=z\in Z(\mathfrak{g})$. □

4. A proof of Lemma 3.2

We shall prove Lemma 3.2, so we assume that $C_{n-2}(u)\in Z(\mathfrak{so}_{n-2})$ and that it satisfies (1.2) in this section. We first write $L(C_n(u))\phi_{\tau_\lambda}$ explicitly.

For $Y_1,\dots,Y_p\in\mathfrak{k}$, the action of $Y_1\cdots Y_p\in U(\mathfrak{k})$ on ϕ_{τ_λ} is given by

$$L(Y_1\cdots Y_p)\phi_{\tau_\lambda}(g)=(Y_1\cdots Y_p)^{\mathrm{opp}}\,\phi_{\tau_\lambda}(g),\quad (Y_1\cdots Y_p)^{\mathrm{opp}}:=(-Y_p)\cdots(-Y_1).$$

Note that, as we have noticed, the symbol τ_λ is omitted. The map "opp" satisfies

$$[A,B]^{\mathrm{opp}}=-[A^{\mathrm{opp}},B^{\mathrm{opp}}].$$

By the assumption, $C_{n-2}(u)$ and $C_{n-2}(u)^{\mathrm{opp}}$ are elements of $Z(\mathfrak{so}_{n-2})$. Since the shifted Harish-Chandra map $\gamma_{\mathfrak{u}}$ does not depend on the choice of $\mathfrak{u}$,

$$\begin{aligned}\gamma_{\mathfrak{u}}(C_{n-2}(u))&=\prod_{p=1}^{\lfloor (n-2)/2\rfloor}(u^2-T_p^2)=\gamma_{\mathfrak{u}}(C_{n-2}(u))^{\mathrm{opp}}\\&=\gamma_{\overline{\mathfrak{u}}}(C_{n-2}(u)^{\mathrm{opp}})=\gamma_{\mathfrak{u}}(C_{n-2}(u)^{\mathrm{opp}}).\end{aligned}$$

Therefore, $C_{n-2}(u)^{\mathrm{opp}}=C_{n-2}(u)$. Analogously, we have $\Omega_{n-2}^{\mathrm{opp}}=\Omega_{n-2}$. By the definition (1.1) of $C_n(u)$, we have the following lemma:

Lemma 4.1. *Let $\phi_{\tau_\lambda}(g)=\sum_{Q\in GT(\lambda)}c(Q;g)\,Q$ be an element of $C^\infty_{\tau_\lambda}(K\backslash G)$. The action of $C_n(u)$ on it is given by*

$$\begin{aligned}&L(C_n(u))\,\phi_{\tau_\lambda}(g)\\&=\sum_{Q\in GT(\lambda)}\Bigg[\left\{-\left(L(H)-\frac{n-2}{2}\right)^2+u^2-\sum_{i=1}^{n-2}L(X_i)^2\right\}c(Q;g)\,C_{n-2}(u)\,Q\\&\qquad+\sum_{i=1}^{n-2}L(X_i)\left(L(H)-\frac{n-5}{2}\right)c(Q;g)\,[A_{n-1,i},C_{n-2}(u)]\,Q\\&\qquad-2\sum_{i=1}^{n-2}L(X_i)c(Q;g)\,A_{n-1,i}C_{n-2}(u)\,Q\\&\qquad+\frac12\sum_{i=1}^{n-2}L(X_i)c(Q;g)\,[\Omega_{n-2},[A_{n-1,i},C_{n-2}(u)]]\,Q\\&\qquad-\frac12\sum_{i,j=1}^{n-2}L(X_i)L(X_j)c(Q;g)\,[A_{n-1,i},[A_{n-1,j},C_{n-2}(u)]]\,Q\Bigg].\end{aligned}\tag{4.1}$$

In the proof of lemmas below, we treat the case when $\ell\neq 0$. The difference between these cases and the case when $\ell=0$ is only one point: $a_{n-2,-\ell}(\sigma_{n-2,\ell}Q)=-a_{n-2,\ell}(Q)$ if $\ell\neq0$, but $a_{n-2,-0}(\sigma_{n-2,0}Q)=a_{n-2,0}(Q)$ if n is even and $\ell=0$. If you modify this point, you get the proof of the latter case.

Next, we see the relationship between $\varpi_{-\ell}\varpi_\ell$ and $C(u_\ell)$.

Lemma 4.2. *There exists a non-zero constant $d_{\lambda,\ell}$ which does not depend on the $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_{n-3}$ parts of $Q = (\boldsymbol{q}_1, \ldots, \boldsymbol{q}_{n-3}, \boldsymbol{q}_{n-2}) \in GT(\lambda)$ such that*

$$\varpi_{-\ell}\varpi_\ell Q = -d_{\lambda,\ell}\, C_{n-2}(u_\ell)\, Q. \tag{4.2}$$

Proof. By the definition of the Gelfand-Tsetlin basis and ϖ_ℓ, the action of $\varpi_{-\ell}\varpi_\ell$ on $Q \in GT(\lambda)$ is given by

$$\begin{aligned} \varpi_{-\ell}\varpi_\ell Q &= a_{n-2,-\ell}(\sigma_{n-2,\ell}Q)\sigma_{n-2,-\ell}\, a_{n-2,\ell}(Q)\,\sigma_{n-2,\ell}Q = -a_{n-2,\ell}(Q)^2\, Q \\ &= -d_{\lambda,\ell} \prod_{1\le |i| \le \lfloor (n-2)/2 \rfloor} (l_{n-2,\ell} + l_{n-3,i})\, Q, \end{aligned} \tag{4.3}$$

where $d_{\lambda,\ell}$ is a constant which does not depend on the $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_{n-3}$ parts of $Q = (\boldsymbol{q}_1, \ldots, \boldsymbol{q}_{n-3}, \boldsymbol{q}_{n-2}) \in GT(\lambda)$.

On the other hand, $C_{n-2}(u)$ acts on Q by a scalar, since $Q = (\boldsymbol{q}_1, \ldots, \boldsymbol{q}_{n-3}, \boldsymbol{q}_{n-2})$ is contained in the irreducible representation of $SO(n-2)$ with highest weight $\boldsymbol{q}_{n-3}$ (cf. Remark 2.4) and $C_{n-2}(u)$ is an element of $Z(\mathfrak{so}_{n-2})$.

Let us calculate this scalar. By the assumption,

$$\gamma_{\mathfrak{u}}(C_{n-2}(u)) = \prod_{i=1}^{\lfloor (n-2)/2 \rfloor} (u^2 - T_i^2).$$

The "rho" of $\mathfrak{so}_{n-2}$ is $\rho_{\mathfrak{so}_{n-2}} := \frac{1}{2}\sum_{i=1}^{\lfloor (n-2)/2 \rfloor}(n-2-2i)e_i$. It follows that

$$C_{n-2}(u)\, Q = \prod_{i=1}^{\lfloor (n-2)/2 \rfloor} \left\{ u^2 - \left(q_{n-3,i} + \frac{n-2-2i}{2} \right)^2 \right\} Q. \tag{4.4}$$

When $n = 2m+1$ is odd and $i > 0$, then $q_{n-3,i} + (n-2-2i)/2 = l_{n-3,i} - 1/2$ and $l_{n-3,-i} = 1 - l_{n-3,i}$. Therefore

$$\begin{aligned} \prod_{1\le |i| \le \lfloor (n-2)/2 \rfloor} (l_{n-2,\ell} + l_{n-3,i}) &= \prod_{i=1}^{m-1} (l_{n-2,\ell} + l_{n-3,i})(l_{n-2,\ell} - l_{n-3,i} + 1) \\ &= \prod_{i=1}^{m-1} \left\{ \left(l_{n-2,\ell} + \frac{1}{2} \right)^2 - \left(q_{n-3,i} + \frac{n-2-2i}{2} \right)^2 \right\}. \end{aligned} \tag{4.5}$$

When $n = 2m$ is even and $i > 0$, then $q_{n-3,i} + (n-2-2i)/2 = l_{n-3,i}$ and $l_{n-3,-i} = -l_{n-3,i}$. Therefore

$$\begin{aligned} \prod_{1\le |i| \le \lfloor (n-2)/2 \rfloor} (l_{n-2,\ell} + l_{n-3,i}) &= \prod_{i=1}^{m-1} (l_{n-2,\ell} + l_{n-3,i})(l_{n-2,\ell} - l_{n-3,i}) \\ &= \prod_{i=1}^{m-1} \left\{ (l_{n-2,\ell})^2 - \left(q_{n-3,i} + \frac{n-2-2i}{2} \right)^2 \right\}. \end{aligned} \tag{4.6}$$

Then (4.2) follows from (4.3), (4.4), (4.5), (4.6) and Definition 3.1. If $d_{\lambda,\ell}$ is not zero, then the lemma is proved.

Consider the case when $d_{\lambda,\ell}$ in (4.3) is zero. By the definition of $a_{n-2,\ell}(Q)$ and (4.3), $d_{\lambda,\ell}$ is zero if and only if one of the following conditions is satisfied:

(1) $1 < \ell \le \lfloor n/2 \rfloor$ and $\lambda_\ell = \lambda_{\ell-1}$.
(2) $-\lfloor n/2 \rfloor + 1 \le \ell \le -1$ and $\lambda_{|\ell|} = \lambda_{|\ell|+1}$.
(3) $n = 2m+1$ is odd, $\lambda_{m-1} = -\lambda_m$ and $\ell = -m+1, -m$.

In the case (1), $q_{n-3,\ell-1} = \lambda_\ell$ since $\lambda_{\ell-1} \geq q_{n-3,\ell-1} \geq \lambda_\ell$. By Definition 3.1,

$$q_{n-3,\ell-1} + \{n-2-2(\ell-1)\}/2 = \lambda_\ell + n/2 - \ell = u_\ell.$$

In the case (2), $q_{n-3,|\ell|} = \lambda_{|\ell|}$ since $\lambda_{|\ell|} \geq q_{n-3,|\ell|} \geq \lambda_{|\ell|+1}$. By Definition 3.1,

$$q_{n-3,|\ell|} + (n-2-2|\ell|)/2 = (\lambda_{|\ell|} + n/2 - |\ell|) - 1 = -u_\ell$$

In the case (3), $\lambda_{m-1} = q_{2m-2,m-1} = -\lambda_m$ since $\lambda_{m-1} \geq q_{2m-2,m-1} \geq -\lambda_m$. The numbers u_{-m+1}, u_{-m} and $q_{n-3,m-1} + \{n-2-2(m-1)\}/2$ are

$$\begin{aligned} &u_{-m+1} = 1 - \{\lambda_{m-1} + (2m+1)/2 - m + 1\} = -(\lambda_{m-1} + 1/2), \\ &u_{-m} = 1 - \{\lambda_m + (2m+1)/2 - m\} = -\lambda_m + 1/2 \quad \text{and} \\ &q_{n-3,m-1} + \frac{2m+1-2-2(m-1)}{2} = q_{2m-2,m-1} + 1/2 = -u_{-m+1} = u_{-m}. \end{aligned}$$

In every case, we get $C_{n-2}(u_\ell)Q = 0$ by (4.4).

On the other hand, if $d_{\lambda,\ell}$ in (4.3) is zero, then $\varpi_{-\ell}\varpi_\ell Q = 0$. Therefore, if we relpace $d_{\lambda,\ell}$ by a non-zero constant, then (4.2) holds. □

In the reminder of this section, we show that (2.7) and $d_{\lambda,\ell}\times$(4.1) are identical when $u = u_\ell$. We first show that the terms which do not contain $L(X_i)$ in these are identical.

Lemma 4.3. *For* $Q \in GT(\lambda)$,

$$\left(L(H) - l_{n-2,\ell} - \lfloor\frac{n-1}{2}\rfloor\right)\left(L(H) + l_{n-2,\ell} - \lfloor\frac{n-2}{2}\rfloor\right)\varpi_{-\ell}\varpi_\ell\, Q = -d_{\lambda,\ell}\left\{\left(L(H) - \frac{n-2}{2}\right)^2 - u_\ell^2\right\} C_{n-2}(u_\ell)\, Q. \tag{4.7}$$

Proof. By Definition 3.1 and

$$\lfloor\frac{n-1}{2}\rfloor = \begin{cases} \frac{n-1}{2} & \text{if } n \text{ is odd} \\ \frac{n-2}{2} & \text{if } n \text{ is even,} \end{cases} \qquad \lfloor\frac{n-2}{2}\rfloor = \begin{cases} \frac{n-3}{2} & \text{if } n \text{ is odd} \\ \frac{n-2}{2} & \text{if } n \text{ is even,} \end{cases}$$

we have

$$-l_{n-2,\ell} - \lfloor\frac{n-1}{2}\rfloor = -\frac{n-2}{2} - u_\ell, \quad l_{n-2,\ell} - \lfloor\frac{n-2}{2}\rfloor = -\frac{n-2}{2} + u_\ell. \tag{4.8}$$

Therefore, this lemma follows from (4.2). □

Next, we check the terms containing $L(X_i)\,L(X_j)$, $1 \leq i \leq j \leq n-2$. We know X_i and X_j commute. Moreover,

$$\begin{aligned} [A_{n-1,i}, [A_{n-1,j}, C_{n-2}(u)]] &= [A_{j,i}, C_{n-2}(u)] + [A_{n-1,j}, [A_{n-1,i}, C_{n-2}(u)]] \\ &= [A_{n-1,j}, [A_{n-1,i}, C_{n-2}(u)]], \end{aligned}$$

since $A_{j,i} \in \mathfrak{so}_{n-2}$ and $C_{n-2}(u) \in Z(\mathfrak{so}_{n-2})$ by the assumption. Therefore, what we should show is the following lemma:

Lemma 4.4. *For* $Q \in GT(\lambda)$,

$$\begin{aligned} &[\varpi_{-\ell}, A_{n-1,i}][\varpi_\ell, A_{n-1,j}]Q + [\varpi_{-\ell}, A_{n-1,j}][\varpi_\ell, A_{n-1,i}]Q \\ &\qquad = -d_{\lambda,\ell}\{2\delta_{i,j}\, C_{n-2}(u) + [A_{n-1,i}, [A_{n-1,j}, C_{n-2}(u)]]\}Q. \end{aligned} \tag{4.9}$$

Proof. By (4.2), we have

$$\begin{aligned}&[A_{n-1,i},\,[A_{n-1,j},\,-d_{\lambda,\ell}\,C_{n-2}(u_\ell)]]Q\\&=[A_{n-1,i},\,[A_{n-1,j},\,\varpi_{-\ell}\varpi_\ell]]Q\\&=[A_{n-1,i},\,[A_{n-1,j},\,\varpi_{-\ell}]]\varpi_\ell Q+[A_{n-1,j},\,\varpi_{-\ell}][A_{n-1,i},\,\varpi_\ell]Q\\&\qquad+[A_{n-1,i},\,\varpi_{-\ell}][A_{n-1,j},\,\varpi_\ell]Q+\varpi_{-\ell}[A_{n-1,i},\,[A_{n-1,j},\,\varpi_\ell]]Q.\end{aligned}$$

As we remarked in Remark 2.6, $\varpi_\ell Q$ is identified with the $V_{\lambda+e_\ell}$ component of $\tau_{\widetilde{\lambda}}(A_{n,n-1})Q$. Therefore, $[A_{n-1,i},\,[A_{n-1,j},\,\varpi_{\pm\ell}]]Q$ is identified with the $V_{\lambda+e_{\pm\ell}}$ component of

$$\begin{aligned}&[A_{n-1,i},\,[A_{n-1,j},\,A_{n,n-1}]]Q=-[A_{n-1,i},A_{n,j}]Q=-\delta_{i,j}\,A_{n,n-1}Q\\&=-\delta_{i,j}\sum_k a_{n-2,k}(Q)\sigma_{n-2,k}Q=-\delta_{i,j}\sum_k\varpi_kQ,\end{aligned}$$

namely, identified with $-\delta_{i,j}\,\varpi_{\pm\ell}$. Then we get

$$\begin{aligned}&[A_{n-1,i},\,[A_{n-1,j},\,\varpi_{-\ell}]]\varpi_\ell Q=\varpi_{-\ell}[A_{n-1,i},\,[A_{n-1,j},\,\varpi_\ell]]Q\\&=-\delta_{i,j}\,\varpi_{-\ell}\varpi_\ell Q=\delta_{i,j}\,d_{\lambda,\ell}C_{n-2}(u_\ell)Q.\end{aligned}$$

It follows that

$$\begin{aligned}&[A_{n-1,i},\,\varpi_{-\ell}][A_{n-1,j},\,\varpi_\ell]Q+[A_{n-1,j},\,\varpi_{-\ell}][A_{n-1,i},\,\varpi_\ell]Q\\&=-2\delta_{i,j}\,d_{\lambda,\ell}C_{n-2}(u_\ell)Q-d_{\lambda,\ell}\,[A_{n-1,n-2},[A_{n-1,n-2},C_{n-2}(u_\ell)]]Q,\end{aligned}$$

so (4.9) holds. □

Finally, we check the terms containing $L(X_i)$, $i=1,\dots,n-2$.

Consider the actions of $M\simeq SO(n-2)$ on $\mathfrak{n}_0=\mathbb{R}\text{-span}\{X_i\,|\,i=1,2,\dots,n-2\}$ and on $\mathbb{R}\text{-span}\{A_{n-1,i}\,|\,i=1,2,\dots,n-2\}$. We can find elements $m_i\in M$ $(1\le i\le n-2)$ such that

$$\mathrm{Ad}(m_i)X_{n-2}=X_i\qquad\text{and}\qquad\mathrm{Ad}(m_i)A_{n-1,n-2}=A_{n-1,i}.$$

If Y_1,Y_2 are M-invariant elements in $U(\mathfrak{g})$, then

$$\mathrm{Ad}(m_i)(X_{n-2}Y_1A_{n-1,n-2}Y_2)=X_iY_1A_{n-1,i}Y_2.$$

By the definition (2.5) of ϖ_ℓ, its action commutes with $m\in M$. Morover, Ω_{n-2} is M-invariant; so is $C_{n-2}(u)$ by the assumption. It follows that, if we can show the terms containing $L(X_{n-2})$ in (2.7) and $d_{\lambda,\ell}\times$(4.1) are identical for $u=u_\ell$, then the terms containing $L(X_i)$, $i=1,\dots,n-3$, are also identical. Therefore, the next lemma will complete the proof of Lemma 3.2.

Lemma 4.5. *For $Q\in GT(\lambda)$,*

$$\begin{aligned}&\left(L(H)-l_{n-2,\ell}-\lfloor\frac{n-3}{2}\rfloor\right)\varpi_{-\ell}[\varpi_\ell,A_{n-1,n-2}]Q\\&\quad+\left(L(H)+l_{n-2,\ell}-\lfloor\frac{n-2}{2}\rfloor\right)[\varpi_{-\ell},A_{n-1,n-2}]\varpi_\ell Q\\&=d_{\lambda,\ell}\Bigg\{\left(L(H)-\frac{n-5}{2}\right)[A_{n-1,n-2},C_{n-2}(u_\ell)]\\&\qquad-2A_{n-1,n-2}C_{n-2}(u_\ell)+\frac{1}{2}[\Omega_{n-2},[A_{n-1,n-2},C_{n-2}(u_\ell)]]\Bigg\}Q.\end{aligned}\tag{4.10}$$

Proof. By (4.2) and (4.8), the difference of both sides of (4.10) is

$$\begin{aligned}
&\left(L(H)-u_\ell-\frac{n-4}{2}\right)\varpi_{-\ell}[\varpi_\ell, A_{n-1,n-2}]Q\\
&+\left(L(H)+u_\ell-\frac{n-2}{2}\right)[\varpi_{-\ell}, A_{n-1,n-2}]\varpi_\ell Q\\
&+\left(L(H)-\frac{n-5}{2}\right)[A_{n-1,n-2},\varpi_{-\ell}\varpi_\ell]\,Q\\
&-2A_{n-1,n-2}\varpi_{-\ell}\varpi_\ell\,Q+\frac{1}{2}[\Omega_{n-2},[A_{n-1,n-2},\varpi_{-\ell}\varpi_\ell]]\,Q\\
&=\left(u_\ell+\frac{1}{2}\right)\varpi_{-\ell}[A_{n-1,n-2},\varpi_\ell]Q+\left(-u_\ell+\frac{3}{2}\right)[A_{n-1,n-2},\varpi_{-\ell}]\varpi_\ell Q \qquad (4.11)\\
&-2A_{n-1,n-2}\varpi_{-\ell}\varpi_\ell\,Q+\frac{1}{2}[\Omega_{n-2},[A_{n-1,n-2},\varpi_{-\ell}\varpi_\ell]]Q.
\end{aligned}$$

We shall show that this is zero. For simplicity, we denote

$$\begin{aligned}
&A:=A_{n-1,n-2}, && \Omega_{n-2}:=\Omega, && a_\ell(Q):=a_{n-2,\ell}(Q),\\
&a_j(Q):=a_{n-3,j}(Q), && \sigma_\ell:=\sigma_{n-2,\ell} && \sigma_j:=\sigma_{n-3,j},\\
&l_\ell:=l_{n-2,\ell}\quad\text{and} && l_j:=l_{n-3,j}.
\end{aligned}$$

By the definitions of ϖ_ℓ and the Gelfand-Tsetlin basis,

$$\varpi_\ell Q=a_\ell(Q)\sigma_\ell Q, \qquad AQ=\sum_j a_j(Q)\sigma_j Q$$

for $Q\in GT(\lambda)$. By the definition of $a_\ell(Q)$ and $a_j(Q)$, we have

$$\frac{a_\ell(Q)a_j(\sigma_\ell Q)}{a_j(Q)a_\ell(\sigma_j Q)}=\frac{l_\ell+l_{-j}}{l_\ell+l_{-j}-1}\quad\text{and}\quad a_{-\ell}(\sigma_\ell Q)=-a_\ell(Q).$$

It follows that

$$\begin{aligned}
\varpi_{-\ell}[A,\varpi_\ell]Q&=\sum_j a_{-\ell}(\sigma_j\sigma_\ell Q)\{a_\ell(Q)a_j(\sigma_\ell Q)-a_j(Q)a_\ell(\sigma_j Q)\}\sigma_j Q\\
&=-\sum_j\frac{a_\ell(\sigma_j Q)^2a_j(Q)}{l_\ell+l_{-j}-1}\sigma_j Q. \qquad (4.12)
\end{aligned}$$

By analogous calculations, we obtain

$$[A,\varpi_{-\ell}]\varpi_\ell Q=\sum_j\frac{a_\ell(Q)^2a_j(Q)}{l_\ell+l_j}\sigma_j Q, \qquad (4.13)$$

$$A\varpi_{-\ell}\varpi_\ell Q=-\sum_j a_\ell(Q)^2a_j(Q)\sigma_j Q, \qquad (4.14)$$

$$[A,\varpi_{-\ell}\varpi_\ell]Q=\sum_j\{a_\ell(\sigma_j Q)^2-a_\ell(Q)^2\}a_j(Q)\sigma_j Q. \qquad (4.15)$$

Since the "rho" of $\mathfrak{so}_{n-2}$ is $\rho_{\mathfrak{so}_{n-2}}=\frac{1}{2}\sum_{i=1}^{\lfloor(n-2)/2\rfloor}(n-2-2i)e_i$, Ω acts on Q by the scalar

$$-|\boldsymbol{q}_{n-3}+\rho_{\mathfrak{so}_{n-2}}|^2+|\rho_{\mathfrak{so}_{n-2}}|^2=-\sum_{i=1}^{\lfloor\frac{n-2}{2}\rfloor}\left\{\left(\frac{l_i-l_{-i}}{2}\right)^2-|\rho_{\mathfrak{so}_{n-2}}|^2\right\}.$$

It follows that

$$\begin{aligned}&[\Omega, [A, \varpi_{-\ell}\varpi_\ell]]Q \\ &\quad = \sum_j \{a_\ell(\sigma_j Q)^2 - a_\ell(Q)^2\} a_j(Q) \left\{ \left(\frac{l_j - l_{-j}}{2}\right)^2 - \left(\frac{l_j - l_{-j}}{2} + 1\right)^2 \right\} \sigma_j Q \\ &\quad = \sum_j \{a_\ell(Q)^2 - a_\ell(\sigma_j Q)^2\} a_j(Q)(l_j - l_{-j} + 1)\sigma_j Q.\end{aligned}$$

By (4.12), (4.13), (4.14), (4.15) and

$$\frac{a_\ell(\sigma_j Q)^2}{a_\ell(Q)^2} = \frac{(l_\ell + l_j + 1)(l_\ell + l_{-j} - 1)}{(l_\ell + l_j)(l_\ell + l_{-j})},$$

the coefficient of $\sigma_j Q$ in (4.11) is

$$\begin{aligned}&- \left(u_\ell + \frac{1}{2}\right) \frac{a_\ell(\sigma_j Q)^2 a_j(Q)}{l_\ell + l_{-j} - 1} + \left(-u_\ell + \frac{3}{2}\right) \frac{a_\ell(Q)^2 a_j(Q)}{l_\ell + l_j} \\ &\quad + 2a_\ell(Q)^2 a_j(Q) + \frac{1}{2}\{a_\ell(Q)^2 - a_\ell(\sigma_j Q)^2\} a_j(Q)(l_j - l_{-j} + 1) \\ &= \frac{a_\ell(Q)^2 a_j(Q)}{(l_\ell + l_j)(l_\ell + l_{-j})} \\ &\quad \times \left\{ - \left(u_\ell + \frac{1}{2}\right)(l_\ell + l_j + 1) + \left(-u_\ell + \frac{3}{2}\right)(l_\ell + l_{-j}) + 2(l_\ell + l_j)(l_\ell + l_{-j}) \right. \\ &\qquad \left. + \frac{1}{2}\Big((l_\ell + l_j)(l_\ell + l_{-j}) - (l_\ell + l_j + 1)(l_\ell + l_{-j} - 1)\Big)(l_j - l_{-j} + 1) \right\}\end{aligned}$$

If n is odd, then $u_\ell = l_\ell + 1/2$ and $l_{-j} = 1 - l_j$. In this case, the term in the braces is

$$-(l_\ell + 1)(l_\ell + l_j + 1) + (-l_\ell + 1)(l_\ell - l_j + 1) + 2(l_\ell + l_j)(l_\ell - l_j + 1) + 2(l_j)^2 = 0.$$

If n is even, then $u_\ell = l_\ell$ and $l_{-j} = -l_j$. In this case, the term in the braces is

$$-(l_\ell + \frac{1}{2})(l_\ell + l_j + 1) + (-l_\ell + \frac{3}{2})(l_\ell - l_j) + 2(l_\ell + l_j)(l_\ell - l_j) + \frac{1}{2}(2l_j + 1)^2 = 0.$$

Therefore, (4.11) is zero. □

5. Pfaffian

When $n = 2m$ is even, there is another generator of $Z(\mathfrak{so}_{2m})$, which is called Pfaffian. In this section, we obtain the Iwasawa decomposition of this element and relate it to the K-type shift operator P_0.

For a set $\{i_1, i_2, \ldots, i_{2k}\}$ of $2k$ different positive integers, define the Pfaffian $\mathrm{Pf}_{2k}(i_{2k}, i_{2k-1}, \ldots, i_1)$ inductively by

$$\begin{aligned}&\mathrm{Pf}_2(i_2, i_1) = A_{i_2, i_1}, \\ &\mathrm{Pf}_{2k}(i_{2k}, i_{2k-1}, \ldots, i_1) = \sum_{j=1}^{2k-1} (-)^{j+1} A_{i_{2k}, i_j} \mathrm{Pf}_{2k-2}(i_{2k-1}, \ldots, \widehat{i_j}, \ldots, i_1),\end{aligned}$$

and define

$$\mathbb{PF}_{2k} = \mathrm{Pf}_{2k}(2k, 2k-1, \ldots, 1).$$

Lemma 5.1. (1) *For every permutation $\sigma \in S_{2k}$,*

$$\mathrm{Pf}_{2k}(i_{\sigma(2k)}, i_{\sigma(2k-1)}, \dots, i_{\sigma(1)}) = \mathrm{sgn}\sigma\, \mathrm{Pf}_{2k}(i_{2k}, i_{2k-1}, \dots, i_1). \tag{5.1}$$

(2) *$\mathbb{PF}_{2m}$ is an element of $Z(\mathfrak{so}_{2m})$.*

Proof. (1) It is enough to show that $\mathbb{PF}_{2k}$ satisfies (5.1). We will show it by induction on k.

If $k = 1$, then $\mathbb{PF}_2 = A_{2,1}$ is alternative under the action of S_2. Assume that (5.1) holds for $k-1$. Then by the definition

$$\mathbb{PF}_{2k} = \sum_{i=1}^{2k-1} (-)^{i+1} A_{2k,i} \mathrm{Pf}_{2k-2}(2k-1, \dots, \hat{\imath}, \dots, 1)$$

and the hypothesis of induction, we can easily show that (5.1) holds for adjacent transpositions $\sigma = (j, j+1)$, $j = 1, \dots, 2k-2$.

Next, consider the case when σ is the transposition $(2k-1, 2k)$. By definition,

$$\begin{aligned}
\mathbb{PF}_{2m} &= A_{2m,2m-1}\mathrm{Pf}_{2m-2}(2m-2, \dots, 1) \\
&\quad + \sum_{i=1}^{2m-1} \sum_{j=i+1}^{2m-2} (-)^{i+1}(-)^{j} A_{2m,i}A_{2m-1,j}\mathrm{Pf}_{2m-4}(2m-2, \dots, \hat{\jmath}, \dots, \hat{\imath}, \dots, 1) \\
&\quad + \sum_{i=1}^{2m-1} \sum_{j=1}^{i-1} (-)^{i+1}(-)^{j-1} A_{2m,i}A_{2m-1,j}\mathrm{Pf}_{2m-4}(2m-2, \dots, \hat{\imath}, \dots, \hat{\jmath}, \dots, 1) \\
&= A_{2m,2m-1}\mathrm{Pf}_{2m-2}(2m-2, \dots, 1) \\
&\quad + \sum_{1 \le i < j \le 2m-1} (-)^{i+j+1}(A_{2m,i}A_{2m-1,j} - A_{2m,j}A_{2m-1,i}) \\
&\qquad\qquad \times \mathrm{Pf}_{2m-4}(2m-2, \dots, \hat{\jmath}, \dots, \hat{\imath}, \dots, 1).
\end{aligned} \tag{5.2}$$

By replacing m with k, we get $\mathrm{Pf}_{2k}(2k-1, 2k, 2k-2, \dots, 1) = -\mathbb{PF}_{2k}$. Since $\mathbb{PF}_{2k}$ is alternative under the action of all adjacent transpositions in S_{2k}, it satisfies (5.1).

(2) By (1), it is enough to show that $\mathbb{PF}_{2m}$ and $A_{2m,2m-1}$ commute, but this is clear from (5.2). □

Let us consider the Iwasawa decomposition of $\mathbb{PF}_{2m}$. Substitute

$$H = \sqrt{-1}A_{2m,2m-1}, \qquad X_i = A_{2m-1,i} + \sqrt{-1}A_{2m,i} \quad (i = 1, 2, \dots, 2m-2)$$

into (5.2). Then we get

$$\begin{aligned}
\sqrt{-1}\mathbb{PF}_{2m} &= H\,\mathbb{PF}_{2m-2} + \sum_{i=1}^{2m-2} (-)^{i+1} X_i\, \mathrm{Pf}_{2m-2}(2m-1, \dots, \hat{\imath}, \dots, 1) \\
&\quad + \sum_{1 \le i < j \le 2m-1} (-)^{i+j}[A_{2m-1,i}, A_{2m-1,j}] \\
&\qquad\qquad \times \mathrm{Pf}_{2m-4}(2m-2, \dots, \hat{\jmath}, \dots, \hat{\imath}, \dots, 1) \\
&= H\,\mathbb{PF}_{2m-2} + \sum_{i=1}^{2m-2} (-)^{i+1} X_i\, \mathrm{Pf}_{2m-2}(2m-1, \dots, \hat{\imath}, \dots, 1) \\
&\quad + \sum_{1 \le i < j \le 2m-1} (-)^{i+j} A_{j,i}\, \mathrm{Pf}_{2m-4}(2m-2, \dots, \hat{\jmath}, \dots, \hat{\imath}, \dots, 1).
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
&(2m-2)\mathbb{PF}_{2m-2}\\
&= \sum_{i=1}^{2m-2} (-)^i \operatorname{Pf}_{2m-2}(i, 2m-2, \dots, \hat{i}, \dots, 1)\\
&= \sum_{i=1}^{2m-2} (-)^i \sum_{j=i+1}^{2m-2} (-)^j A_{i,j} \operatorname{Pf}_{2m-4}(2m-2, \dots, \hat{j}, \dots, \hat{i}, \dots, 1)\\
&\quad + \sum_{i=1}^{2m-2} (-)^i \sum_{j=1}^{i-1} (-)^{j-1} A_{i,j} \operatorname{Pf}_{2m-4}(2m-2, \dots, \hat{i}, \dots, \hat{j}, \dots, 1)\\
&= -2 \sum_{1\le i<j\le 2m-2} (-)^{i+j} A_{j,i} \operatorname{Pf}_{2m-2}(2m-2, \dots, \hat{j}, \dots, \hat{i}, \dots, 1).
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&[A_{2m-1,i}, \mathbb{PF}_{2m-2}]\\
&= [A_{2m-1,i}, (-)^i \operatorname{Pf}_{2m-2}(i, 2m-2, \dots, \hat{i}, \dots, 1)]\\
&= [A_{2m-1,i}, (-)^i \sum_{j=i+1}^{2m-2} (-)^j A_{i,j} \operatorname{Pf}_{2m-4}(2m-2, \dots, \hat{j}, \dots, \hat{i}, \dots, 1)]\\
&\quad + [A_{2m-1,i}, (-)^i \sum_{j=1}^{i-1} (-)^{j+1} A_{i,j} \operatorname{Pf}_{2m-4}(2m-2, \dots, \hat{i}, \dots, \hat{j}, \dots, 1)\\
&= (-)^i \sum_{j=i+1}^{2m-2} (-)^j A_{2m-1,j} \operatorname{Pf}_{2m-4}(2m-2, \dots, \hat{j}, \dots, \hat{i}, \dots, 1)]\\
&\quad + (-)^i \sum_{j=1}^{i-1} (-)^{j+1} A_{2m-1,j} \operatorname{Pf}_{2m-4}(2m-2, \dots, \hat{i}, \dots, \hat{j}, \dots, 1)\\
&= (-)^i \operatorname{Pf}_{2m-2}(2m-1, \dots, \hat{i}, \dots, 1).
\end{aligned}
$$

Therefore, we get the following Iwasawa decomposition of $\mathbb{PF}_{2m}$:

Proposition 5.2.

$$
\sqrt{-1}\mathbb{PF}_{2m} = (H-m+1)\,\mathbb{PF}_{2m-2} - \sum_{i=1}^{2m-2} X_i\,[A_{2m-1,i}, \mathbb{PF}_{2m-2}]. \tag{5.3}
$$

Let us calculate the action of $\mathbb{PF}_{2m}$ on $C^\infty_{\tau_\lambda}(K\backslash G)$. We use the Cartan subalgebra $\mathfrak{h} = \mathfrak{t_m} \oplus \mathfrak{a}$, its basis $H, T_1, \dots, T_{m-1}$ and the dual basis $\{\alpha, e_1, \dots, e_{m-1}\}$ defined in §1.

By (5.3) and (3.2),

$$
\sqrt{-1}\gamma_{\mathfrak{n}}(\mathbb{PF}_{2m}) = H\mathbb{PF}_{2m-2}.
$$

By induction on m, the image of the (shifted) Harish-Chandra map is

$$
(\sqrt{-1})^m \gamma(\mathbb{PF}_{2m}) = HT_1 \cdots T_{m-2}T_{m-1}. \tag{5.4}
$$

Suppose that $Q = (\boldsymbol{q}_1, \dots, \boldsymbol{q}_{2m-2})$ is a Gelfand-Tsetlin base of the representation V_λ of $SO(2m-1)$. Then Q is contained in the irreducible representation of $SO(2m-$

2) whose highest weight is $\boldsymbol{q}_{2m-3}$. By (5.4), the image of Harish-Chandra map of $(\mathbb{PF}_{2m-2})^{\mathrm{opp}}$ is

$$(\sqrt{-1})^{m-1}\gamma_{\mathfrak{u}}((\mathbb{PF}_{2m-2})^{\mathrm{opp}}) = (-)^{m-1}T_{m-1}\cdots T_1.$$

It follows that $(\mathbb{PF}_{2m-2})^{\mathrm{opp}}$ acts on Q by the scalar

$$\begin{aligned}(\sqrt{-1})^{m-1}&(q_{2m-3,1}+m-2)\cdots(q_{2m-3,m-2}+1)q_{2m-3,m-1}\\ &= (\sqrt{-1})^{m-1}\left(\prod_{i=1}^{m-1} l_{2m-3,i}\right).\end{aligned}$$

Since

$$a_{2m-2,0}(Q) = \sqrt{-1}\frac{\prod_{i=1}^{m-1} l_{2m-3,i}\prod_{i=1}^{m} l_{2m-1,i}}{\prod_{i=1}^{m-1} l_{2m-2,k}(l_{2m-2,k}-1)},$$

there exists a constant d_λ, which depends on λ but not on $Q \in GT(\lambda)$, such that

$$-d_\lambda(\sqrt{-1})^m\left(\prod_{i=1}^{m-1} l_{2m-3,i}\right)Q = a_{2m-2,0}(Q)Q = \varpi_0 Q.$$

We have proved the following proposition:

Proposition 5.3. *For $\phi_{\tau_\lambda}(g) = \sum_{Q\in GT(\lambda)} c(Q;g)Q \in C^\infty_{\tau_\lambda}(K\backslash G)$, the action of $\mathbb{PF}_{2m}$ is given by*

$$\begin{aligned}&d_\lambda L(\mathbb{PF}_{2m})\phi_{\tau_\lambda}(g)\\ &=\sum_{Q\in GT(\lambda)}\left\{(L(H)-m+1)c(Q;g)\varpi_0 Q + \sum_{i=1}^{2m-2} L(X_i)c(Q;g)[\varpi_0, A_{2m-1,i}]Q\right\}\\ &=P_0\phi_{\tau_\lambda}(g).\end{aligned}$$

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