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A Set Theoretical Semantics for a Subsystem of Linear Logic

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Abstract. In [**2**], Yamaguchi et al. proposed a naïve semantics of a sub set of linear logic. However, the meanings of additive disjunction and conjunction are opposite to intuition. This note modifies the semantics proposed in the previous work. And it is pointed out that the cause of constructing misleadable semantics is on the interpretation of an exponential connective.

1. Introduction

At the end of 20th century, Girard introduced a sub-structural logic named linear logic[**6**] which has great power of expression.

Girard claimed that linear logic is a resource conscious logic which means that linear logic emphasizes the role of formulas as resources, instead of emphasizing truth in classical or intuitionistic logic[**7**]. As the idea of resource is emphasized, the number of the same propositions is important. In classical or intuitionistic logic, conjunction of two same propositions such as $A \wedge A$ equals to one of the propositions A logically. In contrast to this, multiplicative conjunction $\otimes$ is introduced in linear logic, such that multiplicative conjunction of two same proposition such as $A \otimes A$ holds two of them. $A \otimes A$ and A are logically different. In the sense of sequent calculus, it is necessary that number of propositions of both sides of the sequent delimiter $\vdash$ must be balanced to prove the sequent. Thus, neither $A \otimes A \vdash A$ nor $A \vdash A \otimes A$ holds. This resource consciousness is supported by omitting free usage of some structural rules in proof. Particularly, weakening rule, which adds any formula to the consequent, and contraction rule, which eliminates one of duplicated formulas in the antecedent are only allowed to those formulas with certain modal connectives. These connectives are of-course (!) and why-not (?), and are called exponential connectives. In addition to multiplicative conjunction, additive conjunction, whose symbols is $\&$, is also introduced in linear logic. Additive conjunction is similar to the conjunction of classical logic in the sense that $A \mathbin{\&} A$ logically equals to A. And in full linear logic, there are some more connectives: multiplicative disjunction $⅋$ which is dual of $\otimes$, additive disjunction $\oplus$ which is dual of $\&$, linear negation $(\cdot^{\perp})$, and linear implication $\multimap$ which is called entailment.

Since it focuses on resources, linear logic has found many applications in computer science. For example, some computational models such as Petri nets, counter machines and Turing machines are naturally encoded to linear logic[**5**]. Moreover there are some programming languages with linear logical features: ACL[**4**] which is based upon process algebra, captures simple notions of asynchronous communication by identifying the send and read primitives with two complementary linear logical connectives. Linear logic is familiar to process algebra in some way such as associating communication to cut elimination associating it to inference rule of entailment. Yamaguchi et al. introduced inductive synthesis of process expression based on inductive inference on linear logic[**1**]. Some process algebra interpretations of linear logic employ the idea that each proposition is associated to atomic message. Thus, number of each proposition represents multiplicity of associating message.

Girard gave some semantics of linear logic. One of these semantics associates formula to value with phase space[**3, 6**]. Phase space is a pair $(M, \perp)$ of commutative monoid M and its subset $\perp$. In this semantics, multiplicity of proposition is not clearly interpreted. Though some other semantics are also proposed, most of them focuses proof or cut elimination instead of value of formula, such as proof net by Lafont and game semantics by Blass[**3**].

Yamaguchi et al. proposed an inductive inference algorithm on a subset of linear logic[**2**]. They also proposed a naïve semantics and persisted that the proposed algorithm holds soundness and completeness on that semantics. The semantics consists of set of multiset which represents each formula and the inclusion relation of set associates with entailment relation i.e. sequent delimiter $\vdash$. However, this semantics sometimes maps a formula with $\oplus$ to the trivial empty set. This causes not only that the semantics is trivial but also that the semantics is misleadable while it is intended to be simple. This note proposes a simple semantics and points out the reason why the previous work is confusing.

This note is organized as follows. In the next section, logical system is defined. The target logic is a sub system of intuitionistic linear logic. And a naïve semantics of the logic is introduced with a map from logical formula into a set of multiset of atomic symbols. And then the soundness of logical system is shown. In section 3, the difference between proposed semantics and previous work is explained.

2. Definition and Notation

In this note, the target logic system is a sub-system of intuitionistic linear logic. Comparing with the full linear logic system[**6**], some logical connectives are omitted. The omitted connectives are two exponential connectives (why-not ? and of-course !), linear negation $\cdot^{\perp}$, linear entailment $\multimap$ and multiplicative disjunction $\parr$.

Definition 1 (Formula). *Let $\mathcal{A}$ be the set of symbols. The $\mathcal{F}$ is defined as the least set which satisfies following conditions:*

- $\mathbf{1} \in \mathcal{F}$, ($\mathbf{1}$ is a logical constant of this system.)
- $\mathcal{A} \subset \mathcal{F}$
- *When* $A, B \in \mathcal{F}$, *then* $(A \,\&\, B) \in \mathcal{F}$
- *When* $A, B \in \mathcal{F}$, *then* $(A \oplus B) \in \mathcal{F}$
- *When* $A, B \in \mathcal{F}$, *then* $(A \otimes B) \in \mathcal{F}$
- *When* $A \in \mathcal{F}$, *then* $(!A) \in \mathcal{F}$

A member of $\mathcal{F}$ *is called formula. Especially, a member of* A *is sometimes called atomic formula. In order to simplify the formula expressions, parentheses will be abbreviated when it's not ambiguous under the condition that the strength of connectivity are* $! > \otimes > \&, \oplus$, *and that* $\&$, $\oplus$ *and* $\otimes$ *are left associative and* $!$ *is right associative.*

In the following of this note, each of uppercase Latin letters such as A, B or C represents a formula. And each of upper case Greek letters such as Γ or Δ represents multiset of formulas.

The inference rule is defined in the style of sequent calculus. A sequent is an expression which is separated by a $\vdash$. The left side of $\vdash$ is a multiset of formula. Thus, the left side may be empty, may contain multiple occurrences of the same formula, but the order of the formulas is not concerned. The right side of $\vdash$ is restricted to one formula, this restriction derives that the target system is intuitionistic.

Definition 2 (Inference Rule). *A proof tree is a tree structured graph, each of whose nodes is a sequent, each of whose leaves is initial sequent given as follows, and each of whose edges are represented by a horizontal line between sequences matching one of patterns as follows:*

$$A \vdash A \ (\textit{initial})$$

$$\vdash \mathbf{1} \ (\textit{initial}) \qquad \frac{\Gamma \vdash X}{\mathbf{1}, \Gamma \vdash X} \ (\mathbf{1}\ \textit{left})$$

$$\frac{A, \Gamma \vdash C}{A \,\&\, B, \Gamma \vdash C} \ (\&\ \textit{left}) \qquad \frac{B, \Gamma \vdash C}{A \,\&\, B, \Gamma \vdash C} \ (\&\ \textit{left}) \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \,\&\, B} \ (\&\ \textit{right})$$

$$\frac{A, \Gamma \vdash C \quad B, \Gamma \vdash C}{A \oplus B, \Gamma \vdash C} \ (\oplus\ \textit{left}) \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \oplus B} \ (\oplus\ \textit{right}) \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \oplus B} \ (\oplus\ \textit{right})$$

$$\frac{A, B, \Gamma \vdash C}{A \otimes B, \Gamma \vdash C} \ (\otimes\ \textit{left}) \qquad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B} \ (\otimes\ \textit{right})$$

In the figure above, the name of inference rule is indicated in the parentheses right at the horizontal line. The notation $!\Gamma$ of the ! right rule represents a multiset which contains only formula in the form of $!B$ as its member.

As a semantics of the logic system above, each formula represents a set of multisets of atomic symbols. It is intended that each atomic symbol is a member of $\mathcal{A}$.

In order to represent multiplicity of proposition, a formula only with $\otimes$ is associated with a multiset of proposition. And a formula with $\otimes$, $\oplus$ and $\&$ represents a possible set of the multisets. For interpreting $\otimes$, an operator on two sets of multisets is needed.

Definition 3. *Let each of X and Y be a set of multiset, $\otimes_{\mathcal{I}}$ is an operator defined as follows:*

$$X \otimes_{\mathcal{I}} Y = \{\Gamma \uplus \Delta \mid \Gamma \in X, \Delta \in Y\},$$

where $\uplus$ denotes summation of multisets.

To distinguish set and multiset, a set is represented with incorporating braces, and a multiset is represented with incorporating brackets. Union $\cup$ and intersection $\cap$ of sets are used normally. For example, let $P, Q, R \in \mathcal{A}$, and $E = \{[P], [Q]\}$ and $F = \{[P, R]\}$ then $E \cup F = \{[P], [Q], [P, R]\}$ and $E \otimes_{\mathcal{I}} F = \{[P, P, R], [P, Q, R]\}$.

Lemma 4. *Let each of A, B and C is a set of multiset, when $A \subseteq B$ then $A \otimes_{\mathcal{I}} C \subseteq B \otimes_{\mathcal{I}} C$.*

Since $A \otimes_{\mathcal{I}} C = \{\Gamma \uplus \Delta \mid \Gamma \in A, \Delta \in C\}$, $B \otimes_{\mathcal{I}} C = \{\Lambda \uplus \Delta \mid \Lambda \in B, \Delta \in C\}$ and each $\Gamma \in A$ is also a member of B, thus, trivial.

On these preparation, the Interpretation function is defined.

Definition 5 (Semantics). *Let $\mathcal{I}$ be a map from each formula to a set of multiset of atomic symbols, defined as follows:*

- $\mathcal{I}(\mathbf{1}) = \{[\,]\}$ *(i.e. $\mathbf{1}$ represents the set of empty multiset.)*
- $\mathcal{I}(A) = \{[A]\}$ *where $A \in \mathcal{A}$*
- $\mathcal{I}(A \,\&\, B) = \mathcal{I}(A) \cap \mathcal{I}(B)$
- $\mathcal{I}(A \oplus B) = \mathcal{I}(A) \cup \mathcal{I}(B)$
- $\mathcal{I}(A \otimes B) = \mathcal{I}(A) \otimes_{\mathcal{I}} \mathcal{I}(B)$

In order to interpret the left hand side of a sequent, the interpretation function $\mathcal{I}$ is expanded to receive a multiset of formulas.

- $\mathcal{I}([\,]) = \{[\,]\}$

- $\mathcal{I}([A] \uplus \Gamma) = \mathcal{I}(A) \otimes_{\mathcal{I}} \mathcal{I}(\Gamma)$

THEOREM 6 (SOUNDNESS OF LOGIC SYSTEM). *If there exists a proof tree of a sequent $\Gamma \vdash A$, then $\mathcal{I}(\Gamma) \subseteq \mathcal{I}(A)$ holds.*

The existence of a proof tree of $\Gamma \vdash A$ is assumed. $\mathcal{I}(\Gamma) \subseteq \mathcal{I}(A)$ is proved by structural induction on the construction such proof tree as follows.

- For initial sequent $A \vdash A$

  Since $\mathcal{I}([A]) = \mathcal{I}(A) \otimes_{\mathcal{I}} \mathcal{I}([\,]) = \mathcal{I}(A) \otimes_{\mathcal{I}} \{[\,]\} = \mathcal{I}(A)$, $\mathcal{I}([A]) \subseteq \mathcal{I}(A)$ holds. Indeed, they are equal.

- For initial sequent $\vdash \mathbf{1}$

  Since $\mathcal{I}([\,]) = \{[\,]\} = \mathcal{I}(\mathbf{1})$, $\mathcal{I}([\,]) \subseteq \mathcal{I}(\mathbf{1})$.

- For $\mathbf{1}$ left rule

  $\mathcal{I}([\mathbf{1}] \uplus \Gamma) = \mathcal{I}(\mathbf{1}) \otimes_{\mathcal{I}} \mathcal{I}(\Gamma) = \{[\,]\} \otimes \mathcal{I}(\Gamma) = \mathcal{I}(\Gamma)$, and $\mathcal{I}(\Gamma) \subseteq \mathcal{I}(X)$ is the inductive assumption. Therefore, $\mathcal{I}([\mathbf{1}] \uplus \Gamma) \subseteq \mathcal{I}(X)$ holds.

- For $\&$ left rule

  $\mathcal{I}([A \,\&\, B] \uplus \Gamma) = \mathcal{I}(A \,\&\, B) \otimes_{\mathcal{I}} \mathcal{I}(\Gamma) = (\mathcal{I}(A) \cap \mathcal{I}(B)) \otimes_{\mathcal{I}} \mathcal{I}(\Gamma)$. And $\mathcal{I}(A) \cap \mathcal{I}(B) \subseteq \mathcal{I}(A)$. Thus, $\mathcal{I}([A \,\&\, B] \uplus \Gamma) = (\mathcal{I}(A) \cap \mathcal{I}(B)) \otimes_{\mathcal{I}} \mathcal{I}(\Gamma) \subseteq \mathcal{I}(C)$ holds where $\mathcal{I}([A] \uplus \Gamma) = \mathcal{I}(A) \otimes_{\mathcal{I}} \mathcal{I}(\Gamma) \subseteq \mathcal{I}(C)$ is the inductive assumption. This holds in the case of $B, \Gamma \vdash C$ being the antecedent, similarly.

- For $\&$ right rule

  Since $\mathcal{I}(A \,\&\, B) = \mathcal{I}(A) \cap \mathcal{I}(B)$ and both of $\mathcal{I}(\Gamma) \subseteq \mathcal{I}(A)$ and $\mathcal{I}(\Gamma) \subseteq \mathcal{I}(B)$ are the inductive assumptions, $\mathcal{I}(\Gamma) \subseteq \mathcal{I}(A \,\&\, B)$ holds.

- For $\oplus$ left rule

  Since $\mathcal{I}([A \oplus B] \uplus \Gamma) = (\mathcal{I}(A) \cup \mathcal{I}(B)) \otimes_{\mathcal{I}} \mathcal{I}(\Gamma)$ and both of $\mathcal{I}(A) \otimes_{\mathcal{I}} \mathcal{I}(\Gamma) \subseteq \mathcal{I}(C)$ and $\mathcal{I}(B) \otimes_{\mathcal{I}} \mathcal{I}(\Gamma) \subseteq \mathcal{I}(C)$ are the inductive assumptions, $\mathcal{I}([A \oplus B] \uplus \Gamma) \subseteq \mathcal{I}(C)$.

- For $\oplus$ right rule

  Since $\mathcal{I}(A) \subseteq \mathcal{I}(A) \cup \mathcal{I}(B) = \mathcal{I}(A \oplus B)$ and $\mathcal{I}(\Gamma) \subseteq \mathcal{I}(A)$ is the inductive assumption, $\mathcal{I}(\Gamma) \subseteq \mathcal{I}(A \oplus B)$ holds. This holds in the case of $\Gamma \vdash B$ being the antecedent, similarly.

- For $\otimes$ left rule

  Since $\mathcal{I}([A \otimes B] \uplus \Gamma) = \mathcal{I}(A) \otimes_{\mathcal{I}} \mathcal{I}(B) \otimes_{\mathcal{I}} \mathcal{I}(\Gamma)$, $\mathcal{I}([A] \uplus [B] \uplus \Gamma) \subseteq \mathcal{I}(C)$ is the inductive assumption. and $\otimes_{\mathcal{I}}$ is associative, $\mathcal{I}([A \otimes B] \uplus \Gamma) \subseteq \mathcal{I}(C)$ holds.

- For $\otimes$ right rule

  $\mathcal{I}(\Gamma \uplus \Delta) = \mathcal{I}(\Gamma) \otimes_{\mathcal{I}} \mathcal{I}(\Delta)$ and both of $\mathcal{I}(\Gamma) \subseteq \mathcal{I}(A)$ and $\mathcal{I}(\Delta) \subseteq \mathcal{I}(B)$ are the inductive assumptions. Since $\mathcal{I}(\Gamma) \otimes_{\mathcal{I}} \mathcal{I}(\Delta) \subseteq \mathcal{I}(\Gamma) \otimes_{\mathcal{I}} \mathcal{I}(B) \subseteq \mathcal{I}(A) \otimes_{\mathcal{I}} \mathcal{I}(B) = \mathcal{I}(A \otimes B)$, Thus, $\mathcal{I}(\Gamma \uplus \Delta) \subseteq \mathcal{I}(A \otimes B)$

Therefore, $\mathcal{I}(\Gamma) \subseteq \mathcal{I}(A)$ holds where a proof tree of a sequent $\Gamma \vdash A$ exists.

3. Discussion

At first, the notion in the previous work of Yamaguchi et al.[**2**] should be corrected. The main idea in constructing the semantics is similar to the previous work: each formula is associated with a set of multiset of atomic formula. In contrast to the previous work, the direction of the inclusion relation is reversed and of-course connective is omitted. The inference rules related to the of-course connective is as follows.

$$\frac{\Gamma \vdash B}{!A, \Gamma \vdash B}\,(!\ \text{increase}) \qquad \frac{!A, !A, \Gamma \vdash B}{!A, \Gamma \vdash B}\,(!\ \text{decrease})$$

$$\frac{A, \Gamma \vdash B}{!A, \Gamma \vdash B}\,(!\ \text{left}) \qquad \frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\,(!\ \text{right})$$

A naïve meaning of $!A$ is arbitrary number of A connected with $\otimes$. In fact, $!A \vdash \mathbf{1}$ holds and $!A \vdash A \otimes X$ holds if $!A \vdash X$ is assumed as follows:

$$\frac{\vdash \mathbf{1}}{!A \vdash \mathbf{1}} \qquad \dfrac{\dfrac{\dfrac{A \vdash A}{!A \vdash A} \quad !A \vdash X}{!A, !A \vdash A \otimes X}}{!A \vdash A \otimes X}$$

Here, $\mathbf{1}$ is the identity element of $\otimes$ such that both $A \otimes \mathbf{1} \vdash A$ and $A \vdash A \otimes \mathbf{1}$ hold. And thus, in the previous work, $\mathcal{I}(!A)$ is mapped into a set of all multisets which contains zero or more A i.e. $\{[\ ], [A], [A, A], [A, A, A], \ldots\}$. But in this case, the direction of inclusion relation interpreting $\vdash$ have to be $\supseteq$, because $!A \vdash A$ holds where $\mathcal{I}(A) = \{[A]\}$. This was the original mistake of the previous work. In order to reverse the direction of inclusion relation, $\&$ and $\oplus$ are interpreted by $\cup$ and $\cap$ respectively (they are also reversed). By duality, some properties discussed in [**2**] looks hold. However, as $\&$ is (additive) conjunction, it is misleadable when $\&$ interpreted as union instead of intersection.

In this note, the soundness of logic system is shown. For completeness, the opposite side of righteousness, there exists counter examples such that $\mathcal{I}(\Gamma) \subseteq \mathcal{I}(A)$ holds but $\Gamma \vdash A$ has no proof tree. For instance, $(A \otimes B) \,\&\, (A \otimes C) \vdash A \otimes (B \,\&\, C)$ has no proof tree. The intuitive explanation of no-existence of the proof tree is as follows. There are two possibilities in the inference rule of the lowest node of the proof tree. If $\otimes$ right is applied, since the left hand side of this sequence has only one formula, A or $B \,\&\, C$ in the right hand side must be entailed by empty. This does not work. And if $\&$ left is applied, though $A \otimes B$ or $A \otimes C$ must entail $A \otimes (B \,\&\, C)$, C or B is lacked in the left hand side. Thus, there are no proof of $(A \otimes B) \,\&(A \otimes C) \vdash A \otimes (B \,\&\, C)$. However, $\mathcal{I}((A \otimes B) \,\&(A \otimes C)) = \{[A, B]\} \cap \{[A, C]\} = \varnothing$ and $\mathcal{I}(A \otimes (B \,\&\, C)) = \{[A]\} \otimes_{\mathcal{I}} (\{[B]\} \cap \{[C]\}) = \{[A]\} \otimes_{\mathcal{I}} \varnothing = \varnothing$. As both sides are empty, $\mathcal{I}((A \otimes B) \,\&(A \otimes C)) \subseteq \mathcal{I}(A \otimes (B \,\&\, C))$. Therefore the completeness of logical system does not hold.

This trial for constructing naïve semantics clarifies the difficulty of interpreting of-course connective. It is known that the meaning of exponential connectives are sometimes difficult. As Girard noted in [**3**], the semantics of exponential gave in the original paper[**6**] is ad hoc.

It is also well known that the interpretation of multiplicative disjunction $\invamp$ is hard to explain. Most trite explanation is that $\invamp$ is dual of $\otimes$. But in this note, linear negation is also omitted. Thus, duality is difficult to treat. Entailment is sometimes explained as consumption relation and is defined by $\invamp$ and negation. Maybe there is some hint to interpret multiplicative disjunction.

4. Conclusion and future works

This note proposed a naïve semantics of a sub set of intuitionistic linear logic. In the proposed semantics, similar to the previous work, each formula is mapped into a set of multiset of atomic formula. However, compared to the previous work, it is more straightforward because additive conjunction and disjunction are interpreted into intersection and union respectively. It can be said that the proposed semantics maps each formula into the possible set of quantity of resources that present simultaneously.

As many connectives are omitted in this note, some semantics should be found in the future which successfully interpret more connectives from the point of view that each proposition represents associating resource.

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