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Joint distributions of numbers of trials and returns to the origin until random walk reaches at absorbing states

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Abstract. Joint distributions of the numbers of trials and returns to the origin until, starting at the origin, a random walk reaches at absorbing states, are obtained for a sequence of independently and identically distributed $\{-1,1\}$-valued random variables. The corresponding results for an exchangeable sequence of $\{-1,1\}$-valued random variables defined by Pólya urn sampling are also obtained from de Finetti's theorem. Numerical examples of maximum likelihood estimates are given.

1. Introduction

Random walk is an interesting example of Markov chain and is widely used in probabilistic phenomena (e.g. Feller [5], Blom, Holst and Sandell [2] and Ross [7]). Exact discrete distribution theory in a sequence of independent or dependent trials such as Markov dependent trials has been developed by the method of conditional probability generating function (pgf) and recently in a sequence of exchangeable trials has been investigated with applications (e.g. Aki [1], Inoue, Aki and Hirano [6] and references therein). It is known that Pólya urn sampling generates exchangeable random variables. In this paper, using the method of conditional pgf, we study joint distributions of the numbers of trials and returns to the origin until, starting at the origin, a random walk based on a sequence of independently and identically distributed $\{-1,1\}$-valued random variables reaches at absorbing states. Further when the random walk takes one step based on an exchangeable sequence of $\{-1,1\}$-valued random variables defined by Pólya urn sampling, we also study the corresponding joint distributions. These distributions contain some parameters. We give numerical examples of maximum likelihood estimates.

Let $\xi_1, \xi_2, \ldots$ be $\{-1,1\}-$valued independently and identically distributed (iid) random sequence with $P(\xi_i = 1) = p,\ P(\xi_i = -1) = q\ (1 < p < 1,\ q = 1-p),\ i = 1, 2, \ldots.$ We set $X_n = \xi_1 + \cdots + \xi_n,\ X_0 = 0$. Then, $\{X_n, n = 0, 1, 2, \ldots\}$ is a random walk, starting at the origin, based on iid sequence. If $\xi_1, \xi_2, \ldots$ are exchangeable, then, $\{X_n, n = 0, 1, 2, \ldots\}$ is a random walk based on exchangeable sequence. Here, a sequence of random variables $\{\xi_1, \xi_2, \ldots\}$ is exchangeable if for each n the distribution of $(\xi_1, \xi_2, \ldots, \xi_n)$ is invariant under permutations.

2. Joint distribution

2.1. Iid case

First we consider iid case. Let d be a positive integer. Let W be the number of trials until the random walk firstly reaches at d or $-d$ and let N be the number of time periods that the random walk reenters the origin. Namely,

$$\begin{aligned} W &= \min\{n; X_n = d \text{ or } X_n = -d\}, \\ N &= \text{number of } \{m; X_m = 0 \text{ and } 1 \leqq m < W\}. \end{aligned}$$

We are studying the joint pgf of W and N. For the derivation of it, we use the method of conditional pgf (e.g. Ebneshahrashoob and Sobel [4]). The technique has been used by many researchers for solving complicated problems (e.g., see Aki [1], Inoue, Aki and Hirano [6] and references therein).

Let $\phi_i(t,s)$ be the conditional pgf of W and N given that $X_k = i$ is obseved. Namely, define the conditional pgf of W and N by

$$\phi_i = E[t^W s^N | X_k = i], \text{ for } i = -d+1, ..., d-1, \ \phi = \phi_0.$$

Then the following recurrence relation holds;

$$(*) : \begin{cases} \phi_i & = pt\phi_{i+1} + qt\phi_{i-1}, \ (i = -d+2, ..., -2 \text{ or } i = 2, ..., d-1) \\ \phi_{-1} & = pts\phi + qt\phi_{-2} \\ \phi & = pt\phi_1 + qt\phi_{-1} \\ \phi_1 & = pt\phi_2 + qts\phi \\ \phi_{d-1} & = pt + qt\phi_{d-2} \\ \phi_{-(d-1)} & = pt\phi_{-(d-2)} + qt. \end{cases}$$

First, from $(*)$ we have the relation $A_2\phi = p^2t^2\phi_2 + q^2t^2\phi_{-2}$ where $A_2 = 1 - 2pqt^2s$. Next, by substituting $\phi_2 = pt\phi_3 + qt\phi_1$ and $\phi_{-2} = pt\phi_{-1} + qt\phi_{-3}$ its relation, we have $A_3\phi = p^3t^3\phi_3 + q^3t^3\phi_{-3}$ where $A_3 = A_2 - pqt^2$. Similarly, $A_4\phi = p^4t^4\phi_4 + q^4t^4\phi_{-4}$ holds where $A_4 = A_3 - pqt^2A_2$. By repeating the substitutions like above, we obtain a recurrence relation of A_n. Namely, define the recurrence relation of $A_n(s,t)$ by

$$\begin{aligned} A_1(s,t) &= 1, \\ A_2(s,t) &= 1 - 2pqt^2s, \\ A_n(s,t) &= A_{n-1}(s,t) - pqt^2A_{n-2}(s,t), \ n = 3, 4, \ldots, d. \end{aligned}$$

Then, from the above constitution of $A_n(s,t)$ we have the next theorem.

Theorem 1. *The joint pgf of W and N is given by*

$$\phi(t,s)=\frac{(p^d+q^d)t^d}{A_d(s,t)},\quad d\geqq 2,$$

where

$$A_d(s,t)=\frac{1-2pqt^2s}{\sqrt{1-4pqt^2}}\{\xi(t)^{d-1}-\eta(t)^{d-1}\}-\frac{pqt^2}{\sqrt{1-4pqt^2}}\{\xi(t)^{d-2}-\eta(t)^{d-2}\},$$

$$\xi(t)=\frac{1+\sqrt{1-4pqt^2}}{2},$$
$$\eta(t)=\frac{1-\sqrt{1-4pqt^2}}{2}.$$

PROOF. From the recurrence relation of $A_n(s,t)$, the result is easily checked. □

From the definition of $\phi(t,s)$, the marginal pgfs of W and N are immediately given by $\phi(t,1)$ and $\phi(1,s)$, respectively.

Corollary 1. *When $d=2$, the relation $W=2N+2$ holds with probability* 1.

PROOF. Set $d=2$. By expanding $\phi(t,s)$, we have the general term $(p^2+q^2)(2pq)^k s^k t^{2k+2}$. This completes the proof. □

After expanding $\phi(t,s)$ with respect to t and s, we have the joint probability $P(W=k,N=\ell)$ of W and N by taking the coefficient of $t^k s^\ell$.

For example, when $d=5$, we have the following results;

$$\phi(t,s)=\frac{(p^5+q^5)t^5}{1-(2pqs+3pq)t^2+(4p^2q^2s+p^2q^2)t^4}$$

and

$$\begin{aligned}&P(W=2k+1,\ N=\ell)\\&=\{p^{k+3}(1-p)^{k-2}+p^{k-2}(1-p)^{k+3}\}\sum_{j=2}^{[\frac{k}{2}]+1}\sum\frac{(k-j)!}{n_1!n_2!n_3!}2^{n_1}3^{n_2}(-1)^{n_3}\binom{n_3}{i}4^i,\\&\quad k=2,3,\ldots,\ \ell=0,1,2,\ldots,k-2,\end{aligned}$$

where the second summation is over all nonnegative integers n_1,n_2,n_3 and i such that $n_1+n_2+n_3=k-j$, $n_1+n_2=k-2j+2$ and $n_1+i=\ell$, $i\leqq n_3$.

Here [a] denotes the largest integer not exceeding a.

We can also obtain the means, variances, covariance and correlation coefficient by differentiating the joint pgf. For example, when $d = 5$ and $p = q = 1/2$, $E(W) = 25, E(N) = 4, E(WN) = 180$, $Var(W) = 400, Var(N) = 20$, $\rho(W, N) = 2/\sqrt{5}$.

2.2. Marginal distributions

In this subsection, we study the two marginal distributions of W and N. First, we study the pgf $\phi(t, 1) = (p^d + q^d)t^d/A_d(1, t)$ of W. After some algebra, the denominator $A_d(1, t)$ of $\phi(t, 1)$ can be written as follows;

$$A_d(1,t) = \frac{1}{2^d} \sum_{i=0}^{d} \binom{d}{i} \{(\sqrt{1-4pqt^2}\ )^i + (-\sqrt{1-4pqt^2}\ )^i\}.$$

After expanding $\phi(t, 1)$ with respect to t, we have the probability mass function $P(W = k)$ of W by taking the coefficient of t^k.

For example, when $d = 5$, we have the following result;

$$P(W = 2k+1) = \{p^{k+3}(1-p)^{k-2} + p^{k-2}(1-p)^{k+3}\}c_5(k),\ \ k = 2, 3, \ldots,$$

where

$$c_5(k) = \sum_{j=2}^{[\frac{k}{2}]+1} 5^{k-j}(k-j)!(-1)^{j-2} \frac{1}{(k-2j+2)!(j-2)!}.$$

REMARK 1. *Feller [5] gives the generating function for the duration of the game in the classical ruin problem, that is, the restricted random walk with absorbing barriers at* 0 *and* a *and the initial position* z $(0 < z < a)$. *Hence it may coincide with* $\phi(t, 1)$ *when* $a = 2d$ *and* $z = d$.

Next, we study the marginal distribution of N. Here, the geometric distribution, to be denoted by $G(p)$, is defined as the distribution of the number of failures preceding the first success in a sequence of Bernoulli trials with success probability p.

For $\xi(t)$ and $\eta(t)$ in Theorem 1, we set $\xi = \xi(1)$ and $\eta = \eta(1)$. If $p \neq 1/2$, then after some algebra we have

$$\phi(1, s) = \frac{p_d}{1 - q_d s}$$

where

$$p_d = \frac{(\xi - \eta)(\xi^d + \eta^d)}{\xi^d - \eta^d} \quad \text{and} \quad q_d = \frac{2\xi\eta(\xi^{d-1} - \eta^{d-1})}{\xi^d - \eta^d}.$$

Noting $\eta < \xi$, it is easy to see that $0 < p_d < 1$, $0 < q_d < 1$ and $p_d + q_d = 1$. Therefore we have the following corollary.

Corollary 2. *If $p \neq 1/2$, then the random variable N follows $G(p_d)$.*

If $p = 1/2$, then we have $\phi(1, s) = \frac{d}{d-(d-1)s}$. Therefore we have the following corollary.

Corollary 3. *If $p = q = 1/2$, then the random variable N follows $G(\frac{1}{d})$.*

3. Exchangeable case

Next we consider the exchangeable case. The usual Pólya urn sampling scheme is the following. An urn initially contains α red balls and β black balls. A ball is randomly drawn and is replaced together with one additional ball of same color. Suppose that we repeat drawing balls in the same manner. The sequence of random variables $\xi_i,\ i = 1, 2, \ldots$ is defined as $\xi_i = 1$ if the drawing ball is red at the ith trial, $\xi_i = -1$ otherwise. Then, the sequence $\xi_1, \xi_2, \ldots$ is exchangeable. We set $X_n = \xi_1 + \cdots + \xi_n,\ X_0 = 0$. Then, $\{X_n, n = 0, 1, 2, \ldots\}$ is a random walk, starting at the origin, based on an exchangeable sequence.

Now, we set $Z_i = (\xi_i + 1)/2,\ i = 1, 2, \ldots$. Then

$$Z_i = \begin{cases} 1 & \text{if } i\text{th ball is red,} \\ 0 & \text{otherwise.} \end{cases}$$

The de Finetti measure of $Z_1, Z_2, \ldots$ is known to be $Be(\alpha, \beta)$ (see e.g. Durrett [3], 238-239 and Remark 2 in Aki[1]) where $Be(\alpha, \beta)$ is the beta distibution with probability density function

$$\pi(p) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1-p)^{\beta-1}.$$

For the random walk based on the Pólya urn sampling, we define

$$\begin{aligned} W_e &= \min\{n; X_n = d \text{ or } X_n = -d\}, \\ N_e &= \text{number of } \{m; X_m = 0 \text{ and } 1 \leqq m < W_e\}. \end{aligned}$$

From de Finetti's theorem, the joint pgf of W_e and N_e based on Pólya sampling is given by

$$\varphi_e(t, s) = \int_0^1 \phi(t, s)\pi(p)dp.$$

Here, the marginal pgfs of W_e and N_e are $\varphi_e(t, 1)$ and $\varphi_e(1, s)$, respectively.

Further we have

$$P(W_e = k, N_e = \ell) = \int_0^1 P(W = k, N = \ell)\pi(p)dp,$$

$$P(W_e = n) = \int_0^1 P(W = n)\pi(p)dp.$$

For example, when $d = 5$, the joint distribution of W_e and N_e and the marginal distribution of W_e are as follows;

The joint distribution of W_e and N_e:

$$P(W_e = 2k+1, N_e = \ell)$$
$$= \frac{(\alpha+k-3)\cdots\alpha(\beta+k-3)\cdots\beta}{(\alpha+\beta+2k)\cdots(\alpha+\beta)} g_5(\alpha,\beta) \sum_{j=2}^{[\frac{k}{2}]+1} \sum \frac{(k-j)!}{n_1!n_2!n_3!} 2^{n_1} 3^{n_2} (-1)^{n_3} \binom{n_3}{i} 4^i,$$
$$k = 2, 3, \ldots, \quad \ell = 0, 1, 2, \ldots, k-2,$$

where

$$g_5(\alpha,\beta) = (\alpha+k+2)\cdots(\alpha+k-2) + (\beta+k+2)\cdots(\beta+k-2),$$

and the second summation is over all nonnegative integers n_1, n_2, n_3 and i such that $n_1 + n_2 + n_3 = k - j$, $n_1 + n_2 = k - 2j + 2$ and $n_1 + i = \ell$, $i \leqq n_3$.

The marginal distribution of W_e:

$$P(W_e = 2k+1) = c_5(k)\frac{(\alpha+k-3)\cdots\alpha(\beta+k-3)\cdots\beta}{(\alpha+\beta+2k)\cdots(\alpha+\beta)} g_5(\alpha,\beta), \; k = 2, 3, \ldots.$$

REMARK 2. *The relation in Corollary 4 holds for the exchangeable sequence.*

4. Examples of parametric estimation

4.1. Iid case

When $d = 5$, the following data are simulated by setting $p = 0.3$ in the iid case:

$$[5,12], [7,7], [9,7], [11,5], [13,7], [15,2], [17,2], [19,1], [21,4], [23,1], [25,1], [31,1].$$

The sample size is $n = 50$. Here $[k, j]$ means that the number of observations of $W = k$ is j. Then the likelihood function of p is

$$L(p) = \{P(W_{i_1} = 5)\}^{12}\{P(W_{i_2} = 7)\}^7\{P(W_{i_3} = 9)\}^7 \cdots \{P(W_{i_{12}} = 31)\}^1.$$

Maximizing the likelihood function with respect to p, we obtain the maximum likelihood estimate of p as $\hat{p} = 0.2825$.

4.2. Exchangeable case

For $d = 2$, we have

$$P(W_e = 2k) = \frac{2^{k-1}\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta+2k)}$$

$$\times\{\Gamma(\alpha+k+1)\Gamma(\beta+k-1)+\Gamma(\alpha+k-1)\Gamma(\beta+k+1)\},$$
$$k=1,2,\ldots.$$

The following data are simulated by setting $\alpha = 2$ and $\beta = 2$ in the exchangeable case: $2, 2, 6, 14, 2, 2, 4, 4, 2, 2$. The sample size is $n = 10$. Then the likelihood function of α and β is

$$L(\alpha,\beta) = P(W_{e1} = 2)P(W_{e2} = 2)P(W_{e3} = 2)\cdots P(W_{e10} = 2).$$

Maximizing the likelihood function with respect to α and β, we obtain the maximum likelihood estimates of α and β as $\hat{\alpha} = 6.061$ and $\hat{\beta} = 6.061$. The result is not so good because the sample size 10 for this problem may be small. Setting $\beta = 2$, assume that the parameter to be estimated is α only. Then, we get the maximum likelihood estimate $\hat{\alpha} = 2.475$.

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