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Accurate numerical integration algorithm for the Kepler motion

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Abstract. The two and three-dimensional discrete Kepler motions were presented in [**5, 6**] which conserve all of the constants of motion including the Runge-Lenz vectors. Sequences of points described by the discrete Kepler motions exactly lie in the continuous orbits of the Kepler motions. However, the time evolution of the discrete Kepler motions is different from that of the continuous Kepler motions. In this Letter new formulas are presented which describe the time when the two and three-dimensional continuous Kepler motions arrive at the points located by the discrete Kepler motions. As an application a time adjustment of the discrete Kepler motions is performed. Consequently an accurate numerical integration algorithm for the Kepler motions is designed.

1. Introduction

In [**5**] two of the authors presented a discrete system

$$\frac{Q_k^{(j+1)} - Q_k^{(j)}}{\tau^{(j,j+1)}} = \frac{1}{2}\frac{P_k^{(j+1)} + P_k^{(j)}}{\left(Q_1^{(j)}\right)^2 + \left(Q_2^{(j)}\right)^2},$$
$$\frac{P_k^{(j+1)} - P_k^{(j)}}{\tau^{(j,j+1)}} = h_{2d}\frac{Q_k^{(j+1)} + Q_k^{(j)}}{\left(Q_1^{(j)}\right)^2 + \left(Q_2^{(j)}\right)^2},\ k = 1, 2,\ j = 0, 1, \cdots, \tag{1}$$

where $\tau^{(j,j+1)}$ is a difference of the time variable $t^{(j)}$ defined by

$$\tau^{(j,j+1)} \equiv t^{(j+1)} - t^{(j)} > 0, \tag{2}$$

and $P_k^{(j)}$, $Q_k^{(j)}$ are values of P_k, Q_k at the discrete time $t^{(j)}$, respectively. h_{2d} is the value of the energy. These constants are the same as those in [**5**] up to a constant factor. In the limit $\left(P_k^{(j)}, Q_k^{(j)}\right) \to (p_k, q_k)$, $k = 1, 2$, as $\tau^{(j,j+1)} \to 0$, the two-dimensional discrete Kepler motion (1) goes to

$$\frac{dq_k}{dt} = \frac{p_k}{q_1^2 + q_2^2},\ \frac{dp_k}{dt} = \frac{2h_{2d}q_k}{q_1^2 + q_2^2},\ k = 1, 2. \tag{3}$$

Eqs. (3) do not look the Hamiltonian system of the two-dimensional Kepler motion. However, through the inverse of well-known canonical transformation by

Levi-Civita (cf. [**7**], p. 21)

$$x = q_1 q_2,\ y = \frac{1}{2}(q_1^2 - q_2^2),\ p_x = \frac{p_1 q_2 + p_2 q_1}{q_1^2 + q_2^2},\ p_y = \frac{p_1 q_1 - p_2 q_2}{q_1^2 + q_2^2}, \tag{4}$$

the Hamiltonian system (3) leads to that describing the two-dimensional Kepler motion in the xy-plane. Consequently, (1) can be regarded as the discrete analogue of the two-dimensional Kepler motion. The discrete system (1) is named the *two-dimensional discrete Kepler motion* in [**5**].

In a manner similar to the above, the *three-dimensional discrete Kepler motion* is derived in [**6**]. It is

$$\begin{aligned}
\frac{Q_k^{(j+1)} - Q_k^{(j)}}{\tau^{(j,j+1)}} &= \frac{1}{8}\frac{P_k^{(j+1)} + P_k^{(j)}}{\left(Q_1^{(j)}\right)^2 + \left(Q_2^{(j)}\right)^2 + \left(Q_3^{(j)}\right)^2 + \left(Q_4^{(j)}\right)^2},\\
\frac{P_k^{(j+1)} - P_k^{(j)}}{\tau^{(j,j+1)}} &= h_{3d}\frac{P_k^{(j+1)} + P_k^{(j)}}{\left(Q_1^{(j)}\right)^2 + \left(Q_2^{(j)}\right)^2 + \left(Q_3^{(j)}\right)^2 + \left(Q_4^{(j)}\right)^2},\\
\tau^{(j,j+1)} &\equiv t^{(j+1)} - t^{(j)} > 0,\ k = 1, \cdots, 4,\ j = 0, 1, \cdots,
\end{aligned} \tag{5}$$

where h_{3d} is the value of the corresponding energy. In the limit $\left(P_k^{(j)}, Q_k^{(j)}\right) \to (p_k, q_k)$, $k = 1, \cdots, 4$, as $\tau^{(j,j+1)} \to 0$, the three-dimensional discrete Kepler motion (5) goes to

$$\frac{dq_k}{dt} = \frac{1}{4}\frac{p_k}{q_1^2 + q_2^2 + q_3^2 + q_4^2},\ \frac{dp_k}{dt} = \frac{2h_{3d} q_k}{q_1^2 + q_2^2 + q_3^2 + q_4^2},\ k = 1, \cdots, 4. \tag{6}$$

Eqs. (6) do not seem the Hamiltonian system of the three-dimensional Kepler motion. Through the inverse of the Kustaanheimo-Stiefel (KS) canonical transformation [**4, 7**]

$$\begin{aligned}
&x = q_1^2 - q_2^2 - q_3^2 + q_4^2,\ y = 2(q_1 q_2 - q_3 q_4),\ z = 2(q_1 q_3 + q_2 q_4),\\
&p_x = \frac{1}{2}\frac{p_1 q_1 - p_2 q_2 - p_3 q_3 + p_4 q_4}{q_1^2 + q_2^2 + q_3^2 + q_4^2},\ p_y = \frac{1}{2}\frac{p_1 q_2 + p_2 q_1 - p_3 q_4 - p_4 q_3}{q_1^2 + q_2^2 + q_3^2 + q_4^2},\\
&p_z = \frac{1}{2}\frac{p_1 q_3 + p_2 q_4 + p_3 q_1 + p_4 q_2}{q_1^2 + q_2^2 + q_3^2 + q_4^2},\ p_1 q_4 - p_2 q_3 + p_3 q_4 - p_4 q_1 = 0,
\end{aligned} \tag{7}$$

the Hamiltonian system (6) leads to that describing the three-dimensional Kepler motion in the xyz-plane.

The discrete Kepler motions (1) and (5) preserve all constants of motion such as the Hamiltonians, the angular momenta and the Runge-Lenz vectors [**5, 6**]. As a result, each point described by (1) and (5) exactly traces the orbits of the continuous Kepler motions providing the initial points in phase spaces are the same, namely, $\left(P_k^{(0)}, Q_k^{(0)}\right) \equiv \left(p_k(t^{(0)}), q_k(t^{(0)})\right)$. However, the time evolution of the discrete Kepler motions is different from that of the continuous Kepler motions in the sense in which the discrete Kepler motions locate the points $\left(P_k^{(j)}, Q_k^{(j)}\right)$ which are different from the points $(p_k(t^{(j)}), q_k(t^{(j)}))$ located by the continuous Kepler motions in the same orbits. For example, when $h_{2d} < 0$, the two-dimensional Kepler discrete motion (1) describes points in the elliptic orbit of the two-dimensional Kepler motion (4). As time goes, the distance between the points $(P_k^{(j)}, Q_k^{(j)})$ and $\left(p_k^{(j)}, q_k^{(j)}\right)$ becomes larger. In the case $h_{3d} < 0$, the same is true in the three-dimensional Kepler discrete motion (5). Any transformation of the time variables for the discrete Kepler motions has not been known which adjusts the points in the same orbits.

The purpose of this article is to solve the following open problem: Find the time adjust formulas with the discrete Kepler motions (1) and (5) so that there is no difference in time from the actual time evolution of the continuous Kepler motions (3) and (6). By using these formulas with the discrete Kepler motions, not only orbit but also the time variable of the continuous Kepler motions are then simulated accurately.

In Section 2, we present time adjustment formulas for the two-dimensional and three-dimensional Kepler motions. To this end the actual time is described when the continuous Kepler motions reach the points located by the discrete Kepler motions. By integrating differential equations the time adjustment formulas are then derived. We show that the discrete Kepler motions round the ellipses just one time as the discrete step-sizes go to infinity. Then accurate numerical integration algorithms for the Kepler motions are presented. Some numerical examples illustrate that the points located by the discrete and continuous Kepler motions exactly coincide in the same orbits.

2. Time adjustment-formulas for discrete Kepler motions

In this section we present the time adjustment formulas and algorithms of the two-dimensional and three-dimensional discrete Kepler motion. First, we describe how to derive the time adjustment formula for the two-dimensional Kepler motion. We start with the explicit form of two-dimensional discrete Kepler motion (1),

$$\begin{aligned}
P_k^{(j+1)} &= \frac{\left(2(R^{(j)})^4 + h_{2d}(\tau^{(j,j+1)})^2\right) P_k^{(j)} + 4h_{2d}\tau^{(j,j+1)}(R^{(j)})^2 Q_k^{(j)}}{2\left(R^{(j)}\right)^4 - h_{2d}\left(\tau^{(j,j+1)}\right)^2}, \\
Q_k^{(j+1)} &= \frac{\left(2(R^{(j)})^4 + h_{2d}(\tau^{(j,j+1)})^2\right) Q_k^{(j)} + 2\tau^{(j,j+1)}(R^{(j)})^2 P_k^{(j)}}{2\left(R^{(j)}\right)^4 - h_{2d}\left(\tau^{(j,j+1)}\right)^2}, \\
\left(R^{(j)}\right)^2 &\equiv \left(Q_1^{(j)}\right)^2 + \left(Q_2^{(j)}\right)^2, \ k = 1, 2, \ j = 0, 1, \cdots.
\end{aligned} \tag{8}$$

Here, the value of energy h_{2d} is expressed as that of the Hamiltonian

$$h_{2d} = \frac{1}{2}\frac{\left(P_1^{(j)}\right)^2 + \left(P_2^{(j)}\right)^2}{\left(Q_1^{(j)}\right)^2 + \left(Q_2^{(j)}\right)^2} - \frac{2K^2}{\left(Q_1^{(j)}\right)^2 + \left(Q_2^{(j)}\right)^2}. \tag{9}$$

The two-dimensional discrete Kepler motion (1) is the discrete system which is derived by using the energy preserving method [**2, 3**]. The two-dimensional discrete Kepler motion (1) does not only conserve the Hamiltonian (9), but all of the constants of motion, namely the Hamiltonian, an angular momentum and the Runge-Lenz vector [**5**]. By using the inverse of the Levi-Civita transformation (4), a point

$$\left(X^{(j+1)}, Y^{(j+1)}, P_X^{(j+1)}, P_Y^{(j+1)}\right) \tag{10}$$

is determined which describes a motion in the XY-plane.

To present the time adjustment formula, we consider the actual time when the two-dimensional continuous Kepler motion arrives at the point described by the discrete Kepler motion. Let us set

$$\left(P_k^{(j)}, Q_k^{(j)}\right)_2 = \left(p_k(t^{(j)}), q_k(t^{(j)})\right)_2 \tag{11}$$

for some $t^{(j)}$. Here the suffix "2" indicates that the suffix k runs from 1 to 2. The point $\left(P_k^{(j+1)}, Q_k^{(j+1)}\right)_2$ is determined by (8) and lies in the orbit of the two-dimensional Kepler motion (3) for the initial values (11). Let t_{j+1} be the time when the point $(p_k, q_k)_2$ of (3) arrives at the point $\left(P_k^{(j+1)}, Q_k^{(j+1)}\right)_2$. Namely,

$$p_1(t_{j+1}) = P_1^{(j+1)}, p_2(t_{j+1}) = P_2^{(j+1)}, q_1(t_{j+1}) = Q_1^{(j+1)}, q_2(t_{j+1}) = Q_2^{(j+1)}. \tag{12}$$

It is clear from (8) that $P_k^{(j+1)}$, $Q_k^{(j+1)}$, $k = 1, 2$ are the functions of $\tau^{(j,j+1)}$. Differentiating $Q_1^{(j+1)}$ with respect to $\tau^{(j,j+1)}$, we have

$$\frac{dQ_1^{(j+1)}}{d\tau^{(j,j+1)}} = \frac{8h_{2d}\tau^{(j,j+1)}(R^{(j)})^4 Q_1^{(j)} + 2(R^{(j)})^2\left(2(R^{(j)})^4 + h_{2d}(\tau^{(j,j+1)})^2\right)P_1^{(j)}}{\left(2(R^{(j)})^4 - h_{2d}(\tau^{(j,j+1)})^2\right)^2}.$$

Note that

$$\frac{dQ_1^{(j+1)}}{d\tau^{(j,j+1)}} = \frac{dq_1(t_{j+1})}{dt_{j+1}}\frac{dt_{j+1}}{d\tau^{(j,j+1)}}. \tag{13}$$

Substituting (3) into (13), we derive

$$\frac{dt_{j+1}}{d\tau^{(j,j+1)}} = \frac{q_1(t_{j+1})^2 + q_2(t_{j+1})^2}{p_1(t_{j+1})} \frac{dQ_1^{(j+1)}}{d\tau^{(j,j+1)}}. \tag{14}$$

It follows from the assumption (12) that (14) leads to the differential equation

$$\begin{aligned}
&\frac{dt_{j+1}}{d\tau^{(j,j+1)}} = \frac{G_2(\tau^{(j,j+1)})}{\left(2(R^{(j)})^4 - (\tau^{(j,j+1)})^2 h_{2d}\right)^3},\\
&G_2(\tau^{(j,j+1)}) \equiv 2(R^{(j)})^4 \left(2(R^{(j)})^4 + (\tau^{(j,j+1)})^2 h_{2d}\right)^2\\
&\qquad +8(\tau^{(j,j+1)})^2 (R^{(j)})^6 (P_R^{(j)})^2\\
&\qquad +8(\tau^{(j,j+1)})(R^{(j)})^6 \left(2(R^{(j)})^4 + (\tau^{(j,j+1)})^2 h_{2d}\right) S^{(j)},\\
&S^{(j)} \equiv P_1^{(j)} Q_1^{(j)} + P_2^{(j)} Q_2^{(j)},\ \left(P_R^{(j)}\right)^2 \equiv \left(P_1^{(j)}\right)^2 + \left(P_2^{(j)}\right)^2.
\end{aligned} \tag{15}$$

It is not hard to integrate (15). The resulting solution is expressed as

$$\begin{aligned}
&t_{j+1}(\tau^{(j,j+1)})\\
&= t^{(j)} + \frac{4h_{2d}\tau^{(j,j+1)}(R^{(j)})^8 + 2\tau^{(j,j+1)}(R^{(j)})^6 (P_R^{(j)})^2 + 8(R^{(j)})^8 S^{(j)}}{h_{2d}\left(2(R^{(j)})^4 - h_{2d}(\tau^{(j,j+1)})^2\right)^2}\\
&-\frac{2h_{2d}\tau^{(j,j+1)}(R^{(j)})^4 + \tau^{(j,j+1)}(R^{(j)})^2 (P_R^{(j)})^2 + 8(R^{(j)})^4 S^{(j)}}{h_{2d}\left(2(R^{(j)})^4 - h_{2d}(\tau^{(j,j+1)})^2\right)}\\
&+\frac{\sqrt{2}}{2}(-h_{2d})^{-3/2}\left(\frac{1}{2}(P_R^{(j)})^2 - h_{2d}(R^{(j)})^2\right)\tan^{-1}\left(\frac{\sqrt{-2h_{2d}}\tau^{(j,j+1)}}{2(R^{(j)})^2}\right).
\end{aligned} \tag{16}$$

Here the discrete time $t^{(j)}$ appears in the right hand side as an integral constant. For any given $t^{(j)}$ and the time interval $\tau^{(j,j+1)} \equiv t^{(j+1)} - t^{(j)}$, $t_{j+1}(\tau^{(j,j+1)})$ in (16) describes the time when the continuous Kepler motion $(p_k(t), q_k(t))_2$ reaches the point $\left(P_k^{(j+1)}, Q_k^{(j+1)}\right)_2$

Furthermore, we consider a property of the elliptic orbit of the two-dimensional discrete Kepler motion (8) where $h_{2d} < 0$. Let us take a limit $\tau^{(j,j+1)} \to \infty$ in (8) so that $t^{(j+1)} \to \infty$. Then,

$$Q_k^{(j+1)} \to -Q_k^{(j)},\ P_k^{(j+1)} \to -P_k^{(j)}. \tag{17}$$

By using the inverse of the Levi-Civita transformation (4) we see

$$X^{(j+1)} \to X^{(j)},\ Y^{(j+1)} \to Y^{(j)},\ P_X^{(j+1)} \to P_X^{(j)},\ P_Y^{(j+1)} \to P_Y^{(j)} \tag{18}$$

as $t^{(j+1)} \to \infty$ in the XY-plane. When $h_{2d} < 0$; moreover, the two-dimensional discrete and continuous Kepler motions satisfy the same initial condition as (11), the point (10) lies in the elliptic orbit of the continuous motion for any $t^{(j+1)}$ and $\tau^{(j,j+1)}$ [**5**]. The behavior (18) implies that (10) goes to the point $\left(X^{(j)}, Y^{(j)}, P_X^{(j)}, P_Y^{(j)}\right)$ in the elliptic orbits as $t^{(j+1)} \to \infty$.

Taking the limit $\tau^{(j,j+1)} \to \infty$ in (16), we have

$$\lim_{\tau^{(j,j+1)}\to\infty} t_{j+1}(\tau^{(j,j+1)}) = t^{(j)} + \frac{\sqrt{2}\pi}{8}(-2h_{2d})^{-3/2}\left(-2h_{2d}(R^{(j)})^2 + (P_R^{(j)})^2\right). \tag{19}$$

Let us note that h_{2d} (9) is a constant of motion of (8) as well as of (3). Substituting (9) to (19) and using $K^2 = G(m_1 + m_2)$, we have

$$\begin{aligned}\lim_{\tau^{(j,j+1)}\to\infty} t_{j+1}(\tau^{(j,j+1)}) &= t^{(j)} + 2\pi K^2(-2h_{2d})^{-3/2}\\ &= t^{(j)} + 2\pi G(m_1+m_2)(-2h_{2d})^{-3/2}.\end{aligned} \tag{20}$$

On the other hand, Kepler ’ s Third Law says

$$G(m_1 + m_2)T_{kepl}^2 = 4\pi^2 L_{kepl}^3, \tag{21}$$

where L_{kepl} is the semimajor axis of the ellipse and T_{kepl} the period of the Kepler motion. There is a relationship (cf. [**1**]) between L_{kepl} and the energy h_{2d} such that

$$L_{kepl} = -\frac{G(m_1+m_2)}{2h_{2d}}. \tag{22}$$

Inserting (22) into (21) we see that the period T_{kepl} satisfies

$$T_{kepl}^2 = \frac{4\pi^2 G^2(m_1+m_2)^2}{(-2h_{2d})^3}. \tag{23}$$

Let us set $t^{(j)} = 0$. We derive from (20) and (23)

$$\lim_{\tau^{(j,j+1)}\to\infty} t_{j+1}(\tau^{(j,j+1)}) = 2\pi G(m_1+m_2)(-2h_{2d})^{-3/2} = T_{kepl}. \tag{24}$$

Eq. (24) shows that the limit is equal to the period of the continuous Kepler motion. We conclude that the point (10) rounds the ellipse just one time as $\tau^{(j,j+1)}$ moves from 0 to ∞.

We now present a new accurate numerical integration algorithm for simulating the two-dimensional Kepler motion for any initial value with a high speed. The new algorithm achieves a time adjustment of the algorithm in [**5**]. One step $j \to j+1$, $j = 0, 1, \cdots$ of the algorithm is defined as follows:

i) the point $\left(P_k^{(j)}, Q_k^{(j)}\right)_2$ and the time $t^{(j)}$ are known.

ii) Set the next discrete time $t^{(j+1)}$.

iii) Compute $(P_k^{(j+1)}, Q_k^{(j+1)})_2$ by the discrete Kepler motion (8).

iv) Replace $t^{(j+1)}$ by $t_{j+1}(\tau^{(j,j+1)})$ in (16).

v) Then $\left(P_k^{(j+1)}, Q_k^{(j+1)}\right)_2$ coincides with the point $\left(p_k(t^{(j+1)}), q_k(t^{(j+1)})\right)_2$ of the continuous Kepler motion (3).

We can choose any $t^{(j+1)}$ in ii) such that $t^{(j)} < t^{(j+1)} < \infty$. The corresponding discrete step-size $\tau^{(j,j+1)}$ can take any positive value. Eq. (16) is quite useful to adjust the time evolution of the discrete Kepler motion (8). We call (16) a *time adjustment formula for the two-dimensional discrete Kepler motion.*

Next, we simply show the time adjustment formula for the three-dimensional Kepler motion. As well as the two-dimensional discrete Kepler motion (1), the three-dimensional discrete Kepler motion (5) is rewritten as the following explicit form.

$$
\begin{aligned}
P_k^{(j+1)} &= \frac{\left(8(R^{(j)})^4 + h_{3d}(\tau^{(j,j+1)})^2\right) P_k^{(j)} + 16 h_{3d}\tau^{(j,j+1)}(R^{(j)})^2 Q_k^{(j)}}{8(R^{(j)})^4 - h_{3d}(\tau^{(j,j+1)})^2},\\
Q_k^{(j+1)} &= \frac{\left(8(R^{(j)})^4 + h_{3d}(\tau^{(j,j+1)})^2\right) Q_k^{(j)} + 2\tau^{(j,j+1)}(R^{(j)})^2 P_k^{(j)}}{8(R^{(j)})^4 - h_{3d}(\tau^{(j,j+1)})^2},\\
(R^{(j)})^2 &\equiv (Q_1^{(j)})^2 + (Q_2^{(j)})^2 + (Q_3^{(j)})^2 + (Q_4^{(j)})^2,\ k = 1, \cdots, 4,\ j = 0, 1, \cdots .
\end{aligned} \tag{25}
$$

The three-dimensional discrete Kepler motion (25) keeps the value of energy

$$
h_{3d} = \frac{1}{8}\frac{(P_1^{(j)})^2 + (P_2^{(j)})^2 + (P_3^{(j)})^2 + (P_4^{(j)})^2}{(Q_1^{(j)})^2 + (Q_2^{(j)})^2 + (Q_3^{(j)})^2 + (Q_4^{(j)})^2} - \frac{K^2}{(Q_1^{(j)})^2 + (Q_2^{(j)})^2 + (Q_3^{(j)})^2 + (Q_4^{(j)})^2}. \tag{26}
$$

Through a similar procedure in the case of the two-dimensional discrete Kepler motion, we give the following *time adjustment formula for the three-dimensional Kepler motion.*

$$
\begin{aligned}
& t_{j+1}(\tau^{(j,j+1)}) \\
&= t^{(j)} + \frac{\sqrt{2}\left((P_R^{(j)})^2 - 8h_{3d}(R^{(j)})^2\right)}{8(-h_{3d})^{3/2}} \tan^{-1}\left(\frac{\sqrt{-2h_{3d}}\tau^{(j,j+1)}}{4(R^{(j)})^2}\right) \\
&\quad + \frac{(R^{(j)})^2 \tau^{(j,j+1)} A^{(j)}}{2h_{3d}\left(8(R^{(j)})^4 - h_{3d}(\tau^{(j,j+1)})^2\right)^2}, \\
& A^{(j)} \equiv 64 h_{3d}(R^{(j)})^6 + 8h_{3d}^2(R^{(j)})^2(\tau^{(j,j+1)})^2 + 8(P_R^{(j)})^2(R^{(j)})^4 \\
&\quad + h_{3d}(P_R^{(j)})^2(\tau^{(j,j+1)})^2 + 32(R^{(j)})^2 S^{(j)} h_{3d}\tau^{(j,j+1)}, \\
& (P_R^{(j)})^2 \equiv (P_1^{(j)})^2 + (P_2^{(j)})^2 + (P_3^{(j)})^2 + (P_4^{(j)})^2,
\end{aligned} \tag{27}
$$

$$S^{(j)} \equiv P_1^{(j)}Q_1^{(j)} + P_2^{(j)}Q_2^{(j)} + P_3^{(j)}Q_3^{(j)} + P_4^{(j)}Q_4^{(j)}.$$

$t_{j+1}(\tau^{(j,j+1)})$ in (27) means the time when the three-dimensional continuous Kepler motion $(p_k(t), q_k(t))_4$ reaches the point $(P_k^{(j+1)}, Q_k^{(j+1)})_4$. The suffix "4" indicates that the suffix k runs from 1 to 4. Moreover, along a similar line of thought in the case of the two-dimensional discrete Kepler motion, we conclude that the point $(P_k^{(j+1)}, Q_k^{(j+1)})_4$ given by the three-dimensional discrete Kepler motion rounds the ellipse just one time as $\tau^{(j,j+1)}$ moves from 0 to ∞.

We describe an accurate numerical integration algorithm for the three-dimensional Kepler motion which achieves a time adjustment of the algorithm in [**6**]. One step $j \to j+1$, $j = 0, 1, \cdots$ of the algorithm is defined as follows:

i) The point $\left(P_k^{(j)}, Q_k^{(j)}\right)_4$ and the time $t^{(j)}$ are known.

ii) Set the next discrete time $t^{(j+1)}$.

iii) Compute $(P_k^{(j+1)}, Q_k^{(j+1)})_4$ by the three-dimensional discrete Kepler motion (25).

iv) Replace $t^{(j+1)}$ by $t_{j+1}(\tau^{(j,j+1)})$ in (27).

v) Then $\left(P_k^{(j+1)}, Q_k^{(j+1)}\right)_4$ coincides with the point $\left(p_k(t^{(j+1)}), q_k(t^{(j+1)})\right)_4$ given by the continuous Kepler motion (6).

Finally, we give numerical examples of the time adjustments of two-dimensional and three-dimensional discrete Kepler motions. In Figure 1 we compare the time change of x coordinate of the two-dimensional Kepler motion with that of the two-dimensional discrete Kepler motion and with that of its time adjustment. For these calculations we set $\left(X^{(0)}, Y^{(0)}, P_X^{(0)}, P_Y^{(0)}\right) = (1, 1, 0, 0.6435942529)$ and $K = 1$ so that $h_{2d} = -0.5$. We also draw a comparison among the time change of x coordinate of the three-dimensional Kepler motion, that of the three-dimensional discrete Kepler motion and that of its time adjustment in Figure 2. For initial values we set

$$(X^{(0)}, Y^{(0)}, Z^{(0)}, P_X^{(0)}, P_Y^{(0)}, P_Z^{(0)}) = (0.5, -0.2, 0.4, -0.2, 0.5, 1.513745015)$$

and $K = 1$ in order that $h_{3d} = -0.2$. In both figures we plot points ($\ell = 1, 2, \cdots$) of one steps $j = 0 \to j = 1$ with mutually different step-size $\tau^{(0,1)} = \ell$ for $\ell = 1, 2, \cdots$. The open circles denote $\left(t_1 - t_0, X^{(1)} - X^{(0)}\right)$ given by adjusted discrete Kepler motions. The dotted curves with asterisks indicate $\left(t^{(1)} - t^{(0)}, X^{(1)} - X^{(0)}\right)$ of the non-adjusted two-dimensional discrete Kepler motions [**5, 6**]. The adjusted discrete Kepler motions exactly trace the original orbits in the whole space-time.

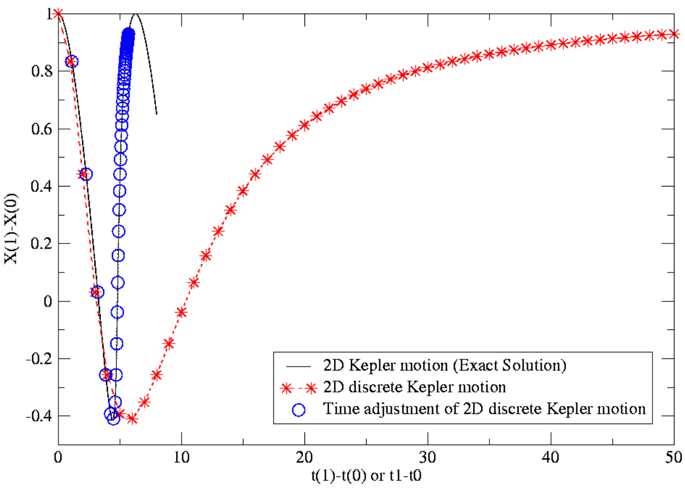

Figure 1. Time adjustment of two-dimensional discrete Kepler motion

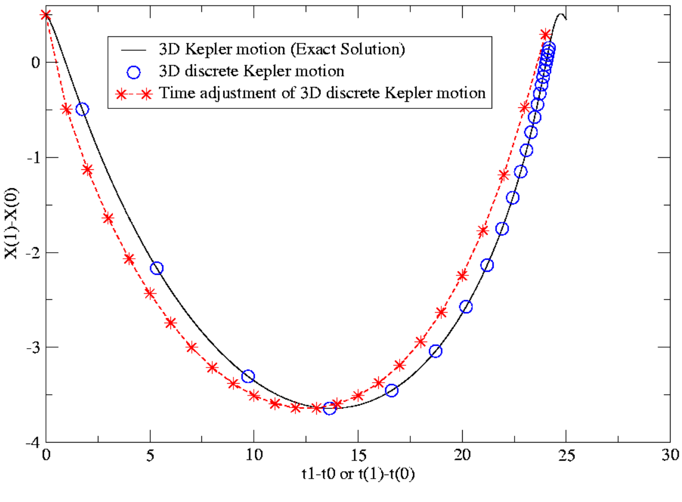

Figure 2. Time adjustment of three-dimensional discrete Kepler motion

3. Concluding remarks

In this article, we present a time adjustment formula for the discrete Kepler motion [**5, 6**]. Let us note that the discrete time variable of the discrete Kepler motion can be arbitrary chosen. Namely the discrete Kepler motion is a numerical integration scheme with a variable step-size. It is obvious that the adjusted discrete Kepler motion presented here also conserves all constants of motion of the continuous Kepler motions including the Runge-Lenz vector.

On the whole space-time the adjusted discrete Kepler motion accurately reproduces the continuous Kepler motion. The time adjustment formula found here says how to choose the correct step-size. Indeed we can compute the correct discrete time $t^{(j+1)}$ by solving the formulas with respect to $t^{(j+1)}$ for any given continuous time t_{j+1}. The new integration algorithm then enables us to know the correct orbits of the Kepler motion with a high speed. This comes from the good property that both the discrete Kepler motion and its time adjustment formula are explicit schemes.

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