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Invariant n-gon relative equilibria of discrete-time $(1+n)$-body problem with small arbitrary masses

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Abstract. The $(1+n)$-body problem with n infinitesimal masses [**3, 4, 11, 13**] is generalized to the general $(1+n)$-body problem (G$(1+n)$-BP) in which n arbitrary non-infinitesimal masses influence a massive central primary. We clarify that the G$(1+n)$-BP has the central configurations similar to those of the $(1+n)$-body problem. We also propose the discrete-time general $(1+n)$-body problem (d-G$(1+n)$-BP), which approximates the orbits of the G$(1+n)$-BP. Moreover, we proved that the d-G$(1+n)$-BP has the same central configurations as the G$(1+n)$-BP. Until the proof in this work, there has been no discrete analog of the G$(1+n)$-BP preserving these central configurations.

1. Introduction

The problem of finding the central configurations of the $(1+n)$-body problem ($(1+n)$-BP) in the plane has been the subject of many papers. One of the reasons why central configurations are interesting is that they allow us to construct exact solutions of the $(1+n)$-BP. It was already pointed out by Laplace and, historically, the problem of central configurations was first formulated in this context. Many researchers considered a restricted version of the $(1+n)$-BP where a central primary is massive and the other n masses are infinitesimal (e.g., see [**3, 4, 11, 13**]). We call this restricted problem the *restricted* $(1+n)$*-body problem* ($R(1+n)$*-BP*). In the R$(1+n)$-BP, n small masses do not influence a massive primary. However, nobody derived the central configurations of the $(1+n)$-BP where a central primary is massive, and the other n masses are arbitrary non-infinitesimal. We call this problem the *general* $(1+n)$*-body problem* ($G(1+n)$*-BP*).

Minesaki gave the discrete-time general three-body problem (d-G3BP) [**7, 8**] which follows the orbits of Lagrangian triangle solutions for the original general three-body problem (G3BP). Minesaki also proved that the discrete-time restricted four-body problem (d-R4BP) yields the correct orbits corresponding to the elliptic relative equilibrium solutions of the original restricted four-body problem (R4BP) [**9**]. These orbits include the orbits of some relative equilibrium solutions already discovered by Baltagiannis and Papadakis [**1**]. Until the proof in [**8, 9**], there has been no discrete analog that preserves the orbits of elliptic relative equilibrium solutions in the G3BP and R4BP. In contrast, high-order symplectic integrators and energy-conserving methods can trace none of such orbits.

The discrete-time general $(1+n)$-body problem (d-G$(1+n)$-BP), as well as the d-G3BP and d-R4BP is a special case of the discrete-time general N-body problem (d-GNBP). Therefore, the d-G$(1+n)$BP can expect to have the same equilibrium solutions as the original G$(1+n)$-body problem (G$(1+n)$-BP). In this paper, we include the following.

1. We present the d-G$(1+n)$-BP as a special case of the discrete-time general N-body problem [**10**]. As non-infinitesimal n masses tend to zero, the d-G$(1+n)$-BP reduces to the discrete-time model corresponding to the R$(1+n)$-BP studied in [**3, 4, 11, 13**].
2. We prove that the d-G$(1+n)$-BP yields the correct trajectories corresponding to equilibrium solutions of the G$(1+n)$-BP. We also numerically check that the d-G$(1+n)$-BP can precisely reproduce these trajectories.

This paper is organized as follows. In Section 2, in the barycentric inertial frame, we obtain the d-G$(1+n)$-BP as a special case of the discrete-time general N-body problem [**10**]. In addition, we rewrite this discrete-time problem as a discrete-time problem in a uniformly rotating reference frame. Next, in Section 3, for any set of non-infinitesimal $n+1$ masses, we analytically clarify that the rewritten d-G$(1+n)$-BP has some configurations of equilibria, each of which is consistent with one of the original G$(1+n)$-BP. In each configuration, n masses except a central massive particle form a rotating tetragon whose size and shape are invariant. Finally, in Section 4, we numerically check that the d-G$(1+n)$-BP can accurately reproduce some equilibrium solutions in the original G$(1+n)$-BP.

2. Discrete-time $(1+n)$-body problem

We consider the G$(1+n)$-BP with a massive mass m_0 and the other n non-infinitesimal masses m_i, $1 \leq i \leq n$ under the action of mutual gravity. In this case we obtain the following model:

$$\frac{d}{dt}\mathbf{q}'_i = \mathbf{v}'_i, \quad \frac{d}{dt}\mathbf{v}'_i = \sum_{k=0}^{i-1} \frac{m_k\,(\mathbf{q}'_k - \mathbf{q}'_i)}{|\mathbf{q}'_k - \mathbf{q}'_i|^3} - \sum_{k=i+1}^{n} \frac{m_k\,(\mathbf{q}'_i - \mathbf{q}'_k)}{|\mathbf{q}'_i - \mathbf{q}'_k|^3}, \quad 0 \leq i \leq n, \quad (1)$$

where $\mathbf{q}'_j \equiv (q'_{j[1]}, q'_{j[2]})$ is the position of m_j, $0 \leq j \leq n$ and $\mathbf{v}'_j \equiv (v'_{j[1]}, v'_{j[2]})$ is the velocity conjugate to $\mathbf{q}'_j$, $0 \leq j \leq n$ in the inertial barycentric frame. Also, $n+1$ position vectors $\mathbf{q}'_j$, $0 \leq j \leq n$ satisfy the constraint $\sum_{j=0}^{n} m_j \mathbf{q}'_j = \mathbf{0}$. Let the G$(1+n)$-BP be rewritten through some variable transformations, and be discretized as a special case of the discrete-time general N-body problem [**10**].

2.1. Discrete-time $(1+n)$-body problem related to inertial frame

We take $\mathbf{q}_{ij} \equiv \left(q_{ij[1]}, q_{ij[2]}\right)$, $0 \leq i < j \leq n$ as the relative position vector from mass m_j to mass m_i. $\mathbf{v}_{ij} \equiv \left(v_{ij[1]}, v_{ij[2]}\right)$ is proportional to the relative momentum vector conjugate to $\mathbf{q}_{ij}$, $0 \leq i < j \leq n$. The vectors $\mathbf{q}_{ij}$ and $\mathbf{v}_{ij}$ in the relative inertial frame are defined as

$$\mathbf{q}_{ij} = \mathbf{q}'_i - \mathbf{q}'_j,\ \mathbf{v}_{ij} = \frac{1}{M}\left(\mathbf{v}'_i - \mathbf{v}'_j\right),\ 0 \leq i < j \leq n, \tag{2}$$

where $M = \sum_{k=0}^{n} m_k$. Also, we define the vectors $\mathbf{Q}_{ij}$ and $\mathbf{V}_{ij}$, $0 \leq i < j \leq n$ as

$$\mathbf{Q}_{ij} \equiv \begin{cases} \left(\dfrac{q_{ij[2]}}{\sqrt{-2q_{ij[1]} + 2|\mathbf{q}_{ij}|}}, \dfrac{1}{2}\sqrt{-2q_{ij[1]} + 2|\mathbf{q}_{ij}|}\right) & \text{for } q_{ij[1]} < 0, \\ \left(\dfrac{1}{2}\sqrt{2q_{ij[1]} + 2|\mathbf{q}_{ij}|}, \dfrac{q_{ij[2]}}{\sqrt{2q_{ij[1]} + 2|\mathbf{q}_{ij}|}}\right) & \text{for } q_{ij[1]} \geq 0, \end{cases} \tag{3a}$$

$$\mathbf{V}_{ij} \equiv \begin{cases} \begin{bmatrix} \dfrac{2\left(v_{ij[1]}q_{ij[2]} - v_{ij[2]}q_{ij[1]} + v_{ij[2]}|\mathbf{q}_{ij}|\right)}{\sqrt{-2q_{ij[1]} + 2|\mathbf{q}_{ij}|}} \\ \dfrac{2\left(v_{ij[1]}q_{ij[1]} + v_{ij[2]}q_{ij[2]} - v_{ij[1]}|\mathbf{q}_{ij}|\right)}{\sqrt{-2q_{ij[1]} + 2|\mathbf{q}_{ij}|}} \end{bmatrix}^{\top} & \text{for } q_{ij[1]} < 0, \\ \begin{bmatrix} \dfrac{2\left(v_{ij[1]}q_{ij[1]} + v_{ij[2]}q_{ij[2]} + v_{ij[1]}|\mathbf{q}_{ij}|\right)}{\sqrt{2q_{ij[1]} + 2|\mathbf{q}_{ij}|}} \\ -\dfrac{2\left(v_{ij[1]}q_{ij[2]} - v_{ij[2]}q_{ij[1]} - v_{ij[2]}|\mathbf{q}_{ij}|\right)}{\sqrt{2q_{ij[1]} + 2|\mathbf{q}_{ij}|}} \end{bmatrix}^{\top} & \text{for } q_{ij[1]} \geq 0, \end{cases} \tag{3b}$$

where the components of the vectors $\mathbf{Q}_{ij}$ and $\mathbf{V}_{ij}$ are the Levi-Civita variables [**6**] related to the relative inertial frame. Substitution of (2) and (3) and their time differentiations into (1) yields the following system:

$$\begin{cases} \dfrac{d}{dt}\mathbf{Q}_{ij} = \dfrac{M}{4}\dfrac{\mathbf{V}_{ij}}{|\mathbf{Q}_{ij}|^2},\ 0 \leq i < j \leq n, & (4a) \\ \displaystyle\sum_{i=0}^{j-1}\mathbf{G}_{ij} - \sum_{i=j+1}^{n}\mathbf{G}_{ji} = \mathbf{0},\ 1 \leq j \leq n, & (4b) \\ \mathbf{\Phi}_{0ij}(\mathbf{Q}) = \mathbf{0},\ 1 \leq i < j \leq n, & (4c) \end{cases}$$

where

$$\mathbf{G}_{ij} \equiv \frac{m_i m_j}{2|\mathbf{Q}_{ij}|^2}\left(\frac{d}{dt}\mathbf{V}_{ij} - \frac{1}{|\mathbf{Q}_{ij}|^4}\left(\frac{M}{4}|\mathbf{V}_{ij}|^2 - 2\right)\mathbf{Q}_{ij}\right)\mathbf{L}(\mathbf{Q}_{ij})^{\top},$$

$$\mathbf{\Phi}_{0ij}(\mathbf{Q}) \equiv \begin{bmatrix} \Phi_{0ij[1]}(\mathbf{Q}) \\ \Phi_{0ij[2]}(\mathbf{Q}) \end{bmatrix}^{\top} \equiv \mathbf{Q}_{0i}\mathbf{L}(\mathbf{Q}_{0i})^{\top} + \mathbf{Q}_{ij}\mathbf{L}(\mathbf{Q}_{ij})^{\top} - \mathbf{Q}_{0j}\mathbf{L}(\mathbf{Q}_{0j})^{\top},$$
$$1 \leq i < j \leq n,$$

and

$$\mathbf{L}(\mathbf{Q}_{ij}) \equiv \begin{bmatrix} Q_{ij[1]} & -Q_{ij[2]} \\ Q_{ij[2]} & Q_{ij[1]} \end{bmatrix}, \ 0 \leq i < j \leq n,$$

is the Levi-Civita matrix [**6**]. This system is the G$(1+n)$-BP described in the Levi-Civita variables related to the relative inertial frame.

Meanwhile, we have already designed an accurate orbital integration method (see the system composed of (13a), (15), and (17) in [**10**]) for the following G$(1+n)$-BP[1] :

$$\begin{cases} \dfrac{d}{dt}\mathbf{Q}_{ij} = \dfrac{M}{4m_i m_j} \dfrac{\mathbf{P}_{ij}}{|\mathbf{Q}_{ij}|^2}, & 0 \leq i < j \leq n, \\ \displaystyle\sum_{i=0}^{j-1} \mathbf{G}_{ij} - \sum_{i=j+1}^{n} \mathbf{G}_{ji} = \mathbf{0}, & 1 \leq j \leq n, \\ \mathbf{\Phi}_{0ij}(\mathbf{Q}) = \mathbf{0}, & 1 \leq i < j \leq n, \end{cases}$$

where $\mathbf{P}_{ij}$ is the momentum conjugate to $\mathbf{Q}_{ij}$, and

$$\mathbf{G}_{ij} \equiv \frac{1}{2|\mathbf{Q}_{ij}|^2}\left(\frac{d}{dt}\mathbf{P}_{ij} - \left(\frac{M}{4m_i m_j}|\mathbf{P}_{ij}|^2 - 2m_i m_j\right)\frac{\mathbf{Q}_{ij}}{|\mathbf{Q}_{ij}|^4}\right)\mathbf{L}^\top(\mathbf{Q}_{ij}), \ 0 \leq i < j \leq n.$$

The G$(1+n)$-BP keeps the value of the following Hamiltonian:

$$h_{\mathrm{LC}} = \sum_{i=0}^{n-1} \sum_{j=i+1}^{n} \left(\frac{M}{8m_i m_j} \frac{|\mathbf{P}_{ij}|^2}{|\mathbf{Q}_{ij}|^2} - \frac{m_i m_j}{|\mathbf{Q}_{ij}|^2} \right). \tag{5}$$

However, if at least one of m_i and m_j is an infinitesimal mass, then we cannot compute $\mathbf{P}_{ij}$ and $\mathbf{Q}_{ij}$ using the above system. Therefore, the above system cannot give the central configurations of the R$(1+n)$-body problem.

In the case of $\mathbf{P}_{ij} = m_i m_j \mathbf{V}_{ij}$, $0 \leq i < j \leq n$, this system is rewritten as the system (4). Even if the system (4) includes some infinitesimal masses, we can integrate it to give $\mathbf{V}_{ij}$ and $\mathbf{Q}_{ij}$. Applying the accurate orbital integration method (see the system (31) in [**10**][1]) for the above system, and subsequently setting

[1] In this work, we set $N = n + 1$ and reduce by one each of subscripts in the system composed of (13a), (15), and (17) in [**10**]. Further, we do not use the Kustaanheimo-Stiefel matrix [**10**] but the Levi-Civita matrix because we consider only the two-dimensional G$(1 + n)$-BP.

$\mathbf{P}_{ij} = m_i m_j \mathbf{V}_{ij}$, we give the following discrete-time model:

$$\begin{cases} \dfrac{\mathbf{Q}_{ij}^{(l+1)} - \mathbf{Q}_{ij}^{(l)}}{\Delta t} = \dfrac{M}{8} \dfrac{|\mathbf{Q}_{ij}^{(l+1)}|^2 + |\mathbf{Q}_{ij}^{(l)}|^2}{|\mathbf{Q}_{ij}^{(l+1)}|^2 |\mathbf{Q}_{ij}^{(l)}|^2} \mathbf{V}_{ij}^{(l+1/2)} \quad 0 \le i < j \le n, & \text{(6a)} \\ \displaystyle\sum_{i=0}^{j-1} \mathbf{G}_{ij}^{(l+1)} - \sum_{i=j+1}^{n} \mathbf{G}_{ji}^{(l+1)} = \mathbf{0}, \ 1 \le j \le n, & \text{(6b)} \\ \mathbf{\Phi}_{0ij}(\mathbf{Q}^{(l+1)}) = \mathbf{0}, \ 1 \le i < j \le n, & \text{(6c)} \end{cases}$$

where Δt is a time step, $\mathbf{Q}_{ij}^{(l)} = \left(Q_{ij[1]}^{(l)}, Q_{ij[2]}^{(l)}\right)$, $\mathbf{V}_{ij}^{(l)} = \left(V_{ij[1]}^{(l)}, V_{ij[2]}^{(l)}\right)$ at the discrete time $t^{(l)} = l\Delta t$, $l = 0, 1, \cdots$,

$$\mathbf{G}_{ij}^{(l+1)} \equiv \frac{m_i m_j}{2|\mathbf{Q}_{ij}^{(l+1/2)}|^2} \left(\frac{\mathbf{V}_{ij}^{(l+1)} - \mathbf{V}_{ij}^{(l)}}{\Delta t} - \frac{1}{|\mathbf{Q}_{ij}^{(l+1)}|^2 |\mathbf{Q}_{ij}^{(l)}|^2} \right.$$
$$\left. \cdot \left(\frac{M}{8} \left(|\mathbf{V}_{ij}^{(l+1)}|^2 + |\mathbf{V}_{ij}^{(l)}|^2 \right) - 2 \right) \mathbf{Q}_{ij}^{(l+1/2)} \right) \mathbf{L} \left(\mathbf{Q}_{ij}^{(l+1/2)} \right)^{\top}, \ 0 \le i < j \le n,$$

and we define the midpoint value $(\bullet)^{(l+1/2)} \equiv \dfrac{(\bullet)^{(l+1)} + (\bullet)^{(l)}}{2}$ of the function $(\bullet)(t)$. The discrete-time system (6) is based on a d'Alembert-type scheme [**2**], so it is second-order accurate. According to the d'Alembert scheme [**2**], the discrete-time system (6) conserves

$$h_{\mathrm{LC}} = \sum_{i=0}^{n-1} \sum_{j=i+1}^{n} m_i m_j \left(\frac{M}{8} \frac{|\mathbf{V}_{ij}|^2}{|\mathbf{Q}_{ij}|^2} - \frac{1}{|\mathbf{Q}_{ij}|^2} \right) \tag{7}$$

to which (5) leads through $\mathbf{P}_{ij} = m_i m_j \mathbf{V}_{ij}$, $0 \le i < j \le n$. In the following, we call the discrete-time system (6) as the *discrete-time general* $(1+n)$*-body problem* (*d-G*$(1+n)$*BP*). Even if at least one of m_i and m_j is an infinitesimal mass, we can compute $\mathbf{V}_{ij}^{(l+1)}$ and $\mathbf{Q}_{ij}^{(l+1)}$ using the problem (6). Thus, the problem can not only give the configurations of the d-G$(1+n)$-BP but also those of the discrete-time restricted $(1+n)$-body problem (d-R$(1+n)$-BP). For each (i,j), $0 \le i < j \le n$, we give $\mathbf{q}_{ij}^{(l)} \equiv (q_{ij[1]}^{(l)}, q_{ij[2]}^{(l)})$ as the relative position vector from mass m_j to mass m_i at the discrete time $t^{(l)}$ and $\mathbf{v}_{ij}^{(l)} \equiv (v_{ij[1]}^{(l)}, v_{ij[2]}^{(l)})$ is proportional to the momentum vector conjugate to $\mathbf{q}_{ij}^{(l)}$. We define the vectors $\mathbf{q}_{ij}^{(l)}$ and $\mathbf{v}_{ij}^{(l)}$ as

$$\mathbf{q}_{ij}^{(l)} = \mathbf{Q}_{ij}^{(l)} \mathbf{L} (\mathbf{Q}_{ij}^{(l)})^{\top}, \ \mathbf{v}_{ij}^{(l)} = \frac{1}{2|\mathbf{Q}_{ij}^{(l)}|^2} \mathbf{V}_{ij}^{(l)} \mathbf{L} (\mathbf{Q}_{ij}^{(l)})^{\top}, \ 0 \le i < j \le n, \ l = 0, 1, 2, \cdots. \tag{8}$$

Eq. (8) is the inverse transformation of (3). Also, in the inertial barycentric frame, we assume that the mass m_i is located at the position $\mathbf{q}_i^{\prime(l)} \equiv (q_{i[1]}^{\prime(l)}, q_{i[2]}^{\prime(l)})$, $0 \le i \le n$

and $\mathbf{v}_i'^{(l)} \equiv (v_{i[1]}'^{(l)}, v_{i[2]}'^{(l)})$ is proportional to the momentum vector conjugate to $\mathbf{q}_i'^{(l)}$, $0 \le i \le n$ at the discrete time $t^{(l)}$, $l = 0, 1, 2, \cdots$. We set vectors $\mathbf{q}_i'^{(l)}$ and $\mathbf{v}_i'^{(l)}$ as

$$\mathbf{q}_i'^{(l)} = \frac{1}{M}\left(\sum_{j=i+1}^{n} m_j \mathbf{q}_{ij}^{(l)} - \sum_{j=0}^{i-1} m_j \mathbf{q}_{ji}^{(l)}\right),\ \mathbf{v}_i'^{(l)} = \sum_{j=i+1}^{n} m_j \mathbf{v}_{ij}^{(l)} - \sum_{j=0}^{i-1} m_j \mathbf{v}_{ji}^{(l)}, \quad 0 \le i \le n,\ l = 0, 1, \cdots. \tag{9}$$

To reduce the redundancy, we can rewrite the d-G$(1+n)$BP (6) using only $\mathbf{Q}_{i,i+1}$ and $\mathbf{V}_{i,i+1}$, $0 \le i \le n$. Here, only these $2n$ vectors are related to the chained position vectors $\mathbf{q}_{i,i+1}$ and the vectors $\mathbf{v}_{i,i+1}$ are proportional to the momentum vectors conjugate to $\mathbf{q}_{i,i+1}$, respectively [**10**]. However, we do not consider such discrete-time system without redundancy because it describes the same motion as the d-G$(1+n)$BP (6).

2.2. Discrete-time $(1+n)$-body problem related to rotating frame

Further, introduce a rotating barycentric frame with the origin at the center of masses. Set the angular velocity of this frame as a constant Ω. We can convert from the vectors $\mathbf{q}_i'^{(l)}$ and $\mathbf{v}_i'^{(l)}$ in the inertial barycentric frame to the vectors $\mathbf{x}_j'^{(l)} \equiv (x_{j[1]}'^{(l)}, x_{j[2]}'^{(l)})$ and $\mathbf{w}_j'^{(l)} \equiv (w_{j[1]}'^{(l)}, w_{j[2]}'^{(l)})$ in this frame as follows:

$$\mathbf{x}_i'^{(l)} = \mathbf{q}_i'^{(l)} R(\Omega l \Delta t), \quad \mathbf{w}_i'^{(l)} = \mathbf{v}_i'^{(l)} R(\Omega l \Delta t), \quad 0 \le i \le n,\ l = 0, 1, \cdots, \tag{10}$$

where

$$R(\theta) \equiv \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Moreover, in a way similar to (2), we set $\mathbf{x}_{ij}^{(l)} \equiv \left(x_{ij[1]}^{(l)}, x_{ij[2]}^{(l)}\right)$ and $\mathbf{w}_{ij}^{(l)} \equiv \left(w_{ij[1]}^{(l)}, w_{ij[2]}^{(l)}\right)$, $0 \le i < j \le n$ as the relative vectors in the rotating frame at the discrete-time $t^{(l)} = l\Delta t$, $l = 0, 1, 2, \cdots$. They are defined as

$$\mathbf{x}_{ij}^{(l)} = \mathbf{x}_i'^{(l)} - \mathbf{x}_j'^{(l)},\ \mathbf{w}_{ij}^{(l)} = \frac{1}{M}\left(\mathbf{w}_i'^{(l)} - \mathbf{w}_j'^{(l)}\right),\ 0 \le i < j \le n,\ l = 0, 1, \cdots. \tag{11}$$

In a way similar to (3), we also put the vectors $\mathbf{X}_{ij}^{(l)}$ and $\mathbf{W}_{ij}^{(l)}$, $0 \le i < j \le n$ as

$$\mathbf{X}_{ij}^{(l)} \equiv \begin{cases} \left(\dfrac{x_{ij[2]}^{(l)}}{\sqrt{-2x_{ij[1]}^{(l)} + 2|\mathbf{x}_{ij}^{(l)}|}}, \dfrac{1}{2}\sqrt{-2x_{ij[1]}^{(l)} + 2|\mathbf{x}_{ij}^{(l)}|} \right) & \text{for } x_{ij[1]}^{(l)} < 0, \\ \left(\dfrac{1}{2}\sqrt{2x_{ij[1]}^{(l)} + 2|\mathbf{x}_{ij}^{(l)}|}, \dfrac{x_{ij[2]}^{(l)}}{\sqrt{2x_{ij[1]}^{(l)} + 2|\mathbf{x}_{ij}^{(l)}|}} \right) & \text{for } x_{ij[1]}^{(l)} \ge 0, \end{cases} \tag{12a}$$

$$\mathbf{W}_{ij}^{(l)} \equiv \begin{cases} \begin{bmatrix} \dfrac{2\left(w_{ij[1]}^{(l)}x_{ij[2]}^{(l)} - w_{ij[2]}^{(l)}x_{ij[1]}^{(l)} + w_{ij[2]}^{(l)}|\mathbf{x}_{ij}^{(l)}|\right)}{\sqrt{-2x_{ij[1]}^{(l)} + 2|\mathbf{x}_{ij}^{(l)}|}} \\ \dfrac{2\left(w_{ij[1]}^{(l)}x_{ij[1]}^{(l)} + w_{ij[2]}^{(l)}x_{ij[2]}^{(l)} - w_{ij[1]}^{(l)}|\mathbf{x}_{ij}^{(l)}|\right)}{\sqrt{-2x_{ij[1]}^{(l)} + 2|\mathbf{x}_{ij}^{(l)}|}} \end{bmatrix}^\top & \text{for } x_{ij[1]}^{(l)} < 0, \\ \begin{bmatrix} \dfrac{2\left(w_{ij[1]}^{(l)}x_{ij[1]}^{(l)} + w_{ij[2]}^{(l)}x_{ij[2]}^{(l)} + w_{ij[1]}^{(l)}|\mathbf{x}_{ij}^{(l)}|\right)}{\sqrt{2x_{ij[1]}^{(l)} + 2|\mathbf{x}_{ij}^{(l)}|}} \\ -\dfrac{2\left(w_{ij[1]}^{(l)}x_{ij[2]}^{(l)} - w_{ij[2]}^{(l)}x_{ij[1]}^{(l)} - w_{ij[2]}^{(l)}|\mathbf{x}_{ij}^{(l)}|\right)}{\sqrt{2x_{ij[1]}^{(l)} + 2|\mathbf{x}_{ij}^{(l)}|}} \end{bmatrix}^\top & \text{for } x_{ij[1]}^{(l)} \ge 0. \end{cases} \tag{12b}$$

Using (3), (8), (10), (11) and (12), we give the relation among $\mathbf{Q}_{ij}^{(l)}$, $\mathbf{X}_{ij}^{(l)}$, $\mathbf{V}_{ij}^{(l)}$ and $\mathbf{W}_{ij}^{(l)}$ as follows:

$$\mathbf{Q}_{ij}^{(l)} = \mathbf{X}_{ij}^{(l)} R\left(-\frac{\Omega}{2} l\Delta t\right), \ \mathbf{V}_{ij}^{(l)} = \mathbf{W}_{ij}^{(l)} R\left(-\frac{\Omega}{2} l\Delta t\right), \quad 0 \le i < j \le n, \ l = 0, 1, \cdots. \tag{13}$$

Using (13), we rewrite the d-G$(1+n)$-BP (6) as the following discrete-time system:

$$\begin{cases} \dfrac{1-s^2}{\Delta t}\left(\mathbf{X}_{ij}^{(l+1)} - \mathbf{X}_{ij}^{(l)}\right) + \dfrac{M}{8}\dfrac{|\mathbf{X}_{ij}^{(l+1)}|^2 + |\mathbf{X}_{ij}^{(l)}|^2}{|\mathbf{X}_{ij}^{(l+1)}|^2|\mathbf{X}_{ij}^{(l)}|^2}\left(\mathbf{W}_{ij}^{(l+1)} - \mathbf{W}_{ij}^{(l)}\right)\mathbf{J} \\ = \dfrac{4s}{\Delta t}\mathbf{X}_{ij}^{(l+1/2)}\mathbf{J} + (1-s^2)\dfrac{M}{8}\dfrac{|\mathbf{X}_{ij}^{(l+1)}|^2 + |\mathbf{X}_{ij}^{(l)}|^2}{|\mathbf{X}_{ij}^{(l+1)}|^2|\mathbf{X}_{ij}^{(l)}|^2}\mathbf{W}_{ij}^{(l+1/2)}, \ 0 \le i < j \le n, & \text{(14a)} \\ \displaystyle\sum_{i=0}^{j-1}\mathbf{H}_{ij}^{(l,l+1)} - \sum_{i=j+1}^{n}\mathbf{H}_{ji}^{(l,l+1)} = \sum_{i=0}^{j-1}\mathbf{I}_{ij}^{(l,l+1)} - \sum_{i=j+1}^{n}\mathbf{I}_{ji}^{(l,l+1)}, \quad 1 \le j \le n, & \text{(14b)} \\ \mathbf{\Phi}_{0ij}(\mathbf{X}^{(l+1)}) = \mathbf{0}, \ 1 \le i < j \le n, & \text{(14c)} \end{cases}$$

where $s = \tan\left(\frac{\Omega\Delta t}{8}\right)$, $\mathbf{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$,

$$\widehat{\mathbf{Q}}_{ij}^{(l+1/2)} = \frac{1}{1+s^2}\left((1-s^2)\mathbf{X}_{ij}^{(l+1/2)} - s\left(\mathbf{X}_{ij}^{(l+1)} - \mathbf{X}_{ij}^{(l)}\right)\mathbf{J}\right),$$

$$\mathbf{H}_{ij}^{(l,l+1)} \equiv \frac{m_i m_j}{2|\widehat{\mathbf{Q}}_{ij}^{(l+1/2)}|^2}\Bigg(\frac{1-s^2}{\Delta t}\left(\mathbf{W}_{ij}^{(l+1)} - \mathbf{W}_{ij}^{(l)}\right) + \frac{2s}{|\mathbf{X}_{ij}^{(l+1)}|^2|\mathbf{X}_{ij}^{(l)}|^2}\left(\frac{M}{8}(|\mathbf{W}_{ij}^{(l+1)}|^2 + |\mathbf{W}_{ij}^{(l)}|^2) - 2\right)\left(\mathbf{X}_{ij}^{(l+1)} - \mathbf{X}_{ij}^{(l)}\right)\mathbf{J}\Bigg)\mathbf{L}\left(\widehat{\mathbf{Q}}_{ij}^{(l+1/2)}\right)^\top,$$

$$\mathbf{I}_{ij}^{(l,l+1)} \equiv \frac{m_i m_j}{2|\widehat{\mathbf{Q}}_{ij}^{(l+1/2)}|^2}\Bigg(\frac{4s}{\Delta t}\mathbf{W}_{ij}^{(l+1/2)}\mathbf{J} + \frac{1-s^2}{|\mathbf{X}_{ij}^{(l+1)}|^2|\mathbf{X}_{ij}^{(l)}|^2}\left(\frac{M}{8}(|\mathbf{W}_{ij}^{(l+1)}|^2 + |\mathbf{W}_{ij}^{(l)}|^2) - 2\right)\mathbf{X}_{ij}^{(l+1/2)}\Bigg)\mathbf{L}\left(\widehat{\mathbf{Q}}_{ij}^{(l+1/2)}\right)^\top,$$
$$0 \le i < j \le n.$$

Through the inverse transformation of (11) and (12), namely,

$$\mathbf{x}_{ij}^{(l+1)} = \mathbf{X}_{ij}^{(l+1)}\mathbf{L}(\mathbf{X}_{ij}^{(l+1)})^\top,\ \mathbf{w}_{ij}^{(l+1)} = \frac{1}{2|\mathbf{X}_{ij}^{(l+1)}|^2}\mathbf{W}_{ij}^{(l+1)}\mathbf{L}(\mathbf{X}_{ij}^{(l+1)})^\top,$$
$$0 \le i < j \le n,\ l = 0, 1, \cdots, \tag{15}$$

$$\begin{cases} \mathbf{x}_i'^{(l+1)} = \dfrac{1}{M}\left(\displaystyle\sum_{j=i+1}^{n} m_j \mathbf{x}_{ij}^{(l+1)} - \sum_{j=0}^{i-1} m_j \mathbf{x}_{ji}^{(l+1)}\right),\ 0 \le i \le n,\ l = 0, 1, \cdots, \\ \mathbf{w}_i'^{(l+1)} = \displaystyle\sum_{j=i+1}^{n} m_j \mathbf{w}_{ij}^{(l+1)} - \sum_{j=0}^{i-1} m_j \mathbf{w}_{ji}^{(l+1)},\ 0 \le i \le n,\ l = 0, 1, \cdots, \end{cases} \tag{16}$$

we can obtain the position $\mathbf{x}_i'^{(l+1)}$ and velocity $\mathbf{w}_i'^{(l+1)}$ of the mass m_i, $i = 0, \cdots, n$ in the rotating barycentric frame at the discrete time $t^{(l+1)} = (l+1)\Delta t$ from the solution of the rewritten d-G$(1+n)$-BP (14), $\mathbf{X}_{ij}^{(l+1)}$ and $\mathbf{W}_{ij}^{(l+1)}$, $0 \le i < j \le n$, $l = 0, 1, 2, \cdots$.

On the other hand, utilizing the transformation similar to (13),

$$\mathbf{Q}_{ij}(t) = \mathbf{X}_{ij}(t)R\left(-\frac{\omega}{2}t\right),\ \mathbf{V}_{ij}(t) = \mathbf{W}_{ij}(t)R\left(-\frac{\omega}{2}t\right),\ 0 \le i < j \le n, \tag{17}$$

we can express the G$(1+n)$-BP (4) as the following system:

$$\begin{cases} \dfrac{d}{dt}\mathbf{X}_{ij} = \dfrac{\omega}{2}\mathbf{X}_{ij}\mathbf{J} + \dfrac{M}{4}\dfrac{1}{|\mathbf{X}_{ij}|^2}\mathbf{W}_{ij},\ 0 \le i < j \le n, & \text{(18a)} \\ \displaystyle\sum_{i=0}^{j-1} \frac{m_i}{2|\mathbf{X}_{ij}|^2}\left(\frac{d}{dt}\mathbf{W}_{ij} - \frac{\omega}{2}\mathbf{W}_{ij}\mathbf{J} - \frac{1}{|\mathbf{X}_{ij}|^4}\left(\frac{M}{4}|\mathbf{W}_{ij}|^2 - 2\right)\mathbf{X}_{ij}\right)\mathbf{L}(\mathbf{X}_{ij})^\top & \\ \displaystyle -\sum_{i=j+1}^{n} \frac{m_i}{2|\mathbf{X}_{ji}|^2}\left(\frac{d}{dt}\mathbf{W}_{ji} - \frac{\omega}{2}\mathbf{W}_{ji}\mathbf{J} - \frac{1}{|\mathbf{X}_{ji}|^4}\left(\frac{M}{4}|\mathbf{W}_{ji}|^2 - 2\right)\mathbf{X}_{ji}\right)\mathbf{L}(\mathbf{X}_{ji})^\top = \mathbf{0}, & \\ \hfill 1 \le j \le n, & \text{(18b)} \\ \mathbf{\Phi}_{0ij}(\mathbf{X}) = \mathbf{0}, \quad 1 \le i < j \le n. & \text{(18c)} \end{cases}$$

In the limit $\omega = \lim_{\Delta t \to 0} 8\tan(\Omega\Delta t/8)/\Delta t$, the rewritten d-G$(1+n)$-BP (14) reduces to the rewritten G$(1+n)$-BP (18).

3. Invariant n-gon equilibria

In this section, we obtain the condition which some invariant n-gon equilibria of the rewritten d-G$(1+n)$-BP (14) satisfy. Also, we give the angular velocity Ω of the rewritten d-G$(1+n)$-BP (14) for the initial condition corresponding to the angular velocity ω of the rewritten G$(1+n)$-BP (18), and clarify that these equilibria of the rewritten d-G$(1+n)$-BP (14) with the Ω accord with those of the G$(1+n)$-BP (18) with the ω.

First, in preparation for obtaining the above conditions, we give the condition for the equilibrium solutions of the rewritten d-G$(1+n)$-BP (14) because the invariant n-gon equilibria are special cases of the equilibrium solutions. This condition is shown in the following lemma.

Lemma 3.1. *(Equilibria of d-G$(1+n)$-BP)* *Let $\mathbf{X}_{ij}^{(l+1)} = \mathbf{X}_{ij}^{(l)} = \boldsymbol{\mathcal{X}}_{ij} = const.$, $\mathbf{W}_{ij}^{(l+1)} = \mathbf{W}_{ij}^{(l)} = \boldsymbol{\mathcal{W}}_{ij} = const.$, $0 \le i < j \le n$, $l = 0, 1, \cdots$ be an equilibrium solution of the rewritten d-G$(1+n)$-BP (14). Then, the positions $\boldsymbol{\mathcal{X}}_{ij}$, $0 \le i < j \le n$ fulfill the following relation:*

$$\begin{cases} \mathbf{0} = \displaystyle\sum_{i=0}^{j-1} \widehat{\boldsymbol{\mathcal{I}}}_{ij} - \sum_{i=j+1}^{n} \widehat{\boldsymbol{\mathcal{I}}}_{ji}, \quad 1 \le j \le n, & \text{(19a)} \\ \mathbf{\Phi}_{0ij}(\boldsymbol{\mathcal{X}}) = \mathbf{0},\ 1 \le i < j \le n, & \text{(19b)} \end{cases}$$

where

$$\widehat{\boldsymbol{\mathcal{I}}}_{ij}=\frac{1-s^2}{2(1+s^2)}\frac{m_im_j}{|\boldsymbol{\mathcal{X}}_{ij}|^2}\left(\frac{128s^2}{1-s^2}\frac{1}{M(\Delta t)^2}|\boldsymbol{\mathcal{X}}_{ij}|^2-\frac{2(1-s^2)}{|\boldsymbol{\mathcal{X}}_{ij}|^4}\right)\boldsymbol{\mathcal{X}}_{ij}\mathbf{L}\,(\boldsymbol{\mathcal{X}}_{ij})^\top,\ 0\le i<j\le n.$$

Moreover, the relation between $\boldsymbol{\mathcal{X}}_{ij}$ *and* $\boldsymbol{\mathcal{W}}_{ij}$ *is expressed by*

$$\boldsymbol{\mathcal{W}}_{ij}=-\frac{16s}{(1-s^2)}\frac{1}{M\Delta t}|\boldsymbol{\mathcal{X}}_{ij}|^2\boldsymbol{\mathcal{X}}_{ij}\mathbf{J},\quad 0\le i<j\le n. \tag{20}$$

Proof. An equilibrium solution $\mathbf{x}_i'$, $\mathbf{w}_i'$, $0\le i\le n$ of the rewritten d-G$(1+n)$-BP (14) must satisfy the condition $\mathbf{x}_i^{\prime(l+1)}=\mathbf{x}_i^{\prime(l)}$, $\mathbf{w}_i^{\prime(l+1)}=\mathbf{w}_i^{\prime(l)}$, $0\le i\le n$, $l=0,1,2,\cdots$. Through (11) and (12), this condition leads to

$$\begin{cases}\mathbf{X}_{ij}^{(l+1)}=\mathbf{X}_{ij}^{(l)}=\boldsymbol{\mathcal{X}}_{ij}=\text{const.},\quad 0\le i<j\le n,\ l=0,1,2,\cdots, & \text{(21a)}\\ \mathbf{W}_{ij}^{(l+1)}=\mathbf{W}_{ij}^{(l)}=\boldsymbol{\mathcal{W}}_{ij}=\text{const.},\quad 0\le i<j\le n,\ l=0,1,2,\cdots. & \text{(21b)}\end{cases}$$

Substituting (21) into (14) yields the following conditional equations for the equilibrium solution:

$$\begin{cases}\mathbf{0}=\dfrac{4s}{\Delta t}\boldsymbol{\mathcal{X}}_{ij}\mathbf{J}+\dfrac{(1-s^2)M}{4}\dfrac{1}{|\boldsymbol{\mathcal{X}}_{ij}|^2}\boldsymbol{\mathcal{W}}_{ij},\quad 0\le i<j\le n, & \text{(22a)}\\ \mathbf{0}=\displaystyle\sum_{i=0}^{j-1}\boldsymbol{\mathcal{I}}_{ij}-\sum_{i=j+1}^{n}\boldsymbol{\mathcal{I}}_{ji},\quad 1\le j\le n, & \text{(22b)}\\ \mathbf{\Phi}_{0ij}(\boldsymbol{\mathcal{X}})=\mathbf{0},\ 1\le i<j\le n, & \text{(22c)}\end{cases}$$

where

$$\boldsymbol{\mathcal{I}}_{ij}=\frac{1-s^2}{2(1+s^2)}\frac{m_im_j}{|\boldsymbol{\mathcal{X}}_{ij}|^2}\left(\frac{4s}{\Delta t}\boldsymbol{\mathcal{W}}_{ij}\mathbf{J}+\frac{1-s^2}{|\boldsymbol{\mathcal{X}}_{ij}|^4}\left(\frac{M}{4}|\boldsymbol{\mathcal{W}}_{ij}|^2-2\right)\boldsymbol{\mathcal{X}}_{ij}\right)\mathbf{L}(\boldsymbol{\mathcal{X}}_{ij})^\top,\ 0\le i<j\le n.$$

Eq. (22a) is equivalent to (20). Substituting (20) into (22b), we give (19a). Then, (22c) is congruent with (19b). Accordingly, the positions $\boldsymbol{\mathcal{X}}_{ij}$, $0\le i<j\le n$ satisfy the relation (19). □

It is clear that a non-degenerate relative equilibrium solution of the G$(1+n)$-BP with $m_0=1$ and $m_i=\epsilon\mu_i$, $1\le i\le n$ converges to one of the R-$(1+n)$-BP as $\epsilon\to 0$ (e.g., [**3**], [**4**], [**11**], [**13**]). Also, (19a) in LEMMA 3.1 gives the relation satisfied by the equilibrium solutions of the d-G$(1+n)$-BP involving in such non-degenerate ones. We show that in the small positive ϵ, the rewritten d-G$(1+n)$-BP (14), as well as the rewritten G$(1+n)$-BP (18), has a non-degenerate equilibrium solution in the following theorem.

Theorem 3.2. *(Coorbital Configurations of d-G$(1+n)$-BP)* *We assume that ϵ is a small positive value, and $0 < \theta_{0i} < \theta_{0j} \leq 2\pi$ for all $1 \leq i < j \leq n$. Let*

$$m_0 = 1, \quad m_i = \epsilon\mu_i, \quad 1 \leq i \leq n, \tag{23}$$

and

$$\begin{cases} \mathbf{X}_{0j}^{(l+1)} = \mathbf{X}_{0j}^{(l)} = \boldsymbol{\mathcal{X}}_{0j} = \kappa \left(-\sin\dfrac{\theta_{0j}}{2}, \cos\dfrac{\theta_{0j}}{2} \right), \ 1 \leq j \leq n, \\ \mathbf{X}_{ij}^{(l+1)} = \mathbf{X}_{ij}^{(l)} = \boldsymbol{\mathcal{X}}_{ij} = \kappa\sqrt{r_{ij}} \left(-\sin\dfrac{\theta_{ij}}{2}, \cos\dfrac{\theta_{ij}}{2} \right), \ 1 \leq i < j \leq n, \end{cases} \tag{24}$$

$$\begin{cases} \mathbf{W}_{0j}^{(l+1)} = \mathbf{W}_{0j}^{(l)} = \boldsymbol{\mathcal{W}}_{0j} = \dfrac{16s}{(1-s^2)\Delta t} \dfrac{\kappa^3}{1+\epsilon\displaystyle\sum_{k=1}^{n}\mu_k} \left(-\cos\dfrac{\theta_{0j}}{2}, -\sin\dfrac{\theta_{0j}}{2} \right), \ 1 \leq j \leq n, \\ \mathbf{W}_{ij}^{(l+1)} = \mathbf{W}_{ij}^{(l)} = \boldsymbol{\mathcal{W}}_{ij} = \dfrac{16s}{(1-s^2)\Delta t} \dfrac{\kappa^3\sqrt{r_{ij}^3}}{1+\epsilon\displaystyle\sum_{k=1}^{n}\mu_k} \left(-\cos\dfrac{\theta_{ij}}{2}, -\sin\dfrac{\theta_{ij}}{2} \right), \ 1 \leq i < j \leq n, \end{cases} \tag{25}$$

be a non-degenerate equilibrium solution of the rewritten d-G$(1+n)$-BP (14) with the angular velocity Ω. Here, $s = \tan(\Omega\Delta t/8)$,

$$\kappa = \frac{(1-s^2)^{1/3}(\Delta t)^{1/3}}{2s^{1/3}} \left(1 + \epsilon\sum_{k=1}^{n}\mu_k \right)^{1/6} \tag{26}$$

and

$$r_{ij} = 2\sin\frac{\theta_{0j} - \theta_{0i}}{2}, \ 1 \leq i < j \leq n. \tag{27}$$

Further, the angles θ_{ij}, $1 \leq i < j \leq n$ are given by

$$\sin(\theta_{ij} - \theta_{0j}) = \frac{\sin(\theta_{0j} - \theta_{0i})}{2\sin\dfrac{\theta_{0j} - \theta_{0i}}{2}}, \quad 1 \leq i < j \leq n. \tag{28}$$

Then, the angles θ_{0j}, $1 \le j \le n$ satisfy the following identities:

$$
\begin{aligned}
0 = &\sum_{i=1}^{j-1} \mu_i \left(1 - \frac{1}{8 \sin^3 \dfrac{\theta_{0j} - \theta_{0i}}{2}} \right) \sin(\theta_{0j} - \theta_{0i}) \\
&- \sum_{i=j+1}^{n} \mu_i \left(1 - \frac{1}{8 \sin^3 \dfrac{\theta_{0i} - \theta_{0j}}{2}} \right) \sin(\theta_{0i} - \theta_{0j}), \ 1 \le j \le n. \qquad (29)
\end{aligned}
$$

Proof. Substitution of (24) into (19b) yields

$$
\cos\theta_{ij} = r_{ij}^{-1} \left(\cos\theta_{0j} - \cos\theta_{0i} \right), \ \sin\theta_{ij} = r_{ij}^{-1} \left(\sin\theta_{0j} - \sin\theta_{0i} \right), \ 1 \le i < j \le n.
$$

Using these relations and some trigonometric function formula, we obtain (27) and (28). Then, using (24), (27) and (28), we rewrite (19a) in LEMMA 3.1 as

$$
\mathbf{0} = \widehat{\boldsymbol{\mathcal{J}}}_{0j} + \sum_{i=1}^{j-1} \widehat{\boldsymbol{\mathcal{J}}}_{ij} - \sum_{i=j+1}^{n} \widehat{\boldsymbol{\mathcal{J}}}_{ij}, \ 1 \le j \le n, \qquad (30)
$$

where

$$
\begin{aligned}
\widehat{\boldsymbol{\mathcal{J}}}_{0j} &= 2s^2 \left(-\cos\theta_{0j}, -\sin\theta_{0j} \right), \quad 1 \le j \le n, \\
\widehat{\boldsymbol{\mathcal{J}}}_{ij} &= \epsilon\mu_i \left(4s^2 \sin\frac{\theta_{0j} - \theta_{0i}}{2} - \frac{1-s^2}{2} \frac{1}{\sin^2 \dfrac{\theta_{0j} - \theta_{0i}}{2}} \right) \left(-\cos\theta_{ij}, -\sin\theta_{ij} \right), \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 1 \le i < j \le n, \\
\widehat{\boldsymbol{\mathcal{J}}}_{ji} &= \epsilon\mu_i \left(4s^2 \sin\frac{\theta_{0i} - \theta_{0j}}{2} - \frac{1-s^2}{2} \frac{1}{\sin^2 \dfrac{\theta_{0i} - \theta_{0j}}{2}} \right) \left(-\cos\theta_{ji}, -\sin\theta_{ji} \right), \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 1 \le j < i \le n.
\end{aligned}
$$

Taking inner product of (30) with $(-\sin\theta_{0j}, \cos\theta_{0j})$, we obtain

$$
\begin{aligned}
0 = &2\epsilon(1 - s^2) \\
&\left(\sum_{i=1}^{j-1} \mu_i \left(1 - \frac{1}{8 \sin^3 \dfrac{\theta_{0j} - \theta_{0i}}{2}} \right) \sin(\theta_{0j} - \theta_{0i}) \right.
\end{aligned}
$$

$$
-\sum_{i=j+1}^{n} \mu_i \left(1 - \frac{1}{8\sin^3 \dfrac{\theta_{0i}-\theta_{0j}}{2}} \right) \sin(\theta_{0i}-\theta_{0j}) \Bigg) , \ 1 \le j \le n.
$$

Accordingly, we can prove that the angles θ_{ij}, $0 \le i < j \le n$ satisfy the identities (29). $\square$

In Section 2.1, we pointed out that d-G$(1+n)$-BP (6) preserves the Hamiltonian of the G$(1+n)$-BP (4), h_{LC} (7). Utilizing this preservation, when the rewritten G$(1+n)$-BP (18) has the same n-gon equilibria (24) as the rewritten d-G$(1+n)$-BP (14), the relation between ω in (18) and Ω in (14) is given by the following lemma.

Lemma 3.3. *(Angular Velocities of G$(1+n)$-BP and d-G$(1+n)$-BP for Equilibria)*
Suppose that

(i) $m_0 = 1$, $m_i = \epsilon\mu_i$, $1 \le i \le n$, *where* ϵ *is small positive.*

(ii) Both of the rewritten d-G$(1+n)$-BP (14) and the rewritten G$(1+n)$-BP (18) have the common set of fixed points

$$
\begin{aligned}
\boldsymbol{\mathcal{X}}_{0j} &= \kappa \left(-\sin\frac{\theta_{0j}}{2}, \cos\frac{\theta_{0j}}{2} \right), \ 1 \le j \le n, \\
\boldsymbol{\mathcal{X}}_{ij} &= \kappa\sqrt{r_{ij}} \left(-\sin\frac{\theta_{ij}}{2}, \cos\frac{\theta_{ij}}{2} \right), \ 1 \le i < j \le n,
\end{aligned}
$$

where κ *and* r_{ij}, $1 \le i < j \le n$ *are defined by (26) and (27), respectively, and the angles* θ_{ij}, $1 \le i < j \le n$ *satisfy (28).*

Then, the angular velocity ω *in the rewritten G$(1+n)$-BP (18) is the following function of the angular velocity* Ω *in the rewritten d-G$(1+n)$-BP (14):*

$$
\omega = \frac{8s}{(1-s^2)\Delta t}, \tag{31}
$$

where $s = \tan\left(\dfrac{\Omega\Delta t}{8}\right)$.

Proof. Consider the positions and velocities of both the G$(1+n)$-BP (4) and the d-G$(1+n)$-BP (6) in the Levi-Civita frame at the discrete time $t^{(l)} = l\Delta t$, $l = 0, 1, \cdots$. According to [**2**], the G$(1+n)$-BP (4) and the d-G$(1+n)$-BP (6) conserve the value of the common h_{LC} (7). Therefore, the Hamiltonian of the

G$(1+n)$-BP (4), $H_{\text{c-LC}}$ and that of the d-G$(1+n)$-BP (6), $H_{\text{d-LC}}$ are given in the following forms.

1. **The Hamiltonian of the d-G$(1+n)$-BP (6)**
   Substitution of (24) and (25) into (13) yields

$$
\begin{aligned}
\mathbf{Q}_{0j}^{(l)} &= \kappa\left(-\sin\frac{\theta_{0j}}{2},\cos\frac{\theta_{0j}}{2}\right)R\left(-\frac{\Omega}{2}l\Delta t\right),1\le j\le n,\\
\mathbf{Q}_{ij}^{(l)} &= \kappa\sqrt{r_{ij}}\left(-\sin\frac{\theta_{ij}}{2},\cos\frac{\theta_{ij}}{2}\right)R\left(-\frac{\Omega}{2}l\Delta t\right),1\le i<j\le n,\\
\mathbf{V}_{0j}^{(l)} &= \frac{16s}{(1-s^2)\Delta t}\frac{\kappa^3}{1+\epsilon\displaystyle\sum_{k=1}^{n}\mu_k}\left(-\cos\frac{\theta_{0j}}{2},-\sin\frac{\theta_{0j}}{2}\right)R\left(-\frac{\Omega}{2}l\Delta t\right),1\le j\le n,\\
\mathbf{V}_{ij}^{(l)} &= \frac{16s}{(1-s^2)\Delta t}\frac{\kappa^3\sqrt{r_{ij}^3}}{1+\epsilon\displaystyle\sum_{k=1}^{n}\mu_k}\left(-\cos\frac{\theta_{ij}}{2},-\sin\frac{\theta_{ij}}{2}\right)R\left(-\frac{\Omega}{2}l\Delta t\right),1\le i<j\le n.
\end{aligned}
\tag{32}
$$

   Using (32), we rewrite the Hamiltonian (7) as

$$
H_{\text{d-LC}}=-\frac{\epsilon}{2\kappa^2}\sum_{j=1}^{n}\mu_j+\frac{2\epsilon^2}{\kappa^2}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\mu_i\mu_j\left(\sin^2\frac{\theta_{0j}-\theta_{0i}}{2}-\frac{1}{8\sin^2\dfrac{\theta_{0j}-\theta_{0i}}{2}}\right).\tag{33}
$$

2. **The Hamiltonian of the G$(1+n)$-BP (4)**
   Substituting $\mathbf{X}_{ij}(t)=\boldsymbol{\mathcal{X}}_{ij}$, $0\le i<j\le n$ in LEMMA 3.3 (ii) into (17), we give

$$
\begin{cases}
\mathbf{Q}_{0j}(t)=\kappa\left(-\sin\dfrac{\theta_{0j}}{2},\cos\dfrac{\theta_{0j}}{2}\right)R\left(-\dfrac{\omega}{2}t\right),\ 1\le j\le n,\\
\mathbf{Q}_{ij}(t)=\kappa\sqrt{r_{ij}}\left(-\sin\dfrac{\theta_{ij}}{2},\cos\dfrac{\theta_{ij}}{2}\right)R\left(-\dfrac{\omega}{2}t\right),\ 1\le i<j\le n.
\end{cases}
\tag{34}
$$

Further, substituting (34) into (4a), we obtain

$$\begin{cases} \mathbf{V}_{0j}(t) = 2\omega \dfrac{\kappa^3}{1+\epsilon \sum_{k=1}^{n} \mu_k} \left(-\cos \dfrac{\theta_{0j}}{2}, -\sin \dfrac{\theta_{0j}}{2} \right) R\left(-\dfrac{\omega}{2} t \right), 1 \le j \le n, \\ \mathbf{V}_{ij}(t) = 2\omega \dfrac{\kappa^3 \sqrt{r_{ij}^3}}{1+\epsilon \sum_{k=1}^{n} \mu_k} \left(-\cos \dfrac{\theta_{ij}}{2}, -\sin \dfrac{\theta_{ij}}{2} \right) R\left(-\dfrac{\omega}{2} t \right), 1 \le i < j \le n. \end{cases}$$

Through the above relations, the Hamiltonian (7) leads to

$$\begin{aligned} H_{\text{c-LC}} = & -\frac{\epsilon}{2\kappa^2} \sum_{j=1}^{n} \left(2 - \omega^2 \frac{\kappa^6}{1 + \epsilon \sum_{k=1}^{n} \mu_k} \right) \mu_j \\ & + \frac{2\epsilon^2}{\kappa^2} \sum_{i=1}^{n} \sum_{j=i+1}^{n} \mu_i \mu_j \left(\omega^2 \frac{\kappa^6 \sin^2 \frac{\theta_{0j} - \theta_{0i}}{2}}{1 + \epsilon \sum_{k=1}^{n} \mu_k} - \frac{1}{8 \sin^2 \frac{\theta_{0j} - \theta_{0i}}{2}} \right). \end{aligned} \tag{35}$$

Because both values of $H_{\text{d-LC}}$ (33) and $H_{\text{c-LC}}$ (35) are equivalent to the value of h_{LC} (7) (see Theorem 1 in [**10**]),

$$\frac{32 s^2}{(1 - s^2)^2} \frac{1}{(\Delta t)^2} = \frac{\omega^2}{2}.$$

Accordingly, we have the relation between ω and Ω given by (31). □

With the help of THEOREM 3.2 and LEMMA 3.3, we find a non-degenerate equilibrium solution for the rewritten G$(1+n)$-BP (18) by using the following theorem.

THEOREM 3.4. *(Coorbital Configurations of G$(1+n)$-BP)* *We assume that* $0 < \theta_{0i} < \theta_{0j} \le 2\pi$ *for all* $1 \le i < j \le n$. *Let*

$$m_0 = 1, \quad m_i = \epsilon \mu_i, \quad 1 \le i \le n,$$

and

$$
\begin{cases}
\mathbf{X}_{0j}(t) = \boldsymbol{\mathcal{X}}_{0j} = \kappa \left(-\sin\dfrac{\theta_{0j}}{2}, \cos\dfrac{\theta_{0j}}{2} \right), \ 1 \le j \le n, \\
\mathbf{X}_{ij}(t) = \boldsymbol{\mathcal{X}}_{ij} = \kappa\sqrt{r_{ij}} \left(-\sin\dfrac{\theta_{ij}}{2}, \cos\dfrac{\theta_{ij}}{2} \right), \quad 1 \le i < j \le n, \\
\mathbf{W}_{0j}(t) = \boldsymbol{\mathcal{W}}_{0j} = 2\omega \dfrac{\kappa^3}{1+\epsilon\sum_{k=1}^n \mu_k} \left(-\cos\dfrac{\theta_{0j}}{2}, -\sin\dfrac{\theta_{0j}}{2} \right), \ 1 \le j \le n, \\
\mathbf{W}_{ij}(t) = \boldsymbol{\mathcal{W}}_{ij} = 2\omega \dfrac{\kappa^3}{1+\epsilon\sum_{k=1}^n \mu_k} \sqrt{r_{ij}^3} \left(-\cos\dfrac{\theta_{ij}}{2}, -\sin\dfrac{\theta_{ij}}{2} \right), \ 1 \le i < j \le n,
\end{cases}
\tag{36}
$$

be a non-degenerate equilibrium solution of the rewritten G$(1+n)$*-BP (18) with the angular velocity* ω. ϵ *is a small positive value, and* κ *is defined by*

$$
\kappa = \omega^{1/3} \left(1 + \epsilon \sum_{k=1}^{n} \mu_k \right)^{1/6}. \tag{37}
$$

Then, the angles θ_{0j}, $1 \le j \le n$ *satisfy the following identities:*

$$
\begin{aligned}
0 = &\sum_{i=1}^{j-1} \mu_i \left(1 - \frac{1}{8\sin^3 \dfrac{\theta_{0j}-\theta_{0i}}{2}} \right) \sin(\theta_{0j}-\theta_{0i}) \\
&- \sum_{i=j+1}^{n} \mu_i \left(1 - \frac{1}{8\sin^3 \dfrac{\theta_{0i}-\theta_{0j}}{2}} \right) \sin(\theta_{0i}-\theta_{0j}), \ 1 \le j \le n.
\end{aligned}
\tag{38}
$$

Proof. Eq. (31) in LEMMA 3.3 describes the relation between the angular velocity Ω in the rewritten d-G$(1+n)$-BP (14) and the angular velocity ω in the rewritten G$(1+n)$-BP (18). Therefore, substitution of (31) into (24) and (25), which describes a coorbital configuration of the d-G$(1+n)$-BP, leads to (36). According to LEMMA 3.3 (ii), the rewritten G$(1+n)$-BP (18) has the same angles θ_{ij}, $0 \le i < j \le n$ as the rewritten d-G$(1+n)$-BP (14), so these angles satisfy (38). □

THEOREM 3.5. *(Identity for Equilibria of G*$(1+n)$*-BP and that of d-G*$(1+n)$*-BP)*
Suppose that

(i) $m_0 = 1$, $m_i = \epsilon\mu_i$, $1 \le i \le n$, *where* ϵ *is small positive.*

(ii) The initial conditions are

$$
\mathbf{Q}_{0j}(0) = \mathbf{Q}_{0j}^{(0)} = \kappa \left(-\sin\frac{\theta_{0j}}{2}, \cos\frac{\theta_{0j}}{2} \right), \ 1 \le j \le n,
$$

$$\mathbf{Q}_{ij}(0) = \mathbf{Q}_{ij}^{(0)} = \kappa\sqrt{r_{ij}}\left(-\sin\frac{\theta_{ij}}{2}, \cos\frac{\theta_{ij}}{2}\right),\ 1 \le i < j \le n,$$

$$\mathbf{V}_{0j}(0) = \mathbf{V}_{0j}^{(0)} = 2\omega\frac{\kappa^3}{1+\epsilon\sum_{k=1}^{n}\mu_k}\left(-\cos\frac{\theta_{0j}}{2}, -\sin\frac{\theta_{0j}}{2}\right),\ 1 \le j \le n,$$

$$\mathbf{V}_{ij}(0) = \mathbf{V}_{ij}^{(0)} = 2\omega\frac{\kappa^3\sqrt{r_{ij}^3}}{1+\epsilon\sum_{k=1}^{n}\mu_k}\left(-\cos\frac{\theta_{ij}}{2}, -\sin\frac{\theta_{ij}}{2}\right),\ 1 \le i < j \le n,$$

where the angular velocity ω is arbitrary, κ is defined by (37), the angles $0 < \theta_{0i} < \theta_{0j} \le 2\pi$, $1 \le i < j \le n$ satisfy the identities (38), and r_{ij} and θ_{ij}, $1 \le i < j \le n$ fulfill (27) and (28), respectively.

Then, for both of the G$(1+n)$-BP (4) and the d-G$(1+n)$-BP (6),

(a) The mass m_0 is nearby the origin and the other masses $m_1, \cdots, m_n$ move near the unit circle whose center is near the origin at any time $t = l\Delta t$, $l = 0, 1, \cdots$.

(b) n masses $m_1, \cdots, m_n$ form a n-gon whose shape and size are invariant independent of time.

Further, both of the G$(1+n)$-BP (4) and the d-G$(1+n)$-BP (6) keep the same shape and size of a n-gon for the common initial condition (ii).

Proof. Using the position vectors in the inertial barycentric frame, we rewrite the coorbital configurations of the rewritten d-G$(1+n)$-BP in THEOREM 3.2 and those of the G$(1+n)$-BP in THEOREM 3.4. It follows that the n-gon given by the d-G$(1+n)$-BP (6) is congruent to that given by the G$(1+n)$-BP (4) for the common special initial condition.

1. **The n-gon configuration given by the d-G$(1+n)$-BP (6) in the inertial barycentric frame**
   In the proof of LEMMA 3.3, we have already given the discrete-time variables $\mathbf{Q}_{ij}^{(l)}$, $\mathbf{P}_{ij}^{(l)}$, $0 \le i < j \le n$, $l = 0, 1, \cdots$ as (32). Using (8) and (9), we rewrite (32) as follows:

$$\mathbf{q}_0'^{(l)} = \frac{\epsilon\kappa^2}{1+\epsilon\displaystyle\sum_{k=1}^{n}\mu_k}\sum_{j=1}^{n}\mu_j\left[-\cos(\theta_{0j}+\Omega l\Delta t),\ -\sin(\theta_{0j}+\Omega l\Delta t)\right],$$

$$\mathbf{v}_0'^{(l)} = \frac{8\epsilon\kappa^2 s}{(1-s^2)\left(1+\epsilon\displaystyle\sum_{k=1}^{n}\mu_k\right)\Delta t}\sum_{j=1}^{n}\mu_j\left[\sin(\theta_{0j}+\Omega l\Delta t),\ -\cos(\theta_{0j}+\Omega l\Delta t)\right],$$

$$\mathbf{q}_i^{\prime(l)} = \kappa^2 \begin{bmatrix} \dfrac{\cos(\theta_{0i}+\Omega l\Delta t)-\epsilon\left(\displaystyle\sum_{j=i+1}^{n}\mu_j r_{ij}\cos(\theta_{ij}+\Omega l\Delta t)\right)+\epsilon\left(\displaystyle\sum_{j=1}^{i-1}\mu_j r_{ji}\cos(\theta_{ji}+\Omega l\Delta t)\right)}{1+\epsilon\displaystyle\sum_{k=1}^{n}\mu_k} \\ \dfrac{\sin(\theta_{0i}+\Omega l\Delta t)-\epsilon\left(\displaystyle\sum_{j=i+1}^{n}\mu_j r_{ij}\sin(\theta_{ij}+\Omega l\Delta t)\right)+\epsilon\left(\displaystyle\sum_{j=1}^{i-1}\mu_j r_{ji}\sin(\theta_{ji}+\Omega l\Delta t)\right)}{1+\epsilon\displaystyle\sum_{k=1}^{n}\mu_k} \end{bmatrix}^{\top},$$

$$\mathbf{v}_i^{\prime(l)} = \kappa^2 \begin{bmatrix} -\dfrac{8s\left(\sin(\theta_{0i}+\Omega l\Delta t)-\epsilon\left(\displaystyle\sum_{j=i+1}^{n}\mu_j r_{ij}\sin(\theta_{ij}+\Omega l\Delta t)\right)+\epsilon\left(\displaystyle\sum_{j=1}^{i-1}\mu_j r_{ji}\sin(\theta_{ji}+\Omega l\Delta t)\right)\right)}{(1-s^2)\Delta t\left(1+\epsilon\displaystyle\sum_{k=1}^{n}\mu_k\right)} \\ \dfrac{8s\left(\cos(\theta_{0i}+\Omega l\Delta t)-\epsilon\left(\displaystyle\sum_{j=i+1}^{n}\mu_j r_{ij}\cos(\theta_{ij}+\Omega l\Delta t)\right)+\epsilon\left(\displaystyle\sum_{j=1}^{i-1}\mu_j r_{ji}\cos(\theta_{ji}+\Omega l\Delta t)\right)\right)}{(1-s^2)\Delta t\left(1+\epsilon\displaystyle\sum_{k=1}^{n}\mu_k\right)} \end{bmatrix}^{\top},$$

$$1 \le i \le n. \tag{39}$$

The relations (39) describe that the d-G$(1+n)$-BP (6) satisfies the property (a) if the condition (i) and (ii) are fulfilled. Since $0 < \theta_{0i} < \theta_{0j} \le 2\pi$, $1 \le i < j \le n$, the property (b) is also met. At $l = 0$, (39) leads to

$$\mathbf{q}_0^{\prime(0)} = \frac{\epsilon\kappa^2}{1+\epsilon\displaystyle\sum_{k=1}^{n}\mu_k}\sum_{j=1}^{n}\mu_j\left[-\cos\theta_{0j}, -\sin\theta_{0j}\right],$$

$$\mathbf{q}_i^{\prime(0)} = \kappa^2 \begin{bmatrix} \dfrac{\cos\theta_{0i} - \epsilon\displaystyle\sum_{j=i+1}^{n}\mu_j r_{ij}\cos\theta_{ij} + \epsilon\sum_{j=1}^{i-1}\mu_j r_{ji}\cos\theta_{ji}}{1+\epsilon\displaystyle\sum_{k=1}^{n}\mu_k} \\ \dfrac{\sin\theta_{0i} - \epsilon\displaystyle\sum_{j=i+1}^{n}\mu_j r_{ij}\sin\theta_{ij} + \epsilon\sum_{j=1}^{i-1}\mu_j r_{ji}\sin\theta_{ji}}{1+\epsilon\displaystyle\sum_{k=1}^{n}\mu_k} \end{bmatrix}^{\top}, \quad 1 \le i \le n,$$

$$\mathbf{v}_0^{\prime(0)} = \frac{8s\epsilon}{(1-s^2)\Delta t}\frac{\kappa^2}{1+\epsilon\displaystyle\sum_{k=1}^{n}\mu_k}\sum_{j=1}^{n}\mu_j\left[\sin\theta_{0j}, -\cos\theta_{0j}\right],$$

$$\mathbf{v}_i^{\prime(0)} = \kappa^2 \begin{bmatrix} -\dfrac{8s\left(\sin\theta_{0i} - \epsilon\displaystyle\sum_{j=i+1}^{n}\mu_j r_{ij}\sin\theta_{ij} + \epsilon\sum_{j=1}^{i-1}\mu_j r_{ji}\sin\theta_{ji}\right)}{(1-s^2)\Delta t\left(1+\epsilon\displaystyle\sum_{k=1}^{n}\mu_k\right)} \\ \dfrac{8s\left(\cos\theta_{0i} - \epsilon\displaystyle\sum_{j=i+1}^{n}\mu_j r_{ij}\cos\theta_{ij} + \epsilon\sum_{j=1}^{i-1}\mu_j r_{ji}\cos\theta_{ji}\right)}{(1-s^2)\Delta t\left(1+\epsilon\displaystyle\sum_{k=1}^{n}\mu_k\right)} \end{bmatrix}^{\top}, \quad 1 \le i \le n,$$

which corresponds to the initial values in the inertial barycentric frame. Through (2) and (3), these values reduce to

$$\begin{aligned}
\mathbf{Q}_{0j}^{(0)} &= \kappa\left(-\sin\frac{\theta_{0j}}{2}, \cos\frac{\theta_{0j}}{2}\right), \quad 1 \le j \le n, \\
\mathbf{Q}_{ij}^{(0)} &= \kappa\sqrt{r_{ij}}\left(-\sin\frac{\theta_{ij}}{2}, \cos\frac{\theta_{ij}}{2}\right), \quad 1 \le i < j \le n, \\
\mathbf{V}_{0j}^{(0)} &= \frac{16s}{(1-s^2)\Delta t}\frac{\kappa^3}{1+\epsilon\displaystyle\sum_{k=1}^{n}\mu_k}\left(-\cos\frac{\theta_{0j}}{2}, -\sin\frac{\theta_{0j}}{2}\right), \quad 1 \le j \le n, \\
\mathbf{V}_{ij}^{(0)} &= \frac{16s}{(1-s^2)\Delta t}\frac{\kappa^3\sqrt{r_{ij}^3}}{1+\epsilon\displaystyle\sum_{k=1}^{n}\mu_k}\left(-\cos\frac{\theta_{ij}}{2}, -\sin\frac{\theta_{ij}}{2}\right), \quad 1 \le i < j \le n.
\end{aligned} \tag{40}$$

2. **The n-gon configuration given by the G(1+n)-BP (4) in the inertial barycentric frame**
   The following $\mathbf{Q}_{ij}(t)$, $\mathbf{V}_{ij}(t)$, $0 \le i < j \le n$ satisfy the G$(1+n)$-BP (4):

$$\begin{aligned}
\mathbf{Q}_{0j}(t) &= \kappa\left(-\sin\left(\frac{\theta_{0j}+\omega t}{2}\right), \cos\left(\frac{\theta_{0j}+\omega t}{2}\right)\right), \quad 1 \le j \le n, \\
\mathbf{Q}_{ij}(t) &= \kappa\sqrt{r_{ij}}\left(-\sin\left(\frac{\theta_{ij}+\omega t}{2}\right), \cos\left(\frac{\theta_{ij}+\omega t}{2}\right)\right), \quad 1 \le i < j \le n, \\
\mathbf{V}_{0j}(t) &= 2\omega\frac{\kappa^3}{1+\epsilon\displaystyle\sum_{k=1}^{n}\mu_k}\left(-\cos\left(\frac{\theta_{0j}+\omega t}{2}\right), -\sin\left(\frac{\theta_{0j}+\omega t}{2}\right)\right), \quad 1 \le j \le n,
\end{aligned}$$

$$\mathbf{V}_{ij}(t)=2\omega\frac{\kappa^3\sqrt{r_{ij}^3}}{1+\epsilon\displaystyle\sum_{k=1}^{n}\mu_k}\left(-\cos\left(\frac{\theta_{ij}+\omega t}{2}\right),-\sin\left(\frac{\theta_{ij}+\omega t}{2}\right)\right),\ 1\leq i<j\leq n. \quad (41)$$

Through (8), (9) and (23), (41) leads to

$$\mathbf{q}_0'(t)=\frac{\epsilon\kappa^2}{1+\epsilon\displaystyle\sum_{k=1}^{n}\mu_k}\sum_{j=1}^{n}\mu_j\left[-\cos(\theta_{0j}+\omega t),-\sin(\theta_{0j}+\omega t)\right],$$

$$\mathbf{q}_i'(t)=\kappa^2\begin{bmatrix}\dfrac{\cos(\theta_{0i}+\omega t)-\epsilon\left(\displaystyle\sum_{j=i+1}^{n}\mu_j r_{ij}\cos(\theta_{ij}+\omega t)\right)+\epsilon\left(\displaystyle\sum_{j=1}^{i-1}\mu_j r_{ji}\cos(\theta_{ji}+\omega t)\right)}{1+\epsilon\displaystyle\sum_{k=1}^{n}\mu_k}\\ \dfrac{\sin(\theta_{0i}+\omega t)-\epsilon\left(\displaystyle\sum_{j=i+1}^{n}\mu_j r_{ij}\sin(\theta_{ij}+\omega t)\right)+\epsilon\left(\displaystyle\sum_{j=1}^{i-1}\mu_j r_{ji}\sin(\theta_{ji}+\omega t)\right)}{1+\epsilon\displaystyle\sum_{k=1}^{n}\mu_k}\end{bmatrix}^{\top},$$

$$1\leq i\leq n,$$

$$\mathbf{v}_0'(t)=\frac{\epsilon\kappa^2\omega}{1+\epsilon\displaystyle\sum_{k=1}^{n}\mu_k}\sum_{j=1}^{n}\mu_j\left[\sin(\theta_{0j}+\omega t),-\cos(\theta_{0j}+\omega t)\right],$$

$$\mathbf{v}_i'(t)=\kappa^2\begin{bmatrix}-\dfrac{\omega\left(\sin(\theta_{0i}+\omega t)-\epsilon\left(\displaystyle\sum_{j=i+1}^{n}\mu_j r_{ij}\sin(\theta_{ij}+\omega t)\right)+\epsilon\left(\displaystyle\sum_{j=1}^{i-1}\mu_j r_{ji}\sin(\theta_{ji}+\omega t)\right)\right)}{1+\epsilon\displaystyle\sum_{k=1}^{n}\mu_k}\\ \dfrac{\omega\left(\cos(\theta_{0i}+\omega t)-\epsilon\left(\displaystyle\sum_{j=i+1}^{n}\mu_j r_{ij}\cos(\theta_{ij}+\omega t)\right)+\epsilon\left(\displaystyle\sum_{j=1}^{i-1}\mu_j r_{ji}\cos(\theta_{ji}+\omega t)\right)\right)}{1+\epsilon\displaystyle\sum_{k=1}^{n}\mu_k}\end{bmatrix}^{\top},$$

$$1\leq i\leq n. \quad (42)$$

Eq. (42) means that the G$(1+n)$-BP (4) fulfills the property (a) under the conditions (i) and (ii). Also, the G$(1+n)$-BP (4) satisfies the property (b) because of $0<\theta_{0i}<\theta_{0j}\leq 2\pi$, $1\leq i<j\leq n$.

At the time $t=0$, the solution of the d-G$(1+n)$-BP, (41) reduces to

$$\mathbf{Q}_{0j}(0)=\kappa\left(-\sin\frac{\theta_{0j}}{2},\cos\frac{\theta_{0j}}{2}\right),\ 1\leq j\leq n,$$

$$\mathbf{Q}_{ij}(0) = \kappa\sqrt{r_{ij}}\left(-\sin\frac{\theta_{ij}}{2}, \cos\frac{\theta_{ij}}{2}\right),\ 1 \le i < j \le n,$$

$$\mathbf{V}_{0j}(0) = 2\omega\frac{\kappa^3}{1+\epsilon\sum_{k=1}^{n}\mu_k}\left(-\cos\frac{\theta_{0j}}{2}, -\sin\frac{\theta_{0j}}{2}\right),\ 1 \le j \le n,$$

$$\mathbf{V}_{ij}(0) = 2\omega\frac{\kappa^3\sqrt{r_{ij}^3}}{1+\epsilon\sum_{k=1}^{n}\mu_k}\left(-\cos\frac{\theta_{ij}}{2}, -\sin\frac{\theta_{ij}}{2}\right),\ 1 \le i < j \le n.$$

Each r.h.s is congruent with that of (41) in the case that the relation between ω and Ω is given by (31) in LEMMA 3.3. This fact corresponds to the condition (ii).

Accordingly, we can clarify that if the G$(1+n)$-BP (4) and the d-G$(1+n)$-BP (6) satisfy the common conditions (i) and (ii), the $n+1$ masses of the G$(1+n)$-BP and those of the d-G$(1+n)$-BP form the same invariant n-gon. $\square$

THEOREM 3.5 describes that the d-G$(1+n)$-BP has the same central configurations as the G$(1+n)$-BP for the common initial conditions. Further, these configurations of both the G$(1+n)$-BP and d-G$(1+n)$-BP with $m_0 = 1$ and $m_i = \epsilon\mu_i$, $1 \le i \le n$ converge to those of both the R$(1+n)$-BP and its discrete version as $\epsilon \to 0$, respectively. Thus, in the case of $\epsilon \to 0$, the d-G$(1+n)$-BP can reproduce the central configurations of the R$(1+n)$-BP in [**3, 4, 11, 13**].

4. Numerical results

Many researchers have devoted to only a special case that n infinitesimal masses do not influence a massive primary. In terms of numerical results, for $n \le 9$, Salo and Yoder [**13**] first gave only central configurations of the G$(1+n)$-BP with equal infinitesimal masses. Further, Cors, Llibre and Olle [**4**] have checked the numerical results of [**13**] and after they have explored bigger values of n up to 15. Casasayas, Llibre and Nunes clarified that there exists only one solution for n large enough if all n infinitesimal masses are equal [**3**]. Cors, Llibre and Olle [**4**] conjectured that there are some central configurations except the regular n-gon only for $n \le 8$ (see Conjecture 6 in [**4**]).

On the other hand, Renner and Sicardy [**11**] generalized the work by Salo and Yode [**13**] to the case of masses with infinitesimal but arbitrary (not necessarily equal) masses. They verified that there always exists a set of masses which defines a central configuration. If n is odd, then for any arbitrary angular separation between the small masses. If n is even, then for given angular separations of the small masses, there is, in general, no set of masses which defines a central configuration.

All of their works are related to the condition that n arbitrary infinitesimal masses do not influence a massive primary. In the preceding section, we first considered the G$(1+n)$-BP under the condition that n arbitrary small masses influence a massive primary. Moreover, we proved that the d-G$(1+n)$-BP has the same central configuration as the G$(1+n)$-BP under the condition. In this section, we compare the results obtained by (i) the discrete-time general $(1+n)$-body problem (d-G$(1+n)$-BP) and (ii) the second-order symplectic method (SI2), both of which are accurate to second order.

In Section 4.1, using (29) for $n = 2, 3, 4$, we numerically give some sets of angles of non-trivial central configurations of the d-G$(1+n)$-BP in which n equal small masses influence a massive primary. We also clarify that the d-G$(1+n)$-BP can correctly compute the equilibrium configurations. In Section 4.2, for $n = 2, 3, 4$, we obtain from (29) some angles of non-trivial central configurations of the d-G$(1+n)$-BP in which $n-1$ equal small masses and a small mass influence a massive primary. We also numerically ascertain that the d-G$(1+n)$-BP exactly compute the equilibrium configurations for a long time interval.

4.1. Central configurations with equal n masses

Let us show that the d-G$(1+n)$-BP accurately reproduces the equilibrium solutions of all masses in the rotating reference frame with a uniform angular velocity, where $m_0 = 1$ and $m_1 = \cdots = m_n = 10^{-8}$. Eq. (29) is the same relation as (2) in [**4**] which the R$(1+n)$-BP satisfies. Therefore, THEOREM 3.5 ensures that the d-G$(1+n)$-BP has the same equilibrium solutions as those of the R$(1+n)$-BP in [**4**]. The computed values (in degrees) of the angles θ_{0i}, $i = 1, \cdots, n$, for the central configurations of the d-G$(1+2)$-BP, d-G$(1+3)$-BP and d-G$(1+4)$-BP (we do not write the trivial solution of equally spaced angles) are given in the following tables.

Table 1. Non-trivial central configuration of d-G$(1+2)$-BP with equal two masses. The central configuration is identified by the strings in the top row.

#	$2E-1$
θ_{01}	60
θ_{02}	360

Table 2. Non-trivial central configurations of d-G$(1+3)$-BP with equal three masses. The central configurations are identified by the strings in the top row.

#	$3E-1$	$3E-2$
θ_{01}	47.3608595705276757	82.4690381114333712
θ_{02}	94.7217191410553653	221.2345190557166856
θ_{03}	360	360

Table 3. Non-trivial central configurations of d-G(1+4)-BP with equal four masses. The central configurations are identified by the strings in the top row.

#	$4E-1$	$4E-2$
θ_{01}	59.9999999999999883	239.6486503921392379
θ_{02}	119.9999999999999906	281.1463711332769163
θ_{03}	239.9999999999999953	318.5022792588623216
θ_{04}	360	360

Tables 1, 2 and 3 in this article correspond to Tables 1, 2 and 3 in [**4**], respectively. The initial conditions for two equilibrium solutions $3E-1$ and $4E-2$ are as follows.

1. Initial conditions of the d-G(1 + 3)-BP for the equilibrium solution $3E-1$

$$
\begin{aligned}
&m_0 = 1,\ m_1 = m_2 = m_3 = 10^{-8},\\
&\mathbf{q}'_0(0) = (-0.0000000159506233,\ -0.0000000173224074),\\
&\mathbf{q}'_1(0) = (0.6773786585173932,\ 0.7356344968700003),\\
&\mathbf{q}'_2(0) = (-0.0823162937159855,\ 0.9966062274514275),\\
&\mathbf{q}'_3(0) = (0.9999999659506244,\ 0.0000000173224072),\\
&\mathbf{v}'_0(0) = (0.0000000173224074,\ -0.0000000159506233),\\
&\mathbf{v}'_1(0) = (-0.7356344968700003,\ 0.6773786585173932),\\
&\mathbf{v}'_2(0) = (-0.9966062274514275,\ -0.0823162937159855),\\
&\mathbf{v}'_3(0) = (-0.0000000173224072,\ 0.9999999659506244).
\end{aligned}
$$

2. Initial conditions of the d-G(1 + 4)-BP for the equilibrium solution $4E-2$

$$
\begin{aligned}
&m_0 = 1,\ m_1 = m_2 = m_3 = m_4 = 10^{-8},\\
&\mathbf{q}'_0(0) = (-0.0000000143699692,\ 0.0000000250666975),\\
&\mathbf{q}'_1(0) = (-0.5053012042164427,\ -0.8629430054946241),\\
&\mathbf{q}'_2(0) = (0.1933160962931250,\ -0.9811364974200589),\\
&\mathbf{q}'_3(0) = (0.7489820802375094,\ -0.6625902243207682),\\
&\mathbf{q}'_4(0) = (0.9999999477033045,\ -0.0000000250666978),\\
&\mathbf{v}'_0(0) = (-0.0000000250666975,\ -0.0000000143699692),\\
&\mathbf{v}'_1(0) = (0.8629430054946241,\ -0.5053012042164427),\\
&\mathbf{v}'_2(0) = (0.9811364974200589,\ 0.1933160962931250),\\
&\mathbf{v}'_3(0) = (0.6625902243207682,\ 0.7489820802375094),\\
&\mathbf{v}'_4(0) = (0.0000000250666978,\ 0.9999999477033045).
\end{aligned}
$$

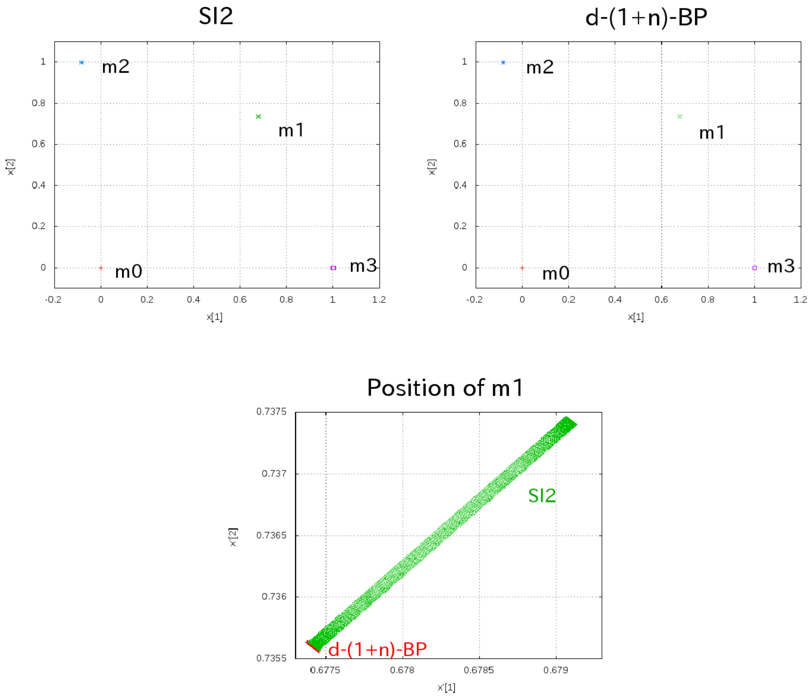

Figure 1. Trajectories of m_0, m_1, m_2, and m_3 in uniformly rotating reference frame $O - x'_{[1]}x'_{[2]}$. Trajectories around equilibria correspond to $3E-1$. Used numerical methods are given at the tops of the upper left and upper right figures. Top of the lower figure means the trajectory of m_1 around its equilibrium computed by SI2 and d-G$(1+n)$-BP methods.

For the d-G$(1+n)$-BP, we introduce a uniformly rotating frame $O - x'_{[1]}x'_{[2]}$ in which the origin stays at the center of mass, and the $x'_{[1]}$ axis passes through the origin and the small mass m_n. Theoretically, all the positions of these masses are fixed in the frame $O - x'_{[1]}x'_{[2]}$. Applying the d-G$(1+n)$-BP and SI2 methods yield all the trajectories of the masses for the configuration $3E-1$ shown in Figure 1 and those for the configuration $4E-2$ shown in Figure 2. We used a fixed time step $\Delta t = 0.1$ for all these methods and integrated over the time interval $0 \leq t \leq 10^4$. The two figures show that the d-G$(1+n)$-BP and SI2 methods seem to reproduce the configurations $3E-1$ and $4E-2$ accurately. In detail, these results indicate that each mass vibrates both in the radius and tangential directions around its own equilibrium for the SI2 method, whereas all masses oscillate only in the tangential direction around their own equilibrium for the d-G$(1+n)$-BP method. For the configurations $2E-1$, $3E-2$ and $4E-1$, we also obtain similar results. Especially, the configuration $2E-1$ corresponds to a Lagrange solution of the G3BP. Minesaki [**8**] had already proved that the d-G3BP [**7**] had the same orbits for the Lagrange equilibrium solutions (e.g., [**5, 12**]) as the original G3BP. Actually, each mass in the d-G3BP can be fixed at its own equilibrium in the original G3BP. Therefore, the d-G$(1+n)$-BP method can compute all of the equilibrium solutions of the Tables 1-3 more correctly than the SI2 method.

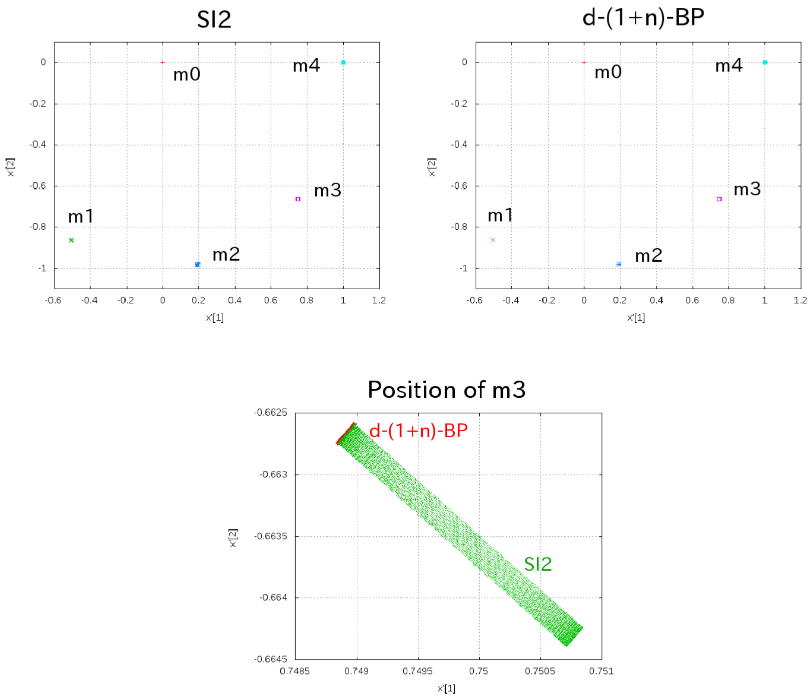

Figure 2. Trajectories of m_0, m_1, m_2, m_3 and m_4 in uniformly rotating reference frame $O-x'_{[1]}x'_{[2]}$. Trajectories around equilibria correspond to $4E-2$. Used numerical methods are given at the tops of the upper left and upper right figures. Top of the lower figure means the trajectory of m_3 around its equilibrium computed by SI2 and d-G$(1+n)$-BP methods.

4.2. Central configurations with arbitrary n masses

In this section, we numerically check that the d-G$(1+n)$-BP can precisely compute some equilibrium solutions, each of which corresponds to an equilibrium point of the original G$(1+n)$-BP in a uniformly rotating reference frame $O-x'_{[1]}x'_{[2]}$. In the case of $m_0 = 1$, $m_1 = \cdots = m_{n-1} = 10^{-8}$ and $m_n = 10^{-10}$, $n = 3, 4$, we numerically compute the angles θ_{0i} in (29) for the central configurations of the d-G(1+3)-BP and d-G(1+4)-BP in Tables 4 and 5. Tables 4 and 5 give the more

Table 4. Non-trivial central configurations of d-G(1+3)-BP with $m_0 = 1$, $m_1 = m_2 = 10^{-8}$ and $m_3 = 10^{-10}$. The central configurations are identified by the strings in the top row.

#	$3A-1$	$3A-2$
θ_{01}	54.8390577888207782	59.8222243848607571
θ_{02}	66.7497713048119662	300.1777756151392288
θ_{03}	360	360

Table 5. Non-trivial central configurations of d-G(1+4)-BP with $m_0 = 1$, $m_1 = m_2 = m_3 = 10^{-8}$ and $m_4 = 10^{-10}$. The central configurations are identified by the strings in the top row.

#	$4A-1$	$4A-2$
θ_{01}	51.4629379775934644	54.6747821275699012
θ_{02}	61.2876368829233543	66.5585870433995897
θ_{03}	71.9068305993430059	300.3496112735630209
θ_{04}	360	360

accurate angles in Figures 5 and 6 in [**11**], respectively. The initial conditions for two equilibrium solutions 3A-1 and 4A-2 are as follows.

1. Initial conditions of the d-G(1 + 3)-BP for the equilibrium solution $3A-1$

$$
\begin{aligned}
&m_0 = 1,\ m_1 = m_2 = 10^{-10},\ m_3 = 10^{-8},\\
&\mathbf{q}_0'(0) = (-0.0000000100970622,\ -0.0000000001736327),\\
&\mathbf{q}_1'(0) = (0.5758751455577946,\ 0.8175376411215560),\\
&\mathbf{q}_2'(0) = (0.3947475270716797,\ 0.9187896191022460),\\
&\mathbf{q}_3'(0) = (0.9999999930970623,\ 0.0000000001736325),\\
&\mathbf{v}_0'(0) = (0.0000000001736327,\ -0.0000000100970622),\\
&\mathbf{v}_1'(0) = (-0.8175376411215560,\ 0.5758751455577946),\\
&\mathbf{v}_2'(0) = (-0.9187896191022460,\ 0.3947475270716797),\\
&\mathbf{v}_3'(0) = (-0.0000000001736325,\ 0.9999999930970623).
\end{aligned}
$$

2. Initial conditions of the d-G(1 + 4)-BP for the equilibrium solution $4A-1$

$$
\begin{aligned}
&m_0 = 1,\ m_1 = m_2 = m_3 = 10^{-10},\ m_4 = 10^{-8},\\
&\mathbf{q}_0'(0) = (-0.0000000101413996,\ -0.0000000002609801),\\
&\mathbf{q}_1'(0) = (0.6230207393690067,\ 0.7822053036503321),
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{q}_2'(0) &= (0.4804127562163634,\ 0.8770425072628336),\\
\mathbf{q}_3'(0) &= (0.3105631150271591,\ 0.9505527466051420),\\
\mathbf{q}_4'(0) &= (0.9999999929747331,\ 0.0000000002609798),\\
\mathbf{v}_0'(0) &= (0.0000000002609801,\ -0.0000000101413996),\\
\mathbf{v}_1'(0) &= (-0.7822053036503321,\ 0.6230207393690067),\\
\mathbf{v}_2'(0) &= (-0.8770425072628336,\ 0.4804127562163634),\\
\mathbf{v}_3'(0) &= (-0.9505527466051420,\ 0.3105631150271591),\\
\mathbf{v}_4'(0) &= (-0.0000000002609798,\ 0.9999999929747331).
\end{aligned}
$$

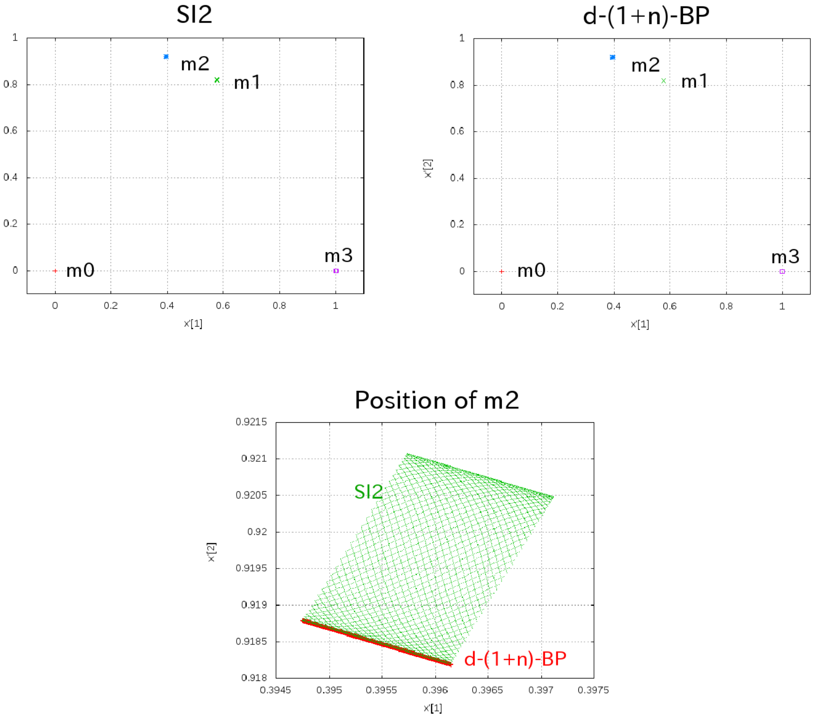

Figure 3. Trajectories of m_0, m_1, m_2, and m_3 in uniformly rotating reference frame $O-x'_{[1]}x'_{[2]}$. Trajectories around equilibria correspond to $3A-1$. Used numerical methods are given at the tops of the upper left and upper right figures. Top of the lower figure means the trajectory of m_2 around its equilibrium computed by SI2 and d-G$(1+n)$-BP methods.

Theoretically, each mass stays at a fixed point in the frame $O-x'_{[1]}x'_{[2]}$. Applying the d-G$(1+n)$-BP and SI2 methods, we obtain all the trajectories of the masses for the configuration $3A-1$ (see Figure 3) and those for the configuration $4A-1$ (see

Figure 4) in the rotating frame $O - x'_{[1]}x'_{[2]}$. Using the fixed $\Delta t = 0.1$, we integrated over the time interval $0 \leq t \leq 10^4$. In Figures 3 and 4, the d-G$(1+n)$-BP and SI2 methods seem to reproduce precisely the configurations $3A-1$ and $4A-1$. However, as in the case of equal n masses, each mass vibrates slightly both in the radius and tangential directions around its own equilibrium for the SI2 method, whereas all masses oscillate inconsiderably only in the tangential direction around their own equilibrium for the d-G$(1+n)$-BP method. Further, similar results are produced for the configurations $3A-2$ and $4A-2$. Thus, the d-G$(1+n)$-BP method can numerically give all of the equilibrium solutions of the Tables 4 and 5 more accurately than the SI2 method. The results given in Sections 4.1 and 4.2 mean that the d-G$(1+n)$-BP method can precisely compute some equilibria in the original G$(1+n)$-BP.

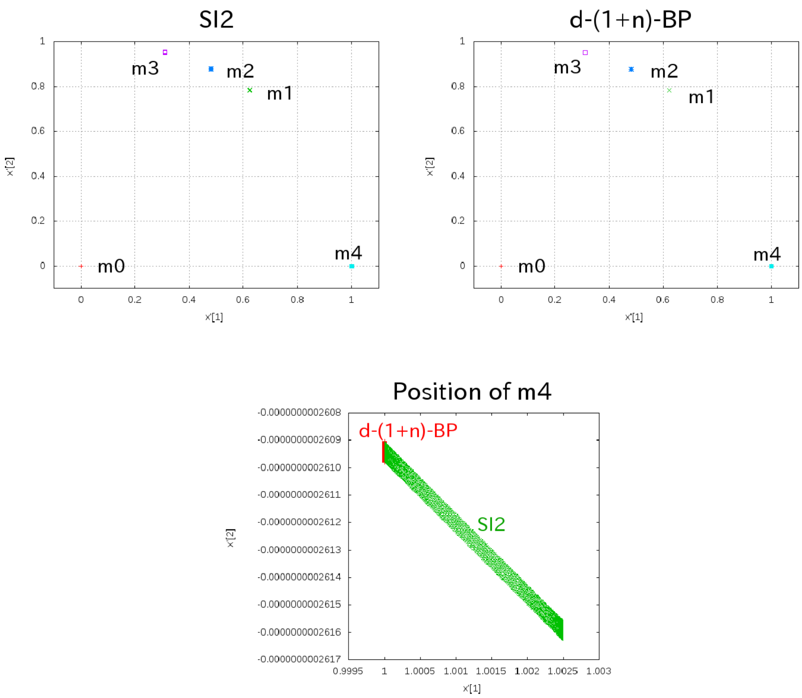

Figure 4. Trajectories of m_0, m_1, m_2, m_3 and m_4 in uniformly rotating reference frame $O - x'_{[1]}x'_{[2]}$. Trajectories around equilibria correspond to $4A-1$. Used numerical methods are given at the tops of the upper left and upper right figures. Top of the lower figure means the trajectory of m_4 around its equilibrium computed by SI2 and d-G$(1+n)$-BP methods.

5. Conclusion

As a special case of the discrete-time general N-body problem [**10**], we designed an integrator for the general $(1+n)$-BP, which we call the d-G$(1+n)$-BP. We analytically clarified for the first time that the d-G$(1+n)$-BP has the same equilibria as the G$(1+n)$-BP in a uniformly rotating frame. These equilibria involve those of the R$(1+n)$-BP in [**3, 4, 11, 13**]. Until the proof in this work, no equilibrium solution of the G$(1+n)$-BP has been known and there has been no discrete analog preserving the equilibrium solutions. Although the d-G$(1+n)$-BP is merely second-order accurate, in the case of $n = 2, 3, 4$, the d-G$(1+n)$-BP can precisely compute these equilibria for a long time interval.

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