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Reaction-diffusion models with a conservation law and pattern formations

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Abstract. A two-component reaction-diffusion system with a mass conservation property is given as a model of the cell polarity. We first review mathematical aspects of the model system. Then based on it, we provide an extended system of three components with the mass conservation, which is regarded as a perturbed system of the two component one when the coupling parameter is small. We show that the system possesses a unique positive constant steady state under a certain condition on the total mass. Then numerical simulations subject to the periodic boundary condition exhibit coexistence of two stable solutions that are the constant steady state and a single spike solution. Moreover, in the transient dynamics Turing-like patterns emerge, though no diffusion driven instability for the constant steady state takes place.

1. Introduction

In a various fields of sciences, including chemistry, biology and ecology, Reaction-diffusion systems play important roles in theoretical studies for pattern formations and dynamics. Those model equations have been extensively studied in mathematical literatures ([**20**], [**18**], [**17**], [**10**], [**24**] and references therein). One of the most characteristic aspects of the reaction-diffusion models is emergence of spatial patterns by Turing mechanism, that is, a uniform steady state of the model equations becomes unstable in the presence of diffusions, which is called diffusion driven instability, and the instability induces spatially structured patterns (see [**23**], [**14**] and [**17**]).

As a model exhibiting the Turing-type instability, Otsuji et al. [**19**] and Ishihara et al [**6**] proposed a mass-conserved reaction-diffusion system:

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1 \dfrac{\partial^2 u}{\partial x^2} - f(u,v), \\ \\ \dfrac{\partial v}{\partial t} = d_2 \dfrac{\partial^2 v}{\partial x^2} + f(u,v), \end{cases} \qquad x \in (0,L), \tag{1}$$

with periodic boundary conditions. Here u stands for the concentration of active protein in the membrane while v for inactive protein in the cytosol in a cell. Unlike the standard activator-inhibitor model, This system has a conservation law such that the total mass is conserved, i.e.,

$$\int_0^L (u(x,t)+v(x,t))dx = \text{constant}.$$

By this conservation the linearized eigenvalue problem around a uniform steady state always allows zero eigenvalue. Nevertheless, for an appropriately chosen f there is a steady state that is stable in the diffusion free system and it becomes unstable by the diffusions. Here the stability is meant by the Lyapunov's sense, which is stated precisely in the end of this section. In [**19**] the authors numerically show that Turing-like wave patterns certainly emerge and a sinusoidal pattern grows to be a spiky pattern in a certain time interval. More interesting simulations in [**19**] exhibit that after a long time those spikes break down until a single spike remains. Namely, the multi-spike pattern can be transiently observed but eventually the spikes disappears except for one spike.

Motivated by this transient dynamics, for a class of functions f including the specific examples in [**19**] and [**6**], mathematical studies have been developed in [**7, 15, 16**]. Those mathematical results show that every solution converges to a set of equilibria and that every stable equilibrium must be constant or unimodal in the case of the periodic boundary conditions (see also [**12**] and [**11**]).

The simple models in [**19**] exhibit an accumulation of biochemical subtract (protein), therefore they call them conceptual models describing the polarization in cells (see also [**13**]). In a realistic situation in cells such a conservation law can be observed in a certain time scale. For instance, proteins would be removed gradually by degradation. In such a case no longer mass conservation holds in a longer time scale. This leads to a problem on how the dynamics is affected by a perturbation to the mass conserved system. A recent study in [**8**] shows that a perturbed system with no conservation property still exhibit the emergence of the Turing-like pattern in transient dynamics, though the final state is a trivial uniform state. In fact, by a small perturbation to the system one can break both the conservation property and the diffusion driven instability so that the perturbed system has a globally asymptotically stable constant steady state. Nevertheless, in transient dynamics the Turing-like wave patterns can be observed.

In this article we propose an extended reaction-diffusion system with three components. This can be regarded as a perturbed system to the above mass conserved system when the coupling constant is small. Unlike the perturbed model mentioned above, our new model still has a conservation property. Our first study for the extended system is to show by a simple computation that a unique constant solution never looses stability in the presence of the diffusion for the total mass in a suitable range. Then we numerically show the emergence of the Turing-like patterns for suitable initial data and that those solutions converge to a stable single spike pattern after a long time. This implies the coexistence of the stable constant steady state and the stable single spike pattern.

We may point out the following implication of our numerics. In a standard theory of the Turing instability a constant steady state, which is stable for spatially uniform perturbation, looses the stability by diffusion and the resulting instability induces wave patterns. The above numerics, however, suggests that for the emergence of Turing-like wave patterns it is not necessary to destabilize a constant steady state by diffusion. This phenomenon is reported in [**8**] for a perturbed system as mentioned before, but in our numerics the emergence of Turing-like wave patterns is observed for not only small values of the coupling constant but also large values. We hope our result would provide a new view into pattern formation phenomena and modeling for them.

In the next section we treat the specific model equations proposed by [**19**] and state mathematical results proved in [**15**] together with new simple results (Propositions 2.2 and Corollary 2.3). In §3 we consider the extended system mentioned above. Based on the (u, v) system in §2, we add w equation coupled with v equation so that the total mass $\int_\Omega (u+v+w)\,dx$ is preserved. When the couple strength is small, this system is a perturbed system of the (u, v) system. We set parameter values for which a unique constant steady state is stable even if in the presence of diffusions. Nonetheless, in numerics, there are solutions which exhibit Turing-like patterns and converges to a single spike after a long transient time. In the final section we summarize the results.

Before concluding the section we state the definition of the stability ([**4**], [**5**]). Let $\boldsymbol{u}(x,t;\boldsymbol{u}_0) = (u(x,t), v(x,t))$ be a solution to (1) with $\boldsymbol{u}(x,0;\boldsymbol{u}_0) = \boldsymbol{u}_0(x) = (u_0(x), v_0(x))$. Choosing an appropriate space X (for instance, $X = H^1(\Omega)\times H^1(\Omega)$ for f satisfying some condition, see [**16**]), we can define the semiflow $S(t)$ in a phase space X (with norm $\|\cdot\|$) by

$$[S(t)\boldsymbol{u}_0](x) = \boldsymbol{u}(x,t;\boldsymbol{u}_0), \quad \boldsymbol{u}_0 \in X.$$

We say an equilibrium $\boldsymbol{u}^*$ is stable if for any $\varepsilon > 0$, there is some $\delta > 0$ such that

$$\|S(t)\boldsymbol{u}_0 - \boldsymbol{u}^*\| < \varepsilon \quad (\forall t > 0), \quad \text{if} \quad \|\boldsymbol{u}_0 - \boldsymbol{u}^*\| < \delta, \quad \boldsymbol{u}_0 \in X.$$

This stability is called the Lyapunov stability. This definition is also applied to the three component case with a slight modification. Throughout the present article we use this notion for the stability.

We remark that as for the conserved system we cannot expect the asymptotic stability for an equilibrium but the above Lyapunov stability works.

2. A mass conserved reaction-diffusion system

We consider the following model equations:

$$\begin{cases} u_t = d\Delta u - g(u) + v, \\ \tau v_t = \Delta v + g(u) - v, \end{cases} \qquad x \in \Omega, \tag{2}$$

with the Neumann boundary condition

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \qquad x \in \partial\Omega, \tag{3}$$

where Ω is a bounded domain of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and d is positive parameter. We assume g is a smooth function and for a constant $M_1 > 0$,

$$0 < g(u) < M_1 \quad (0 < u < \infty), \qquad g(0) = 0. \tag{4}$$

Under this condition there is a unique nonnegative classical solution satisfying

$$(u(x,0), v(x,0)) = (u_0(x), v_0(x)), \quad u_0, v_0 \in C^0(\overline{\Omega}), \quad u_0(x) \geq 0, v_0(x) \geq 0 \ (x \in \overline{\Omega}).$$

Put

$$s := \langle u(\cdot,t)\rangle + \tau\langle v(\cdot,t)\rangle, \qquad \langle \phi \rangle := \frac{1}{|\Omega|}\int_\Omega \phi(x)dx \tag{5}$$

which is conserved for t, where $|\Omega|$ stands for the Lebesgue measure of Ω.

For an appropriate g it is known that the system exhibit a Turing-type instability. For instance,

$$g(u) = \frac{au}{u^2 + b} \tag{6}$$

is a specific example, where a and b are positive constant ([**6**], [**19**]). More precisely, for this specific $g(u)$ with $a = 1, b \leq 1/8, \tau = 1$ and $s \geq 2$, a simple computation tells that there exists a unique positive constant equilibrium and it loses stability for sufficiently small d.

Henceforth we assume

$$\tau d < 1.$$

Put

$$\alpha := \sqrt{\tau(1 - \tau d)}, \quad \beta := \alpha/\tau.$$

We write the equations (2) with the new variable $w = (u + \tau v)/\alpha$ as

$$\begin{cases} u_t = d\Delta u - g(u) - u/\tau + \beta w, \\ \tau w_t = \Delta w - \beta \Delta u, \end{cases} \qquad x \in \Omega. \tag{7}$$

Then (5) turns to be

$$\langle w \rangle = s/\alpha. \tag{8}$$

As seen in [**16**] and [**15**] the system possesses a Lyapunov function defined by

$$\mathcal{E}(u, w) := \int_\Omega \left(\frac{d}{2}|\nabla u|^2 + G(u) + \frac{d}{2}u^2 + \frac{1}{2}(w - \beta u)^2 \right) dx, \qquad G(u) := \int_0^u g(u)du.$$

In fact it is easy to check

$$\frac{d}{dt}\mathcal{E}(u(\cdot, t), w(\cdot, t)) = -\int_\Omega |u_t|^2 dx - \frac{1}{\tau}\int_\Omega |\nabla(w - \beta u)|^2 dx \leq 0.$$

We study the stationary problem of (7) (or (2))

$$\begin{cases} d\Delta u - g(u) - u/\tau + \beta w = 0, \\ \Delta w - \beta \Delta u = 0, \end{cases} \qquad x \in \Omega, \tag{9}$$

under the boundary condition

$$\frac{\partial u}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, \quad x \in \partial\Omega,$$

and the constraint

$$s = \alpha \langle w \rangle.$$

From the second equation of (9) we have

$$w - \beta u = \langle w \rangle - \beta \langle u \rangle = s/\alpha - \beta \langle u \rangle,$$

then applying this to the first equation of (9) yields

$$d\Delta u - g(u) - du + \frac{s}{\tau} - \frac{1 - \tau d}{\tau}\langle u \rangle = 0. \tag{10}$$

Let $u^*(x)$ be any solution to (10) with the Neumann boundary condition. Then

$$(u, v) = (u^*(x), v^*(x)), \quad v^*(x) := -du^*(x) + \frac{s}{\tau} - \frac{1 - \tau d}{\tau}\langle u^* \rangle$$

gives a solution to (2) with (3).

On the profile of stable equilibrium solutions we first introduce the following result ([**15**]):

PROPOSITION 2.1. *Assume $d\tau < 1$ and consider the case Ω is an interval, say $\Omega = (0, L)$. Then any stable equilibrium solution $(u^*(x), v^*(x))$ of* (2) *with the Neumann boundary condition must be monotone, namely both components of the solution are constant or strictly monotone.*

We next consider the cylindrical domain Ω in $\mathbb{R}^n$ $(n \geq 2)$ defined by

$$\Omega = \{x = (x_1, y) : 0 < x_1 < L, \quad y \in \omega\}, \tag{11}$$

where ω is a bounded domain in $\mathbb{R}^{n-1}$ with smooth boundary $\partial\omega$. Let σ_j and $\Phi_j(y)$ be the j-th eigenvalue and the corresponding normalized eigenfunction of the minus Laplacian $-\Delta_y$ in ω with the Neumann boundary condition, namely,

$$-\Delta_y \Phi_j = \sigma_j \Phi_j, \quad y \in \omega, \qquad \frac{\partial \Phi_j}{\partial \nu} = 0, \quad y \in \partial\omega, \quad \int_\omega \Phi_j^2 dy = 1.$$

PROPOSITION 2.2. *Assume $d\tau < 1$ and consider the equations in the cylindrical domain of* (11). *Then any stable solution $(u^*(x_1, y), v^*(x_1, y))$ of* (2) *with the Neumann boundary condition must be monotone in x_1-axis. Moreover, there is a positive number δ such that if the domain satisfies $d\sigma_2 > \delta$, the stable solution is uniform in y variable while the solution is non-uniform in y if $d\sigma_2 < \delta$.*

PROOF. As for the former assertion see Corollary 1.2 in [**15**]. We show the latter part. We let $U^*(x_1)$ be a solution to (10) uniform to y-direction, namely, $U = U^*$ satisfies the equation

$$dU_{x_1 x_1} - g(U) - dU + \frac{s}{\tau} - \frac{1 - \tau d}{\tau L} \int_0^L U(x_1) dx_1 = 0, \quad x \in (0, L),$$

and $U_{x_1} = 0$ $(x = 0, L)$. Assume

$$(u, v) = \left(U^*(x_1), -dU^*(x_1) + \frac{s}{\tau} - \frac{1 - \tau d}{\tau L} \int_0^L U^*(x_1)\, dx_1 \right)$$

is stable in the perturbation uniform in y-direction.

We note

$$d(U^*_{x_1})_{x_1 x_1} - (g'(U^*) + d) U^*_{x_1} = 0. \tag{12}$$

Consider the linearized eigenvalue problem

$$\mathcal{L}(\varphi) := -\left[d\Delta\varphi - \{g'(U^*(x_1)) + d\}\varphi - \frac{1-\tau d}{\tau}\langle\varphi\rangle\right] = \mu\varphi, \tag{13}$$
$$\mathrm{Dom}(\mathcal{L}) = \{\varphi \in H^2(\Omega) : \partial\varphi/\partial\nu = 0 \quad \text{on} \quad \partial\Omega\}.$$

By virtue of Theorem 1.1 in [**15**] it suffices to study the above linearized problem to prove the stability of the solution. In fact, the theorem tells that the number of positive eigenvalues of $\mathcal{L}$ and the linearized operator of (2) coincides, thus the problem can be reduced to that of $\mathcal{L}$.

Applying the the Fourier expansion

$$\varphi = \sum_{j=1}^{\infty} \xi_j(x_1)\Phi_j(y) \qquad (\Phi_1(y) = \frac{1}{|\omega|^{1/2}}),$$

to (13), we have

$$\mathcal{L}_1(\xi_1) := -\left[d(\xi_1)_{x_1x_1} - \{g'(U^*(x_1)) + d\}\xi_1 - \frac{1-\tau d}{L\tau}\int_0^L \xi_1\, dx_1\right] = \mu\xi_1, \tag{14}$$
$$\mathcal{L}_j(\xi_j) := -\left[d(\xi_j)_{x_1x_1} - \{g'(U^*(x_1)) + d + d\sigma_j)\}\xi_j\right] = \mu\xi_j, \quad j \geq 2. \tag{15}$$

Those operators have the domain

$$\mathrm{Dom}(\mathcal{L}_j) := \{\xi \in H^2(0,L) : \xi_{x_1}(0) = \xi_{x_1}(L) = 0\} \quad (j \geq 1).$$

By the stability assumption for $U^*(x_1)$ in $(0,L)$ the least eigenvalue of $\mathcal{L}_1$ is non-negative.

By (12) we have

$$\mathcal{L}_0(U^*_{x_1}) = 0, \quad \mathcal{L}_0(\xi) := -\left[d(\xi)_{x_1x_1} - \{g'(U^*(x_1)) + d\}\xi\right].$$

In view of $U^*_{x_1}(0) = U^*_{x_1}(L) = 0$, $U^*_{x_1}$ is an eigenfunction to zero eigenvalue of $\mathcal{L}_0$ with the Dirichlet boundary condition. By comparison of eigenvalues for the Neumann and Dirichlet conditions ([**2**]) the least eigenvalue of $\mathcal{L}_0$ with the Neumann condition is negative, say $-\delta < 0$. Comparing $\mathcal{L}_0$ and $\mathcal{L}_j$ $(j \geq 2)$, we obtain $-\delta + d\sigma_j$ the least eigenvalue of $\mathcal{L}_j$ with the Neumann boundary condition. Since $0 < \sigma_2 < \sigma_j$ $(j \geq 3)$, we have the desired assertion of the Proposition. □

The next result immediately follows from Proposition 2.2

Corollary 2.3. *Under the condition* $\tau d < 1$ *consider* (2) *with* (3) *in a rectangle domain* $\Omega = \{x = (x_1, x_2) : 0 < x_1 < L_1, \quad 0 < x_2 < L_2\}$. *Let*

$(u^*(x_1, x_2), v^*(x_1, x_2))$ be a stable equilibrium solution which is non-uniform in both x_1 and x_2-axises. Then the maximum of $u^*(x_1, x_2)$ attains at a corner of the domain, where v^* takes the minimum.

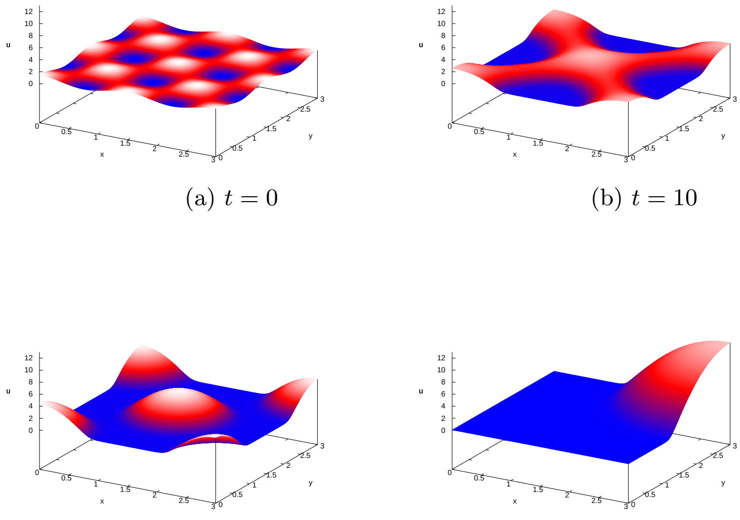
We show a numerical result on time evolution of a solution to (2) with (6) under the boundary condition (3) in a square domain $\Omega = (0, 3) \times (0, 3)$. We set the parameter values

$$\tau = 1, \quad a = 1.0, \quad b = 0.1, \quad d_1 = 0.02, \quad d_2 = 1. \tag{16}$$

Take initial data as

$$u(x, 0) = 1 + \cos(4\pi x)\cos(4\pi y), \quad v(x, 0) = 1 + \cos(2\pi x)\cos(2\pi y). \tag{17}$$

In Figure 1 one can see how the spatial pattern changes and converges to a localized steady state having its peak at a corner.



(a) $t = 0$ (b) $t = 10$

(c) $t = 100$ (d) $t = 1000$

Figure 1. Bird's eye views of u-component of the solution to (2) with (6). The domain is $\Omega = (0, 3) \times (0, 3)$ and (3) is assumed. The parameter values are given by (16) and initial data are by (17). The subfigures are for $t = 0, 10, 100$ and 1000. The solution converges to a localized pattern whose peak exists at a corner.

3. A extended model of mass conserved system

In [**8**] the following perturbed system is considered:

$$\begin{cases} u_t = d_1 u_{xx} - g(u) + v, \\ v_t = d_2 v_{xx} + g(u) - v - \varepsilon v^2, \end{cases} \qquad 0 < x < 2\pi \tag{18}$$

with the periodic boundary conditions and $g(u) = au/(u^2 + b)$ $(a, b > 0)$. In this system

$$\frac{d}{dt}\int_0^L (u(x,t) + v(x,t))dx = -\varepsilon \int_0^L v(x,t)^2 dx \leq 0.$$

Hence for nonnegative solutions

$$\mathcal{E}_a(u,v) := \int_0^L (u+v)dx$$

works as a Lyapunov function and every nonnegative solution converges to a trivial solution $(u, v) = (0, 0)$.

However, transient dynamics is not so simple. With

$$a = 1, \quad b = 0.1, \quad d_1 = 0.001, \quad d_2 = 0.05, \quad \varepsilon = 0.01,$$

a numerical computation shows that the system exhibits a similar dynamics as in the unperturbed system except that the single spike eventually collapses.

This behavior of the solutions seems natural, because the perturbed dynamics could be approximated by the unperturbed one. In this case, however, the unperturbed system allows the global attractor for each fixed $s = \langle u \rangle + \langle v \rangle$ and it has a rich structure for s in some interval and small d_1, while the perturbed system as seen above has a trivial attractor which is globally asymptotically stable. Namely, the structure of the attractors is drastically changed even though ε is taken arbitirarily small.

Another interesting aspect is related to the mechanism of the Turing-like pattern. It is understood that for the onset of the diffusion-driven instability the system needs a steady state which is stable in the diffusion-free system. The perturbed system, however, has no longer such a steady state. Nonetheless, it can exhibits a Turing-like pattern in transient dynamics. This fact implies that it is possible to produce a Turing-like pattern even though the usual diffusion-instability condition is not met in some model equations.

To understand this property more, we propose the next system:

$$\begin{cases} u_t = d_1\Delta u - g(u) + v, \\ v_t = d_2\Delta v + g(u) - v - \varepsilon h(v) + \varepsilon h(w), \\ w_t = d_3\Delta w + \varepsilon h(v) - \varepsilon h(w), \end{cases} \tag{19}$$

where all the coefficients are positive and g is the same as in the previous section or (18). $h(\cdot)$ is assumed to be a Michaelis-Menten type kinetics (see [**9**]) as

$$h(w) := \frac{w}{1+w}.$$

This model combines the (u, v) system and reactions creating a flow from v to w and feedback from w to v. We consider the equations in a bounded interval $\Omega = (0, L)$ with periodic boundary conditions. When ε is small, this system is certainly regarded as a perturbation system of (2) with $\tau = 1$. We, however, are also interested in the dynamical structure of not small ε but large one.

Henceforth we always consider the nonnegative solution to (18). In fact the maximum principle ensure the nonnegativity of the solution with nonnegative initial data (see [**3**]).

First note that the system (19) has a conservation property such as

$$m := \frac{1}{L}\int_0^L (u(x,t) + v(x,t) + w(x,t))\,dx \tag{20}$$

is conserved. It is easy to see that the system has a constant solution which is obtained by solving

$$u = g(v), \quad u + 2v = m, \quad w = v\,. \tag{21}$$

We let $(\overline{u}, \overline{v}, \overline{w})$ be a solution of (21). Linearize the equations around this solution and consider the linearized eigenvalue problem. Then the eigenvalues of the linearized operator are given by the ones of the matrices

$$\begin{pmatrix} -d_1\sigma_j - g'(\overline{u}) & 1 & 0 \\ g'(\overline{u}) & -d_2\sigma_j - 1 - \varepsilon\beta & \varepsilon\beta \\ 0 & \varepsilon\beta & -d_3\sigma_j - \varepsilon\beta \end{pmatrix} \qquad (j = 1, 2, 3, \ldots),$$

where

$$\beta := h'(\overline{v}), \qquad \sigma_1 = 0, \qquad \sigma_{2k} = \sigma_{2k+1} = \frac{4k^2\pi^2}{L^2} \quad (k = 1, 2, \ldots).$$

Hence for each j the eigenvalues of this matrix are solutions of

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0,$$

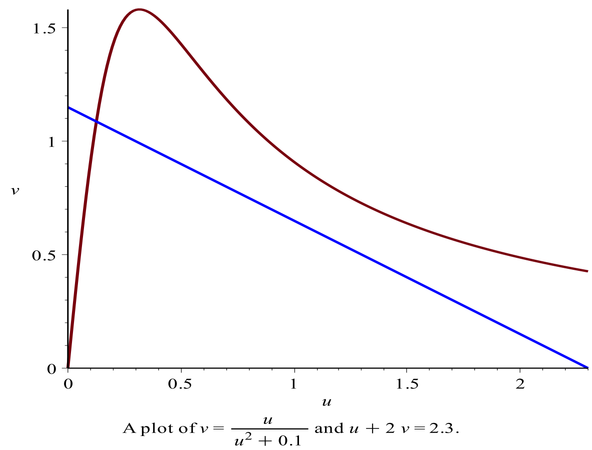


Figure 2. A plot of $v = g(u) = u/(u^2 + 0.1)$ and $u + 2v = 2.3$. There is a unique intersection point where $g'(u) > 0$.

$$
\begin{aligned}
a_1 &:= (d_1 + d_2 + d_3)\sigma_j + g'(\overline{u}) + 1 + 2\varepsilon\beta, \\
a_2 &:= (d_1 d_2 + d_2 d_3 + d_3 d_1)\sigma_j^2 \\
&\quad + \{d_1 + d_3 + (d_2 + d_3)g'(\overline{u}) + \varepsilon\beta(2d_1 + d_2 + d_3)\}\sigma_j + 2\varepsilon\beta g'(\overline{u}) + \varepsilon\beta, \\
a_3 &:= d_1 d_2 d_3 \sigma_j^3 + \{d_1 d_3 + d_2 d_3 g'(\overline{u}) + \varepsilon\beta(d_1 d_3 + d_1 d_2)\}\sigma_j^2 \\
&\quad + \varepsilon\beta\{(d_2 + d_3)g'(\overline{u}) + d_1\}\sigma_j.
\end{aligned}
$$

For $j = 1 (\sigma_1 = 0)$ we have

$$
\lambda = 0, \quad \lambda^2 + (g'(\overline{u}) + 1 + 2\varepsilon\beta)\lambda + \varepsilon\beta(2g'(\overline{u}) + 1) = 0.
$$

If $g'(\overline{u}) > -1/2$, then the constant solution is stable with respect to the spatially uniform perturbation. In other words the solution is stable in the diffusion free equations. If $g'(\overline{u}) > 0$, then for each $j \geq 1$ we can easily check $a_1 a_2 - a_3 > 0$ which leads to negativity of real part for all the eigenvalues λ (by a condition of Routh-Hurwitz, see for instance [**1**]). Hence, destabilization never takes place in the presence of diffusions in this case.

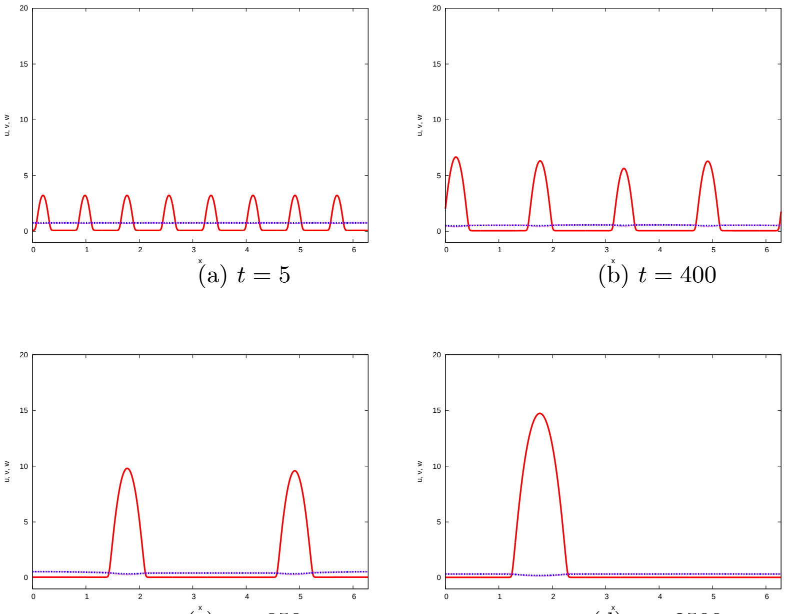
For specific parameter values we show numerical simulations. Take

$$
a = 1, \quad b = 0.1, \quad L = 2\pi, \quad d_1 = 0.001, \quad d_2 = d_3 = 0.05, \quad \varepsilon = 2. \tag{22}
$$

We choose the initial condition as

$$u(x,0) = 2 + 0.2\sin(8\pi x), \quad v(x,0) = 0.2, \quad w(x,0) = 0.1, \tag{23}$$

so that $m = 2.3$ of (20) is met. Then there is a unique constant solution satisfying $g'(\overline{u}) > 0$ (see Figure 2).



(a) $t = 5$ (b) $t = 400$

(c) $t = 850$ (d) $t = 2500$

Figure 3. The snapshots of the profile of the solution to (19) for $t = 5, 400, 850$ and 2500 are displayed. Solid curve indicate u, dashed curve for v and dot one for w. The curves of v and w are too close to distinguish them in the figures. The subfigures consist of (a) :emergence of a spiky pattern, (b) :a transient four-spike pattern, (c) :a transient two-spike pattern, and (d) :a single pattern that is numerically stable. The parameter values and the initial data are stated (22) and (23) respectively.

In Figure 3 the snapshots of the profile of the solution for $t = 5, 400, 850$ and 2500 are shown. Solid curve indicate u, dashed curve for v and dot one for w. The curves of v and w are too close to distinguish them in the figures. In the subfigure (a) a wave is growing to a spiky pattern, and the number of the spikes becomes smaller in (b). After further decrease of the spikes as seen in (c), the pattern settles in a stable single spike (d).

We also performed the similar simulations for not only small parameter values (including $\varepsilon = 0.1$) also large values (for instance, $\varepsilon = 10, 20, 100$). All the simulations exhibit qualitatively similar dynamics to the case $\varepsilon = 2$.

Recall that for any ε the constant solution is stable in the present parameter setting. On the other hand, the basin of the constant solution or the spike solution, that is the set of solutions attracted by each stable solution, is affected by the change of ε. It, however, is difficult to identify the boundary of the basins.

4. Summery

In this article we proposed a new reaction-diffusion model (19) with a conservation property. This model equations consists of three variables and can be regarded as an extended system of the previous two-component system (2). In fact, when the coupling parameter ε is small, then the system is regarded as a perturbed system of (2). In the extended system, even though there is no constant steady state which causes the Turing-type instability, Turing-like wave patterns emerge in transient dynamics. Qualitatively similar dynamics can be observed for a wide range of the parameter ε. Mathematical theory for this dynamical behavior is worth developing and will be our future work.

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