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Pairs of polynomials which satisfy the local functional equations

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Abstract. In this note, we survey the fundamental theorem of prehomogeneous vector spaces which is related to the local functional equations of polynomials and announce a recent result of this field. Especially we introduce that we could recently make series of local functional equations of non-prehomogeneous types.

1. Introduction

Let P and P^* be homogeneous polynomials in n variables of degree d with real coefficients. For a decomposition to connected components $\{\ x \in \mathbb{R}^n |\ P(x) \neq 0\}_{\mathbb{R}} = \bigcup_{i=1}^{\nu} \Omega_i$, put

$$|P(x)|_i := \begin{cases} |P(x)| & x \in \Omega_i \\ 0 & \text{otherwise} \end{cases}, \quad |P^*(y)|_i := \begin{cases} |P^*(y)| & y \in \Omega_i^* \\ 0 & \text{otherwise} \end{cases}$$

It is an interesting problem both in analysis and in number theory to find a condition on P and P^* under which they satisfy a functional equation of the form

$$\widehat{|P(x)|_i^s} := (\text{Fourier transform of } |P(x)|_i^s) = \sum_{j=1}^{\nu} \gamma_{ij}(s) |P^*(y)|_j^{-\frac{n}{d}-s} \tag{1}$$

where $d = \deg\ P = \deg\ \ P^*$ and the gamma-factors $\gamma_{ij}(s)$ are meromorphic functions of s. We use the notation $\hat{f}$ as the Fourier transform of a function f. Indeed, the following classical examples are well known.

Example 1.1. *For a positive definite quadratic form $x_1^2 + \cdots + x_n^2$,*

$$\widehat{(x_1^2 + \cdots + x_n^2)^{s-\frac{n}{2}}} = \pi^{-2s+\frac{n-2}{2}} \Gamma(s)\Gamma(s - \frac{n-2}{2}) \sin \pi(\frac{n}{2} - s)(y_1^2 + \cdots + y_n^2)^{-s}.$$

This local functional equation is related to Epstein zeta function $\zeta(Q, s) := \sum_{x \in \mathbb{Z}^n \setminus \{0\}} \frac{1}{Q(x)^s} = \sum_{m=1}^{\infty} \frac{\sharp\{\ x \in \mathbb{Z}^n\ |\ Q(x) = m\ \}}{m^s}$ *of a positive definite quadratic form* Q. *The function* $\zeta(Q, s)$ *satisfies functional equation*

$$\zeta(Q, \frac{n}{2} - s) = \pi^{\frac{n}{2}-2s} \frac{\Gamma(s)}{\Gamma(\frac{n}{2} - s)} \zeta(Q, s)$$
$$= \pi^{-2s+\frac{n-2}{2}} \Gamma(s)\Gamma(s - \frac{n-2}{2}) \sin \pi(\frac{n}{2} - s)\zeta(Q, s).$$

This gamma factor of the functional equation comes from the Fourier transform $\widehat{(x_1^2 + \cdots + x_n^2)^{s-\frac{n}{2}}}$.

Example 1.2.

$$|\widehat{\det X|^{s-n}} = (2\pi)^{-ns}(2\pi)^{\frac{n(n-1)}{2}} 2^n \cos(\pi\tfrac{s}{2}) \cdots \cos(\pi\tfrac{s-n+1}{2}) \\ \times\Gamma(s)\Gamma(s-1)\cdots\Gamma(s-n+1)|\det Y|^{-s}$$

This example is related to a product of shifted Riemann zeta functions $Z_n(s) := \zeta(s)\zeta(s-1)\cdots\zeta(s-n+1)$, *where* $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$. *Then* $Z_n(s)$ *has the following functional equation:*

$$\zeta(1-s)\zeta(2-s)\cdots\zeta(n-s) = \\ (2\pi)^{-ns}(2\pi)^{\frac{n(n-1)}{2}} 2^n \cos(\pi\tfrac{s}{2}) \cdots \cos(\pi\tfrac{s-n+1}{2})\Gamma(s)\Gamma(s-1)\cdots\Gamma(s-n+1) \\ \times\zeta(s)\zeta(s-1)\cdots\zeta(s-n+1).$$

This gamma factor of the functional equation comes from the Fourier transform above.

Example 1.3. *This example is not Fourier transform, but is similar to Fourier transform. This example is essentially related to the problem to find systematically a pair of polynomials* P, P^* *which satisfy the equation (1).*
Let $\mathcal{S}_n^+$ *denotes the set of positive definite symmetric matrices of size n. It is known that*

$$(a) \quad \int_{\mathcal{S}_n^+} e^{-trXY}(\det X)^{\alpha-\frac{n+1}{2}} dX \\ = (\det Y)^{-\alpha} \prod_{i=1}^{n} \pi^{\frac{i-1}{2}} \Gamma(\alpha - \frac{i-1}{2}) = \left\{ \pi^{\frac{n(n-1)}{4}} \prod_{i=1}^{n} \Gamma(\alpha - \frac{i-1}{2}) \right\} (\det Y)^{-\alpha}$$

for $\alpha > \frac{n-1}{2}$ *and* $y \in \mathcal{S}_n^+$.

We remark that this example is related to multivariate statistics and hyperbolic partial differential equations. This integral is first studied by Whishart [**42**] *in multivariate statistics, and Siegel* [**40**] *applied it in number theory. On the other hand, M.Riesz* [**26**] *found a similar integral formula on the Lorentz cone*

$$\mathcal{LC}_n := \left\{ x \in \mathbb{R}^n;\ x_1 > \sqrt{x_2^2 + \cdots + x_n^2} \right\}$$

in study of the wave equation. Namely, he showed

$$(b) \qquad \begin{aligned} &\int_{\mathcal{LC}_n} e^{-2\langle x,y\rangle}(x_1^2 - x_2^2 - \cdots - x_n^2)^{\alpha - \frac{n}{2}}\,dx \\ &= \pi^{\frac{n-2}{2}}\Gamma(\alpha)\Gamma(\alpha - \tfrac{(n-2)}{2})(y_1^2 - y_2^2 - \cdots - y_n^2)^{-\alpha} \end{aligned}$$

for $\alpha > \frac{n-2}{2}$ *and* $y \in \Lambda_n$. *Indeed, Gårding* [**10**] *considered the formula (a) as an analogue of (b) in his work on certain hyperbolic partial differential equations.*

A beautiful answer of the problem to find a pair of polynomials which satisfies the condition (1) is given by the theory of prehomogeneous vector spaces due to Mikio Sato. We explain in next section the idea of fundamental theorem of prehomogeneous vector spaces related to this problem.

The notion of prehomogeneous vector spaces was defined by M.Sato in 1960's. His original motivation was the construction of fundamental solutions of partial differential equations. The theory of prehomogeneous vector spaces has several aspects.
Number theoretic aspect $\cdots$ Theory of zeta functions.
Analytic aspect $\cdots$ Explicit construction of fundamental solutions of linear partial differential equations with constant coefficients.
Representation theoretic aspect $\cdots$ Invariant theory, infinite dimensional representations of real and p-adic algebraic groups, automorphic representations, plane arrangements.
For classification and the constructions of basic relative invariants of prehomogeneous vector spaces, see [**37**], [**18**], [**27**], [**1**], [**21**], [**22**], [**23**].

2. Fundamental Theorem of Prehomogeneous vector spaces

In this section, we introduce the fundamental theorem of prehomogeneous vector spaces along [**19**]. As we already discussed in section 1, Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ satisfies the following functional equation: $\zeta(1-s) = (2\pi)^{-s}\Gamma(s)2\cos\frac{\pi s}{2}\zeta(s)$. As the reason why $\zeta(s)$ satisfies this functional equation, we can point out the following two facts:
(I) The *Fourier transform* of a complex power of x is again a complex power of y. That is,

$$\int_0^{\infty} x^{s-1} e^{2\pi\sqrt{-1}xy}\,dx = (2\pi)^{-s}\Gamma(s)e^{\frac{\pi\sqrt{-1}s}{2}(\mathrm{sgn} y)}|y|^{-s} \quad (0 < \mathfrak{Re}(s) < 1). \tag{2}$$

(II) *Poisson's summation formula;* that is, for the Fourier transform

$$\hat{f}(y) = \int_{-\infty}^{\infty} f(x) e^{2\pi\sqrt{-1}xy} dx,$$

we have

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n). \tag{3}$$

Indeed, let

$$f(x) = \begin{cases} x^{s-1} & (x > 0) \\ 0 & (x \leq 0) \end{cases}.$$

Taking no account for the convergence and calculating just formally, we have

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=1}^{\infty} n^{s-1} = \zeta(1-s). \tag{4}$$

On the other hand, (I) implies that

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \hat{f}(n) &= (2\pi)^{-s}\Gamma(s) \left(\sum_{n=-\infty}^{-1} (-n)^{-s} e^{\frac{-\pi\sqrt{-1}s}{2}} + \sum_{n=1}^{\infty} n^{-s} e^{\frac{\pi\sqrt{-1}s}{2}} \right) \\ &= (2\pi)^{-s}\Gamma(s) 2\cos\frac{\pi s}{2} \zeta(s). \end{aligned} \tag{5}$$

Hence, by (II), we can see how to obtain the functional equation

$$\zeta(1-s) = (2\pi)^{-s}\Gamma(s) 2\cos\frac{\pi s}{2} \zeta(s). \tag{6}$$

To give a rigorous proof based on this principle, we have to consider the zeta integral

$$I(s, \hat{\varphi}) := \int_0^{\infty} t^{s-1} \sum_{x \in \mathbb{Z} \setminus \{0\}} \hat{\varphi}(tx) dt, \tag{7}$$

where $\hat{\varphi}$ is the Fourier transform of a rapidly decreasing function φ on $\mathbb{R}$ and the integral converges absolutely for $\mathfrak{Re}(s) > 1$. A sketch of the proof can be given as follows: we have

$$\begin{aligned} I(s, \hat{\varphi}) &:= \sum_{n=1}^{\infty} \int_0^{\infty} t^{s-1} \hat{\varphi}(nt) dt + \sum_{n=1}^{\infty} \int_0^{\infty} t^{s-1} \hat{\varphi}(-nt) dt \\ &= \zeta(s) \int_{-\infty}^{\infty} |x|^{s-1} \hat{\varphi}(x) dx. \end{aligned} \tag{8}$$

Poisson's summation formula (II) implies that

$$I(s,\hat{\varphi}) = J(s,\varphi) - \frac{\hat{\varphi}(0)}{s} + \frac{\varphi(0)}{s-1}, \tag{9}$$

where

$$J(s,\varphi) := \int_1^\infty t^{s-1} \sum_{x\in\mathbb{Z}\setminus\{0\}} \hat{\varphi}(tx)dt + \int_1^\infty t^{-s} \sum_{x\in\mathbb{Z}\setminus\{0\}} \varphi(tx)dt. \tag{10}$$

Since $J(s,\varphi)$ is an entire function of s, we see that $I(s,\hat{\varphi})$ extends analytically to a meromorphic function on the whole s-plane with poles of order at most one at $s=0,1$. Moreover, from this form, we see that it satisfies $I(s,\hat{\varphi}) = I(1-s,\varphi)$. Since for each $s\in\mathbb{C}$ there exists φ satisfying $\int_{-\infty}^{\infty} |x|^{s-1}\hat{\varphi}(x)dx \neq 0,\infty$, the zeta function

$$\zeta(s) = (\int_{-\infty}^{\infty} |x|^{s-1}\hat{\varphi}(x)dx)^{-1}\cdot I(s,\hat{\varphi}) \tag{11}$$

extends analytically to a meromorphic function on the whole s-plane. On the other hand $\int_{-\infty}^{\infty} |x|^{s-1}\hat{\varphi}(x)dx$ also extends analytically to a meromorphic function on the whole s-plane. Hence it follows from (I) that

$$\int_{-\infty}^{\infty} |x|^{s-1}\hat{\varphi}(x)dx = (2\pi)^{-s}\Gamma(s)2\cos\frac{\pi s}{2}\int_{-\infty}^{\infty} |x|^{-s}\varphi(x)dx. \tag{12}$$

By this equation and $I(s,\hat{\varphi}) = I(1-s,\varphi)$, we obtain

$$\zeta(1-s) = (2\pi)^{-s}\Gamma(s)2\cos\frac{\pi s}{2}\zeta(s). \tag{13}$$

This is related to Example 1.2 in section 1. Its higher degree version of (I) is easily obtained.

Thus, if we can generalize (I); that is, if we can get a systematic method of finding a polynomial $f(x)$ on a vector space V such that
(I)' the Fourier transform of a complex power $f(x)^s$ is essentially given by a complex power of some polynomial $f^*(x)$ on the dual space V^* of V.
Then it is expected that, as an example, a function of the form

$$\zeta(s,f) = \sum_{x\in\mathbb{Z}^n\setminus\{f^{-1}(0)\}} \frac{1}{|f(x)|^s} \tag{14}$$

satisfies a functional equation. That is to say, it is expected that Dirichlet series (zeta functions) with functional equations will be obtained systematically. For example, let $P(x) = {}^t xAx$ ($x \in \mathbb{R}^n$) be a positive definite quadratic form, and $Q(y) = {}^t yA^{-1}y$ its dual. Then we have

$$\int_{\mathbb{R}^n \setminus \{0\}} |Q(y)|^{s-\frac{n}{2}} \hat{\Phi}(y)dy = \pi^{\frac{n}{2}-2s} \cdot \sqrt{\det P} \cdot \frac{\Gamma(s)}{\Gamma(\frac{n}{2}-s)} \cdot \int_{\mathbb{R}^n \setminus \{0\}} |P(x)|^{-s}\Phi(x)dx, \tag{15}$$

where Φ is rapidly decreasing function on $\mathbb{R}^n$, and we see that Epstein's zeta function

$$\zeta(s, P) = \sum_{x \in \mathbb{Z}^n \setminus \{0\}} \frac{1}{|P(x)|^s} \tag{16}$$

satisfies a functional equation. This is related to Example 1.1 in section 1.

Now we discuss the reason why the Fourier transform of x^s or $P(x)^s$ is again essentially a complex power of some polynomial. In 1961, Mikio Sato found out that the reason is the existence of a big action of a group, and the fact that x or $P(x)$ are relative invariants with respect to that action. Thus he reached the action of prehomogeneous vector spaces. We call a triplet (G, ρ, V) a *prehomogeneous vector space* if $\rho : G \longrightarrow GL(V)$ is a rational representation of a connected algebraic group G on a finite-dimensional vector space V such that V has a dense G-orbit $\rho(G)v_0 = \{\rho(g)v_0 | g \in G\}$ with respect to the Zariski toplology, all defined over $\mathbb{C}$. In general, a G-orbit $\rho(G)v_0$ is called a homogeneous space under the action of the group G. The condition $V = \overline{\rho(G)v_0}$ implies that V is an almost homogeneous space under the action of G. Therefore we call it a prehomogeneous vector space. A relative invariant of (G, ρ, V) is a nonzero rational function $f(x)$ on V satisfying $f(\rho(g)x) = \chi(g)f(x)$ with some constant $\chi(g)$ for each $g \in G$. Then $\chi : G \longrightarrow GL_1$ becomes a rational character of G, and we say that f is a relative invariant corresponding to the character χ. For example, we consider the action of $G = GL_1(\mathbb{C}) = \mathbb{C}^\times$ on $V = \mathbb{C}$ by $\rho(g)x = gx$ ($g \in \mathbb{C}^\times, x \in \mathbb{C}$). Then, since $V \setminus \{0\} = \rho(G) \cdot 1$ is a dense orbit, it follows that (G, ρ, V) is a prehomogeneous vector space, and $f(x) = x$ is a relative invariant corresponding to $\chi(g) = g$. Next, let $SO_n(P) = \{B \in SL_n(\mathbb{C});\ {}^t BAB = A\}$ be the special orthogonal group which stabilizes the quadratic form $P(x) = {}^t xAx$ (${}^t A = A \in GL_n(\mathbb{C})$) on $V = \mathbb{C}^n$. Then $G = GL_1 \times SO_n(P)$ acts on V by $\rho(\alpha, B)x = \alpha Bx$ ($\alpha \in GL_1,\ B \in SO_n(P),\ x \in \mathbb{C}^n$), and $\{x \in V; P(x) \neq 0\}$ is a dense G-orbit in V. Hence (G, ρ, V) is a prehomogeneous vector space, and $P(x)$ is a relative invariant corresponding to the character $\chi(\alpha, B) = \alpha^2$.

Then, what is the principle under which the Fourier transform of a complex power $f(x)^s$ of a relative invariant $f(x)$ on a prehomogeneous vector space is again a complex power of a polynomial $f^*(y)$?

To see this, let us consider the basic properties of relative invariants on prehomogeneous vector spaces.

First, if $f(x)$ is an absolute invariant, i.e., a relative invariant corresponding to

$\chi = 1$, then it is a constant on each orbit $\rho(G)v_0$: $f(x) = c(= f(v_0))$. That is to say, $\{x \in V; f(x) = c\} \supset \rho(G)v_0$. The closure of the left-hand side is given by $\overline{\rho(G)v_0} = V$, so that $f(x)$ becomes a constant function on V.

Next, if f_1 and f_2 are relative invariants corresponding to the same character χ, then f_2/f_1 is an absolute invariant. Hence it is a constant, and we have $f_2(x) = cf_1(x)$ (c is a constant). Thus we see that

(III) *Relative invariants of a prehomogeneous vector space corresponding to the same character are identical up to a constant.*

Conversely, it is known that if relative invariants of a triplet (G, ρ, V) satisfy condition (III), then (G, ρ, V) is a prehomogeneous vector space. In other words, a prehomogeneous vector space is characterized by condition (III).

For simplicity, we consider the case where G is a reductive algebraic group, and where the complement $S = V \setminus \rho(G)v_0$ of a dense G-orbit $\rho(G)v_0$ is the zeros $S = \{x \in V; f(x) = 0\}$ of an irreducible polynomial $f(x)$. Let $d = \deg f$ and $n = \dim V$. Then, it is known that $m = 2n/d$ is a natural number, and that $f(\rho(g)x) = \chi(g)f(x)$ $(g \in G)$ for a character χ satisfying $\det \rho(g)^2 = \chi(g)^m$ $(g \in G)$. Further, the dual (G, ρ^*, V^*) of (G, ρ, V) is also a prehomogeneous vector space satisfying similar properties, and it has an irreducible relative invariant $f^*(y)$ satisfying $f^*(\rho^*(g)y) = \chi(g)^{-1}f^*(y)$.

Now, not worrying about convergence, we consider just formally the Fourier transform

$$\varphi(y) = \int_V f(x)^{s-\frac{n}{d}} \cdot e^{2\pi\sqrt{-1}\langle x,y\rangle} dx \quad (y \in V^*) \tag{17}$$

of $f(x)^{s-\frac{n}{d}}$, which we also consider just formally. Since

$$\begin{aligned} \varphi(\rho^*(g)y) &= \chi(g)^{s-\frac{n}{d}} \cdot \det \rho(g) \cdot \varphi(y) \\ &= \chi(g)^s \varphi(y) \quad (g \in G), \end{aligned} \tag{18}$$

the Fourier transform $\varphi(y)$ of $f(x)^{s-\frac{n}{d}}$ becomes a relative invariant on V^* corresponding to $\chi(g)^s$. On the other hand, $f^*(y)^{-s}$ is also a relative invariant on V^* corresponding to the same character $\chi(g)^s$. Since (G, ρ^*, V^*) is a prehomogeneous vector space, we expect that principle (III) will imply the coincidence of $\varphi(y)$ and $f^*(y)^{-s}$ up to a constant, although we cannot apply (III) because they are not rational functions. That is to say, we expect the existence of a constant c satisfying

$$\int_V f(x)^{s-\frac{n}{d}} \cdot e^{2\pi\sqrt{-1}\langle x,y\rangle} dx = cf^*(y)^{-s}. \tag{19}$$

This is the principle which says that the Fourier transform of a complex power $f(x)^s$ of a relative invariant coincides with a complex power of a polynomial $f^*(y)$. Of course, in order to make their convergence (and so on) meaningful, we need to discuss the details rigorously; for example, we have to regard $f(x)^{s-\frac{n}{d}}$ and $f^*(y)^{-s}$ as distributions. Following this principle, Mikio Sato obtained, in 1961, the fundamental theorem of prehomogeneous vector spaces concerning the Fourier transform of a complex power of a relative invariant. That is to say, suppose that (G, ρ, V) satisfies the conditions above, and that it is defined over $\mathbb{R}$. Then $V_{\mathbb{R}} - S_{\mathbb{R}}$ and $V^*_{\mathbb{R}} - S^*_{\mathbb{R}}$ are decomposed into the same number of connected components: $V_{\mathbb{R}} - S_{\mathbb{R}} = V_1 \cup \cdots \cup V_\ell$ and $V^*_{\mathbb{R}} - S^*_{\mathbb{R}} = V^*_1 \cup \cdots V^*_\ell$, where V_i, V^*_j are $G^+_{\mathbb{R}}$-orbits. Here $G^+_{\mathbb{R}}$ is the connected component containing the unit element of the subgroup $G_{\mathbb{R}}$ which consists of $\mathbb{R}$-rational points of G. Further, for rapidly decreasing functions Φ, Φ^* on $V_{\mathbb{R}}, V^*_{\mathbb{R}}$ respectively, the integrals

$$\int_{V_i} |f(x)|^s \Phi(x) dx \tag{20}$$

and

$$\int_{V^*_j} |f^*(y)|^s \Phi^*(y) dy \tag{21}$$

converge for $\mathrm{Re}(s) > 0$, and can be extended analytically to meromorphic functions on the whole s-plane, which satisfy

$$\int_{V_j} |f(x)|^{s-\frac{n}{d}} \widehat{\Phi^*}(x) dx = \sum_{i=1}^{\ell} a_{ij}(s) \int_{V^*_i} |f^*(y)|^{-s} \Phi^*(y) dy. \tag{22}$$

We also write this equation for simplicity as follows:

$$\widehat{|f(x)|_j^{s-\frac{n}{d}}} = \sum_{i=1}^{\ell} a_{ij}(s) |f^*(y)|_i^{-s} \tag{23}$$

where

$$|f(x)|_i := \begin{cases} |f(x)| \text{ on } x \in V_i \\ \quad 0 \quad \text{otherwise} \end{cases} \text{and } |f^*(x)|_i := \begin{cases} |f^*(x)| \text{ on } x \in V^*_i \\ \quad 0 \quad \text{otherwise} \end{cases}.$$

We call this equation *the fundamental theorem of prehomogeneous vector spaces* , or *local functional equation.*

In section 1, Example 1.1(resp. Example 1.2) comes from a prehomogeneous vector space $(GL(1) \times SO(n), \mathbb{C}^n)$ (resp. $(GL(n), M(n, \mathbb{C}))$). Example 1.1 is the case where the real form is positive definite. Here we can know the local functional equation of general case of its real forms from the theory of prehomogeneous vector spaces as follows:

For a real prehomogeneous vector space $(GL(1, \mathbb{R}) \times SO(p, q), \mathbb{R}^{p+q})$ with basic relative invariant $P^* = P = \sum_{i=1}^{p} x_i^2 - \sum_{j=p+1}^{p+q} x_j^2$,

$$\begin{bmatrix} \widehat{|P|_+^s} \\ \widehat{|P|_-^s} \end{bmatrix} = \pi^{-2s-1}\Gamma(s+1)\Gamma(s+\tfrac{p+q}{2}) \begin{bmatrix} -\sin\pi(s+\frac{q}{2}) & \sin(\frac{\pi p}{2}) \\ \sin(\frac{\pi q}{2}) & -\sin\pi(s+\frac{p}{2}) \end{bmatrix} \begin{bmatrix} |P|_+^{-s-\frac{p+q}{2}} \\ |P|_-^{-s-\frac{p+q}{2}} \end{bmatrix}. \tag{24}$$

For the case of Example 1.3 in section 1, from information of prehomogeneous vector space $(GL(n), 2\Lambda_1, \mathrm{Sym}(n))$ where $\mathrm{Sym}(n)$ is a set of symmetric matrices of size n, Shintani [**39**] got the following local functional equation:

$$\widehat{|\det X|_{(n,0)}^{s-\frac{n+1}{2}}} = (2\pi)^{-ns}\Gamma_{\mathcal{S}_n^+}(s)\sum_{p=0}^{n} e^{(2p-n)\frac{\pi\sqrt{-1}s}{2}} |\det Y|_{(p,n-p)}^{-s} \tag{25}$$

where $\Gamma_{\mathcal{S}_n^+}(s) = \prod_{i=1}^{n} \pi^{\frac{i-2}{2}}\Gamma(s - \dfrac{i-1}{2})$ and $|\det X|_{(p,n-p)}$ is defined as follows:

$$|\det X|_{(p,n-p)} = \begin{cases} |\det X| \text{ on the orbit of signature } (p, n-p) \\ 0 \quad\quad\quad\quad \text{otherwise} \end{cases}.$$

3. Variations of the fundamental theorem of prehomogeneous spaces

F.Sato [**29**] generalizes the fundamental theorem to the partial Fourier transforms with respect to regular subspaces without assuming the reductivity of the group G. Here the partial Fourier transform is defined as follows: Let (G, ρ, V) be a prehomogeneous vector space of the form $(G, \rho, V) = (G, \rho_1 \oplus \rho_2, E \oplus F)$. We call F a regular subspace if there exists a relative invariant $P(x, y)$ for $x \in E, y \in F$ such that $\det\left(\frac{\partial^2 P}{\partial y_i \partial y_j}(x, y)\right)$ is not identically zero with respect to $y \in F$. We assume that the singular set S is a hypersurface. For $V = E \oplus F$, we put $V^* = E \oplus F^*$. For $\varphi \in \mathcal{S}(V_{\mathbb{R}})$, we define the partial Fourier transform $\hat{\varphi}$ of φ with respect to the regular subspace F by

$$\hat{\varphi^*}(x, y^*) = \int_{F_{\mathbb{R}}} \varphi(x, y) e^{2\pi\sqrt{-1}\langle y, y^*\rangle} dy.$$

F.Sato [**29**] proves the fundamental theorem for such partial Fourier transforms. When G is reductive, $E = \{0\}$, and $F = V$, it is nothing but a result from M.Sato [**36**].

A.Gyoja [**11**] proves the fundamental theorem for reductive prehomogeneous vector spaces without assuming the regularity.

In general, it is difficult to determine the explicit forms of functional equations of zeta functions. In other words, the calculation of c-function $c_{ij}(s)$ (see [**18**], p.124, Theorem 4.17) would be difficult. Using M.Sato's idea, M.Kashiwara applied the theory of simple holonomic systems to prehomogeneous vector spaces, and developed an algorithm to calculate $c_{ij}(s)$, which is called microlocal calculus. Although the whole theory of Kashiwara's algorithm is not published yet, M.Kashiwara [**17**] (noted by T.Miwa in Japanese) gives an outline of this method. By using this algorithm, M.Muro and T.Suzuki calculated the explicit forms of $c_{ij}(s)$ for some irreducible regular prehomogeneous vector spaces. For example, see M.Muro [**41**].

The fundamental theorem over p-adic fields is proved by J.Igusa [**12**] in the case that the group G is reductive and the singular set S is an irreducible hypersurface with finite orbits condition. F.Sato [**30**] generalized this result without assuming the reductivity of G nor the irreducibility of S. However, he also assumed some finiteness condition for singular orbits. Examples of the functional equations over p-adic field can be found in J.Igusa [**13**], [**14**] and F.Sato [**30**]. T.Kimura , T.Kogiso and M.Fujinaga [**20**] proved the fundamental theorem over local fields of positive characteristic under certain conditions.

F.Sato and H.Ochiai [**35**] showed that Castling transforms of basic relative invariants for regular prehomogeneous vector spaces also satisfy local functional equations and one can get infinitely sequences of polynomials which satisfy local functional equations.

4. Recent development of the research for local functional equations.

4.1. Non-reductive prehomogeneous vector spaces

In [**15**], H.Ishi and T.Kogiso show that the space associated with sub-Hankel determinant is a non-reductive, regular prehomogeneous vector space, and give the multiplicative Legendre transforms of sub-Hankel determinants. Moreover we observe certain relations between b-functions of polarization of PV-polynomials and b-functions of sub-Hankel determinants, and give some formulae about sub-Hankel determinants whose components are orthogonal polynomials.

Studies for Wishart distributions of homogeneous cones are related to non-reductive regular prehomogeneous vector spaces. Laplace transforms and multiplicative Legendre transforms of basic relative invariants on the associated spaces to homogeneous cones are applied to Wishart distribution in statistics. See [**2**], [**25**], [**8**], [**16**].

4.2. Local functional equations of non-prehomogeneous type

F.Sato and T.Kogiso [**33**] , [**24**] construct polynomials of degree 4 that can not be obtained from prehomogeneous vector spaces, but, for which one can associate local zeta functions satisfying functional equations. Let C_n be the Clifford algebra of the positive definite real quadratic form $v_1^2+\cdots+v_n^2$. For a $C_p\otimes C_q$-module W, we define a homogeneous polynomial $\tilde{P}$ (called a *Clifford quartic form*) of degree 4 on W such that the associated local zeta functions satisfy a functional equation. The Clifford quartic forms $\tilde{P}$ can not be a relative invariants of any prehomogeneous vector space unless $p+q$ and $\dim W$ are small. We also classify the exceptional cases of small dimension, namely, we determine all the prehomogeneous vector spaces with Clifford quartic forms as relative invariant.

4.3. Glueing of Local functional equations

In F.Sato [**31**], the following is studied. Let G be a graph with n vertexes $\{v_1,\ldots,v_n\}$ (without multiple edges) and consider the vector space $\mathrm{Sym}_G(\mathbb{R}) = \{X \in \mathrm{Sym}_n(\mathbb{R}) \mid X_{ij}=0\ (\ v_i \nsim v_j\ )\}$. Denote by $\mathrm{Sym}_G^*(\mathbb{R})$ its dual vector space. With a statistical motivation, Letac and Massam [**25**] calculated explicitly the Gamma integral attached to the cones of positive definite "matrices" in $\mathrm{Sym}_G^*(\mathbb{R})$ and the dual cone in $\mathrm{Sym}_G(\mathbb{R})$ under the condition that G is decomposable. From their result we can derive rather easily the functional equation for the local zeta functions attached to the cones. In this note, we report that the local zeta functions attached to not necessarily definite connected components also satisfy functional equations. The cones for decomposable G are in general not homogeneous and our functional equations can not be obtained from the theory of prehomogeneous vector spaces.

4.4. Local functional equations of polarizations

For a homogeneous rational function $f\in\mathbb{C}(x_1,\ldots,x_n)$, if $f(x)$ satisfies that

$$\phi_f:\mathbb{C}^n\to\mathbb{C}^n,\ \phi_f(x)=\nabla_x\log f(x)=\frac{1}{f(x)}\nabla_x f(x) \tag{26}$$

is birational mapping, then $f(x)$ is called a homaloidal rational function. If a rational function $f^*(x^*)\in\mathbb{C}(x_1^*,\ldots,x_n^*)$ satisfies $f^*(\phi_f(x))=\dfrac{1}{f(x)}$, then f^* is called *the multiplicative Legendre transform* of f. Following [**5**], we call a polynomial f a *homaloidal EKP-polynomial* if f is homaloidal and its multiplicative Legendre transform f^* is also a polynomial. By definition, a regular prehomogeneous vector space has homaloidal relatively invariant polynomials. It is rather difficult to construct homaloidal polynomials that are not relative invariants of prehomogeneous vector spaces and the classification of homaloidal polynomials has been done only for some special cases:

- Cubic homaloidal EKP-polynomials are classified by Etingof- Kazhdan-

Polishchuk ([**6**]).
• Homaloidal polynomials in 3 variables without multiple factors are classified by Dolgachev ([**5**]).
• In [**3**], Bruno determined when a product of linear forms is homaloidal.
All the homaloidal polynomials classified in these works are relative invariants of prehomogeneous vector spaces, and Etingof, Kazhdan and Polishchuk ([**6**], §3.4, Question 1) asked whether homaloidal EKP-polynomials are relative invariants of regular prehomogeneous vector spaces. In [**24**], we show that the Clifford quartic forms are counter examples of degree 4 to the question raised by Etingof, Kazhdan and Polishchuk, since most of Clifford quartic forms([**33**] , [**24**])are non-prehomogeneous as will be shown in [**24**],Theorem 3.2.
In [**34**], we assume $n \geq 2$, let $f(x) = f(x_1, \ldots, x_n)$ be a homaloidal polynomial with $\mathbb{R}$-coefficients and degree of $f = d \geq 2$. For $x, y \in \mathbb{R}^n$, $F(x, y) := \langle \nabla_x f(x), y \rangle$ means a polarization of $f(x)$.
In [**34**], we show that the polarization of prehomogeneous polynomial are also prehomogeneous polynomial. It is difficult that the polarizations of homaloidal polynomials of non-prehomogeneous type are also hamaloidal polynomials of non-prehomogeneous type. However, we show that the ploraizations of Clifford quartic forms are also homaloidal pynomials of non-prehomogeneous type.
Thus the following theorem shows that infinitely many local functional equations of non-prehomogeneous type from polarizations of Clifford quartic forms.

For $i, j = 0, 1$, we put

$$\Omega_i := \ \{x \in \mathbb{R}^n | \mathrm{sgn}(f(x)) = (-1)^i\}, \tag{27}$$

$$\tilde{\Omega}_{i,j} := \{(x, y) \in \mathbb{R}^n \oplus \mathbb{R}^n | x \in \Omega_i, \mathrm{sgn} F(x, y) = (-1)^j\}. \tag{28}$$

Then

$$\Omega := \{x \in \mathbb{R}^n | f(x) \neq 0\} = \Omega_0 \cup \Omega_1, \tag{29}$$

$$\tilde{\Omega} := \{(x, y) \in \mathbb{R}^n \oplus \mathbb{R}^n | f(x)F(x, y) \neq 0\} = \tilde{\Omega}_{0,0} \cup \tilde{\Omega}_{0,1} \cup \tilde{\Omega}_{1,0} \cup \tilde{\Omega}_{1,1}. \tag{30}$$

We define local zeta function $Z_{i,j}(\Phi; s, t)$ as follows:

$$Z_{i,j}(\Phi; s, t) := \int_{\tilde{\Omega}_{i,j}} |f(x)|^s |F(x, y)|^t |H_f(x)| \Phi(x, y) dx dy \tag{31}$$

for a Schwartz Bruhat function $\Phi \in \mathcal{S}(\mathbb{R}^n \oplus \mathbb{R}^n)$, where $H_f(x)$ is hessian of $f(x)$. Since $f(x), F(x, y), H_f(x)$ are all polynomials, these integrals $Z_{i,j}(\Phi; s, t)$ are absolutely convergent for $\mathfrak{Re}(s) > 0, \mathfrak{Re}(t) > 0$ and can be continued to meromorphic functions of $(s, t) \in \mathbb{C}^2$. Let $f^*(x^*)$ be the multiplicative Legendre transform, $F^*(x^*,y^*) := \langle \nabla_{x^*} f^*(y^*), x^* \rangle$ be the polarization of $f^*(x^*)$. For $\phi_f(x) = \dfrac{1}{f(x)} \nabla_x f(x)$, put $\Omega^* := \phi_f(\Omega)$, $\Omega_i^* := \phi_f(\Omega_i)$ $(i = 0, 1)$, and

$$\tilde{\Omega}^*_{i,j} := \{(x^*,y^*) \in \mathbb{R}^n \oplus \mathbb{R}^n | x^* \in \Omega_i^*, \mathrm{sgn} F^*(x^*,y^*) = (-1)^j\}, \tag{32}$$

$$\tilde{\Omega}^* := \{(x^*,y^*) \in \mathbb{R}^n \oplus \mathbb{R}^n | x^* \in \Omega^*, F^*(x^*,y^*) \neq 0\} = \cup_{i,j=0,1} \tilde{\Omega}^*_{i,j}. \tag{33}$$

Then the dual local zeta distribution $Z^*_{i,j}(\Phi; s, t)$ is defined by

$$Z^*_{i,j}(\Phi; s,t) = \int_{\tilde{\Omega}^*_{i,j}} |f^*(y^*)|^s |F^*(x^*,y^*)|^t \Phi(x^*,y^*) dx^* dy^* \tag{34}$$

for $\Phi \in \mathcal{S}(\mathbb{R}^n \oplus \mathbb{R}^n)$.
For generally, $f^*(x^*), F^*(x^*,y^*)$ are rational functions and the convergent area of $Z^*_{i,j}(\Phi; s,t)$ is nontrivial, we assume the following.

Assumption of Convergence (A)
There exists a certain open set D in $\mathbb{C}^2$ such that $(s,t) \in D$, $Z^*_{i,j}(\Phi; s,t)$ $(i, j = 0, 1)$ is absolutely convergent for any $\Phi \in \mathcal{S}(\mathbb{R}^n \oplus \mathbb{R}^n)$.

THEOREM 4.1. *(F.Sato and T.Kogiso) Under the assumption (A), for any $\Phi \in \mathcal{S}(\mathbb{R}^n \oplus \mathbb{R}^n)$, $Z_{i,j}(\Phi; s,t)$ and $Z^*_{i,j}(\hat{\Phi}; s,t)$ satisfiy the following functional equation*

$$\begin{aligned} Z_{i,j}(\hat{\Phi},; s,t) = (d-1) \sum_{k,\ell=0,1} & \gamma_{i+j,k}(t) \gamma_{k,i+dk+\ell}(ds + (d-1)(t+n) - 1) \\ & \times Z^*_{i+dk,\ell}(\Phi; (d-1)s + (d-2)(n+t), -ds - (d-1)(t+n)) \end{aligned} \tag{35}$$

where $\hat{\Phi}$ means the Fourier transform

$$\hat{\Phi}(x,y) = \int_{\mathbb{R}^n \oplus \mathbb{R}^n} \Phi(x^*,y^*) \mathbf{e}[\langle x, x^* \rangle + \langle y, y^* \rangle] dx dy \tag{36}$$

$\mathbf{e}[u] = \exp(2\pi u \sqrt{-1})$. *Indices $i + dk + \ell, i + dk$ are considered as* mod 2, *and*

$$\gamma_{i,j}(s) = (2\pi)^{-(s+1)} \Gamma(s+1) \exp((-1)^{i+j} \pi \sqrt{-1}(s+1)/2). \tag{37}$$

The proof of this theorem is in [**34**].

References

[1] K.Amano, M.Fujigami and T.Kogiso , Construction of irreducible relative invariant of the prehomogeneous vector space $(SL_5 \times GL_4, \Lambda^2(\mathbb{C}) \otimes \mathbb{C}^4)$, *Linear Algebra Appl.* **355** (2002), 215–222.

[2] S. A. Andersson and G. G. Wojnar, *Wishart distributions on homogeneous cones*, J. Theoret. Probab. **17** (2004), 781–818.

[3] Bruno, A., On homaloidal polynomials, *Michigan Math. J.* **55** (2007), 347–354.

[4] J.-L.Clerc, Zeta distributions associated to a representation of a Jordan algebra, *Math.Z.,* **239** (2002), 263–276.

[5] Dolgachev, I., Polar Cremona transformations, *Michigan Math. J.* **48** (2000), 191–202.

[6] Etingof, P., Kazhdan, D. and A. Polishchuk, When is the Fourier transform of an elementary function elementary?, *Selecta Math.* **8** (2002), 27–66.

[7] J,Faraut and Koranyi, *Analysis of symmetric cones,* Oxford University Press, 1994.

[8] P. Graczyk and H. Ishi, *Riesz measures and Wishart laws associated to quadratic maps*, J. Math. Soc. Japan **66** (2014), 317–348.

[9] L.Gårding, Linear hyperbolic partial differential equations with constant coefficients. *Acta Math.*, **85**(1950),1–62.

[10] L.Gårding, Probléme de Cauchy's problem for two totally hyperbolic linear differential equations by means of Riesz i ntegrals, . *Ann. of Math.*, **48**(1947),785–826.

[11] A.Gyoja, Theory of prehomogeneous vector spaces without regularity condition, *Publ. Res. Inst. Math. Sci.,* **27**(1991), No.6, 861–922

[12] J.Igusa, Some results on p-adic complex powers, *Amer. J. Math.,* **106**(1984), No.5, 1013–1032

[13] J.Igusa, On functional equations of complex powers, *Invent. Math.,* **85**(1986), No.1, 1–29

[14] J.Igusa, Zeta distributions associated with some invariants, *Amer. J. Math.,* **109**(1987), No.1, 1–33

[15] H.Ishi and T.Kogiso, Some properties of associated spaces with sub-Hankel determinants. arXiv:1609.01039.

[16] H. Ishi, "Homogeneous cones and their applications to statistics", in Modern methods of multivariate statistics, Travaux en Cours **82**, pp. 135–154, Hermann, 2014.

[17] M.Kashiwara, Microlocal calculus and Fourier transforms of relative invariants of prehomogeneous vector spaces, *Surikaisekikenkyusho Kokyuroku* **238**(1975), No.238, 60–147, noted by Tetsuji Miwa in Japanese

[18] T.Kimura, A classification theory of prehomogeneous vector spaces, *Representations of Lie groups, Kyoto, Hiroshima, Academic Press, Boston, MA* (1988), 223–256

[19] T.Kimura, A , Introduction to prehomogeneous vector spaces. Translated from the 1998 Japanese original by Makoto Nagura and Tsuyoshi Niitani and revised by the author. Translations of Mathematical Monographs, 215. American Mathematical Society, Providence, RI, 2003.

[20] T.Kimura, T.Kogiso and M.Fujinaga, Fundamental theorem of prehomogeneous vector spaces of characteristic p, *Bull. Austral. Math. Soc.,,* **56**(1997), No.2, 331–341

[21] T.Kimura, T.Kogiso and K.Sugiyama, Relative invariants of 2-simple prehomogeneous vector spaces of Type I, *J. Alg.* **308** (2007), no. 2, 445–483.

[22] T.Kogiso, G.Miyabe, M.Kobayashi and T.Kimura, Nonregular 2-simple prehomogeneous vector space of Type I and their relative invariants, *J. Alg.,* **251**(2002), 27–69

[23] T.Kogiso, G.Miyabe, M.Kobayashi and T.Kimura, Relative invariants for some prehomogeneous vector spaces, *Math. of Comp.,* **72**(2002), No.242, 865–889

[24] T.Kogiso and F.Sato, Clifford quartic forms and local functional equations of no prehomogeneous type. J.of Math. Sci., Univ. of Tokyo. **23**(2016), 791–866.

[25] G. Letac and H. Massam, *Wishart distributions for decomposable graphs*, The Annals of Statistics **35** (2007), 1278–1323.

[26] M.Riesz, L'intégrale de Riemann-Liouville et le probème de Cauchy,. *Acta Math.*, **81**(1949), 1–223.

[27] H.Rubenthaler, Espaces prèhomogénes de type parabolique, Lectures on harmonic analysis on Lie groups and related topics (Strasbourg, 1979), *Lectures in Math.,* **14**, *Kinokuniya Book Store, Tokyo*, (1982), 189–221,

[28] F.Sato, Quadratic maps and non-prehomogeneous local functional equations, *Comment. Math. Univ. St. Pauli,* **56**(2007), No.2, 163–184.

[29] F. Sato, Zeta functions in several variables associated with prehomogeneous vector spaces I: functional equations, *Tôhoku Math. J.* **34** (1982), 437–483.

[30] F. Sato, On functional equations of zeta distributions, *Automorphic forms and geometry of arithmetic varieties, Academic Press, Boston, MA,* (1989), 465–508.

[31] F. Sato, Local functional equations associated with decomposable graphs. Josai Mathematical Monographs **6** (2013), pp. 59–69.

[32] F.Sato and T.Kogiso, Representations of Clifford algebras and quartic polynomials with local functional equations, RIMS Kokyuroku, **1617** (2008), 63–82. .

[33] F.Sato and T.Kogiso, Representations of Clifford algebras and local functional equations, *RIMS Kokyuroku-Bessatsu.*, bf B 36, (2012) 53–66.

[34] F.Sato and T.Kogiso, Local functional equations associated with the polarization of homaloidal polynomials, preprint.

[35] F.Sato and H.Ochiai, Castling transforms of prehomogeneous vector spaces and functional equations. Comment. Math. Univ. St. Paul. **40** (1991), no. 1, 61–82., .

[36] M. Sato, Theory of prehomogeneous vector spaces (notes by T.Shintani in Japanese), *Sugaku no Ayumi* **15** (1970), 85–157.

[37] M. Sato and T. Kimura, A classification of irreducible prehomogeneous vector spaces and their invariants, *Nagoya Math. J.* **65** (1977), 1–155.

[38] M.Sato and T.Shintani, On zeta functions associated with prehomogeneous spaces, *Ann. of Math.* **100**(1974), 131–170.

[39] T.Shintani, On zeta functions associated with the vector space of quadratic forms, *J. Math. Soc. Japan* **24**(1972), 132–188.

[40] G.L.Siegel, Über die analytishe theorie der quadratiche Formen. *Ann. of Math.*, **36**(1935), 527–606.

[41] M.Muro, Microlocal analysis and calculations on some relatively invariant hyper functions related to zeta functions associated with the vector spaces of quadratic forms, *Publ. Res. Inst. Math. Sci.***22** (1986), 395–463.

[42] J.Wishart, The generalized product moment distribution in samples from a normal multivariate population. *Biometrika*, **20** A (1928), 32–52.

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