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Intertwining operators between holomorphic discrete series representations

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Abstract. In this article the author presents the results on the explicit construction of the intertwining operator between a holomorphic discrete series representation of some Lie group G and that of some subgroup $G_1 \subset G$. More precisely, we construct a G_1-intertwining projection operator from a representation $\mathcal{H}$ of G onto a representation $\mathcal{H}_1$ of G_1 as a differential operator, in the case $(G, G_1) = (G_0 \times G_0, \Delta G_0)$ and both $\mathcal{H}$, $\mathcal{H}_1$ are of scalar type, and also construct a G_1-intertwining embedding operator from $\mathcal{H}_1$ of G_1 into $\mathcal{H}$ of G as an infinite-order differential operator, in the case $\mathcal{H}$ is of scalar type and $\mathcal{H}_1$ is multiplicity-free under a maximal compact subgroup $K_1 \subset G_1$. In this paper we mainly deal with the case $(G, G_1) = (Sp(1, \mathbb{R}) \times Sp(1, \mathbb{R}), \Delta Sp(1, \mathbb{R}))$ and the cases $(G, G_1) = (SU(s, s), Sp(s, \mathbb{R}))$, $(SU(s, s), SO^*(2s))$.

1. Introduction

First we review the known results on the branching lows when we restrict a representation of a Lie group to some Lie subgroup. Let G be a Lie group, $G_1 \subset G$ be a closed subgroup, and let $\mathcal{H}$ be a representation of G. We consider the restriction $\mathcal{H}|_{G_1}$ of the representation $\mathcal{H}$ to the subgroup G_1. Then in general, it may behave wildly, for example, it may contain continuous spectrums, or its multiplicity becomes infinite. However, Kobayashi and his collaborators found the conditions for $(G, G_1, \mathcal{H})$ such that $\mathcal{H}|_{G_1}$ behaves nicely, for example, it is discretely decomposable ([**4, 5, 6, 8, 15, 16**]), its multiplicity becomes finite or uniformly bounded ([**10, 12, 14**]), or it decomposes multiplicity-freely ([**7, 9**]). Then under such nice conditions, it is expected that we can study the branching low of $\mathcal{H}|_{G_1}$ more in detail and in explicit form. Namely, for a representation $\mathcal{H}_1$ of G_1 which appears abstractly in the decomposition of $\mathcal{H}|_{G_1}$, we want to construct the G_1-intertwining operator between $\mathcal{H}|_{G_1}$ and $\mathcal{H}_1$. Such program was suggested recently by Kobayashi (see [**11**]), and was studied by several people, e.g., Clerc-Kobayashi-Ørsted-Pevzner [**1**], Kobayashi-Pevzner [**17, 18**], Kobayashi-Speh [**19**], Möllers-Oshima [**20**], and Peng-Zhang [**22**].

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In this paper we assume that G and G_1 are of Hermitian type, namely, the corresponding Riemannian symmetric spaces G/K, G_1/K_1 have the natural complex structure and become the Hermitian symmetric spaces, and also assume that the embedding map $G_1/K_1 \hookrightarrow G/K$ is holomorphic. Moreover let $\mathcal{H}$ be a holomorphic discrete series representation of G. Then it is proved by Kobayashi [**7**] that the restriction $\mathcal{H}|_{G_1}$ is discretely decomposable, its multiplicity is finite, and moreover the multiplicity becomes uniformly bounded if (G, G_1) is a symmetric pair. Thus for a representation $\mathcal{H}_1$ of G_1 which appears in the decomposition of $\mathcal{H}|_{G_1}$, the author has aimed to construct the G_1-intertwining operators between $\mathcal{H}|_{G_1}$ and $\mathcal{H}_1$.

Now we summarize the author's recent results. The author constructed the G_1-intertwining operators $\mathcal{H}|_{G_1} \rightleftarrows \mathcal{H}_1$ in the case (G, G_1) is one of

$$\begin{array}{ll} (U(q,s), U(q,s')\times U(s'')), & (SO^*(2s), SO^*(2(s-1))\times SO(2)), \\ (SO^*(2s), U(1,s-1)), & (SO(2,2s), U(1,s)), \\ (E_{6(-14)}, SO(2)\times SO(2,8)) & \end{array}$$

which are given by normal derivatives, constructed the operators $\mathcal{H}|_{G_1} \to \mathcal{H}_1$ in the case (G, G_1) is of the form

$$(G_0 \times G_0, \Delta G_0)$$

when $\mathcal{H}$, $\mathcal{H}_1$ are of scalar type, which gives essentially the same results with Peng-Zhang [**22**], and constructed the operators $\mathcal{H}_1 \to \mathcal{H}|_{G_1}$ in the case (G, G_1) is one of

$$\begin{array}{ll} (Sp(s,\mathbb{R}), Sp(s',\mathbb{R})\times Sp(s'',\mathbb{R})), & (U(q,s), U(q',s')\times U(q'',s'')), \\ (SO^*(2s), SO^*(2s')\times SO^*(2s'')), & (Sp(s,\mathbb{R}), U(s',s'')), \\ (SO^*(2s), U(s',s'')), & (SU(s,s), Sp(s,\mathbb{R})), \\ (SU(s,s), SO^*(2s)), & (SO(2,n), SO(2,n')\times SO(n'')), \\ (E_{6(-14)}, SU(1,5)\times SL(2,\mathbb{R})), & (E_{6(-14)}, SO(2)\times SO^*(10)), \\ (E_{6(-14)}, SU(2,4)\times SU(2)), & (E_{7(-25)}, SL(2,\mathbb{R})\times SO(2,10)), \\ (E_{7(-25)}, SO(2)\times E_{6(-14)}), & (E_{7(-25)}, SO^*(12)\times SU(2)), \\ (E_{7(-25)}, SU(6,2)) & \end{array}$$

when $\mathcal{H}$ are of scalar type and $\mathcal{H}_1$ are multiplicity-free under a maximal compact subgroup $K_1 \subset G_1$, but except for one series of $(SU(2r+1, 2r+1), SO^*(2(2r+1)))$, and the result on $(E_{7(-25)}, SO(2)\times E_{6(-14)})$ holds under the assumption that the conjecture in the author's previous paper [**21**] is true. In this paper we only deal with the case $(G, G_1) = (Sp(1,\mathbb{R})\times Sp(1,\mathbb{R}), \Delta Sp(1,\mathbb{R}))$, $(SU(s,s), Sp(s,\mathbb{R}))$, and $(SU(s,s), SO^*(2s))$, and assume both $\mathcal{H}$, $\mathcal{H}_1$ are of scalar type. The results for other cases will appear in a forthcoming paper.

This paper is organized as follows. In Section 2 we review known facts on holomorphic discrete series representations. In Section 3 the author gives main results for simplest cases, namely for $(G, G_1) = (Sp(1,\mathbb{R}) \times Sp(1,\mathbb{R}), \Delta Sp(1,\mathbb{R}))$ (Theorem 3.1) and for $(G, G_1) = (SU(2,2), Sp(2,\mathbb{R}))$ (Theorem 3.2), and in Section 4 the proofs of these results are given. In Section 5 the author gives the results for $(G, G_1) = (SU(s,s), Sp(s,\mathbb{R}))$, $(SU(s,s), SO^*(2s))$ for general s (Theorem 5.1).

2. Holomorphic discrete series representations

First we review the holomorphic discrete series representations. Let G be a simple Lie group, and $K \subset G$ be a maximal compact subgroup of G. We assume that K has a non-discrete center, and G has a complexification $G^{\mathbb{C}}$. We denote the Lie algebras of G, K and $G^{\mathbb{C}}$ by the corresponding lower case fraktur, $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{g}^{\mathbb{C}}$. Let $\vartheta : G \to G$ be the Cartan involution which fixes K, and we extend it to $\vartheta : G^{\mathbb{C}} \to G^{\mathbb{C}}$ anti-holomorphically. Then we can take an element $z \in \mathfrak{z}(\mathfrak{k})$ in the center of $\mathfrak{k}$ such that the eigenvalues of $ad(z)$ are $+\sqrt{-1}$, 0, $-\sqrt{-1}$. We write the corresponding eigenspace decomposition as $\mathfrak{g}^{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}^-$. Then it is known that there exists a domain $D \subset \mathfrak{p}^+$ such that it is diffeomorphic to the Hermitian symmetric space G/K through the following diagram.

$$\begin{tikzcd} G/K \arrow[r] \arrow[d, dashed, "\wr"'] & G^{\mathbb{C}}/K^{\mathbb{C}}P^- \\ D \arrow[r, hook] & \mathfrak{p}^+ \arrow[u, "\exp"'] \end{tikzcd}$$

Now let (τ, V) be a holomorphic representation of $K^{\mathbb{C}}$, and χ be a suitable character of $\tilde{K}^{\mathbb{C}}$, the universal covering group of $K^{\mathbb{C}}$. We consider the space of holomorphic sections $\Gamma_{\mathcal{O}}(G/K, \tilde{G} \times_{\tilde{K}} (V \otimes \chi^{-\lambda}))$ of the holomorphic line bundle $\tilde{G} \times_{\tilde{K}} (V \otimes \chi^{-\lambda}) \to \tilde{G}/\tilde{K} = G/K$. Then since $G/K \simeq D$ is a contractible complex domain, it is isomorphic to the space of vector-valued holomorphic functions on D.

$$\Gamma_{\mathcal{O}}(G/K, \tilde{G} \times_{\tilde{K}} (V \otimes \chi^{-\lambda})) \simeq \mathcal{O}(D, V).$$

Then under this identification, $\tilde{G}$ acts on $\mathcal{O}(D, V)$ by the form

$$\tau_\lambda(g) f(w) = \chi(\mu(g^{-1}, w))^\lambda \tau(\mu(g^{-1}, w))^{-1} f(g^{-1}w) \qquad (g \in G,\ w \in D),$$

using some smooth map $\mu : \tilde{G} \times D \to \tilde{K}^{\mathbb{C}}$. Then if there exists a Hilbert subspace $\mathcal{H}_\lambda(D, V) \subset \mathcal{O}(D, V)$ such that $\tilde{G}$ acts unitarily on it, then the representation $(\tau_\lambda, \mathcal{H}_\lambda(D, V))$ is called the unitary highest weight representation of $\tilde{G}$. Especially, if the parameter λ is sufficiently large, then this action preserves the explicit inner product

$$\langle f, g\rangle_{\lambda,\tau} := \int_D \big(\tau(B(w)^{-1})f(w), g(w)\big)_\tau \chi(B(w))^{\lambda-p}dw \qquad (f, g \in \mathcal{O}(D, V)),$$

where p is an integer which is determined from $\mathfrak{g}$, and $B : \mathfrak{p}^+ \supset D \to \tilde{K}^{\mathbb{C}}$ is some smooth map. Let $\mathcal{H}_\lambda(D, V)$ be the space of all elements such that the norm given as above is finite. Then the corresponding unitary representation $(\tau_\lambda, \mathcal{H}_\lambda(D, V))$ is called the holomorphic discrete series representation of $\tilde{G}$.

Example 2.1. *We define the Lie group G as*

$$G := \left\{ g \in GL(2r, \mathbb{C}) : g \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix} {}^t g = \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix},\ g \begin{pmatrix} 0 & I_r \\ I_r & 0 \end{pmatrix} = \begin{pmatrix} 0 & I_r \\ I_r & 0 \end{pmatrix} \bar{g} \right\}.$$

Then G is isomorphic to $Sp(r, \mathbb{R})$ via the Cayley transform. The corresponding Hermitian symmetric space G/K is diffeomorphic to

$$D_{Sp(r,\mathbb{R})} = \{w \in \mathrm{Sym}(r, \mathbb{C}) : I - ww^* \textit{ is positive definite.}\}.$$

Let (τ, V) be a representation of $K^{\mathbb{C}} = GL(r, \mathbb{C})$. Then the universal covering group $\tilde{G}$ acts on $\mathcal{O}(D_{Sp(r,\mathbb{R})}, V)$ by

$$\tau_\lambda\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} \right) f(w) := \det(Cw + D)^{-\lambda} \tau\big({}^t(Cw + D)\big) f\big((Aw + B)(Cw + D)^{-1}\big).$$

Then if the parameter λ is sufficiently large, this preserves the inner product

$$\langle f, g\rangle_{\lambda,\tau} := \int_D \big(\tau((I - ww^*)^{-1})f(w), g(w)\big)_\tau \det(I - ww^*)^{\lambda-(r+1)}dw.$$

Example 2.2. *We define the Lie group G as*

$$G := SU(p, q) = \left\{ g \in SL(p + q, \mathbb{C}) : g \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} g^* = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \right\}.$$

Then the corresponding Hermitian symmetric space G/K is diffeomorphic to

$$D_{SU(p,q)} = \{w \in M(p, q; \mathbb{C}) : I - ww^* \textit{ is positive definite.}\}.$$

Then the universal covering group $\tilde{G}$ acts on $\mathcal{O}(D_{SU(p,q)})$ by

$$\tau_\lambda\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} \right) f(w) := \det(Cw + D)^{-\lambda} f\big((Aw + B)(Cw + D)^{-1}\big).$$

Then if the parameter λ is sufficiently large, this preserves the inner product

$$\langle f,g\rangle_\lambda := \int_D f(w)\overline{g(w)}\det(I-ww^*)^{\lambda-(p+q)}dw.$$

Now we consider a reductive subgroup $G_1 \subset G$. Without loss of generality we may assume that G_1 is stable under the Cartan involution ϑ of G. We denote the Cartan decompostion of $\mathfrak{g}_1 = \mathrm{Lie}(G_1)$ under ϑ as $\mathfrak{g}_1 = \mathfrak{k}_1 \oplus \mathfrak{p}_1$. We assume that $\mathfrak{p}_1^{\mathbb{C}} = (\mathfrak{p}_1^{\mathbb{C}} \cap \mathfrak{p}^+) \oplus (\mathfrak{p}_1^{\mathbb{C}} \cap \mathfrak{p}^-)$ holds, and write $\mathfrak{p}_1^+ := \mathfrak{p}_1^{\mathbb{C}} \cap \mathfrak{p}^+$, $\mathfrak{p}_2^+ := (\mathfrak{p}_1^+)^\perp$. Here the orthogonal complement is taken with respect to the restriction of the Killing form of $\mathfrak{g}^{\mathbb{C}}$ on $\mathfrak{p}^+ \times \mathfrak{p}^-$, under the identification $\overline{\mathfrak{p}^+} \simeq \mathfrak{p}^-$ via ϑ. Then G_1/K_1 is diffeomorphic to a bounded domain $D_1 \subset \mathfrak{p}_1^+$, and the embedding map $G_1/K_1 \hookrightarrow G/K$ is holomorphic. Now, since the $\tilde{K}$-finite part $\mathcal{O}(D,V)_{\tilde{K}}$ of $\mathcal{O}(D,V)$ is equal to the space $\mathcal{P}(\mathfrak{p}^+,V)$ of V-valued polynomials on $\mathfrak{p}^+$, and since $\mathfrak{p}^+$ acts on $\mathcal{O}(D,V)_{\tilde{K}} = \mathcal{P}(\mathfrak{p}^+,V)$ by first order differential operators with constant coefficients, the space of $\mathfrak{p}^+$-null vectors is equal to the space of polynomials on $\mathfrak{p}_2^+$.

$$(\mathcal{O}(D,V)_{\tilde{K}})^{\mathfrak{p}_1^+} \simeq \mathcal{P}(\mathfrak{p}_2^+,V).$$

Then since every $(\mathfrak{g}_1,\tilde{K}_1)$-submodule in $\mathcal{O}(D,V)_{\tilde{K}}$ contains some $\mathfrak{p}_1^+$-null vectors, if $(\mathcal{H}_\lambda(D,V)_{\tilde{K}})^{\mathfrak{p}_1^+} \simeq \mathcal{H}_\lambda(D,V)_{\tilde{K}} \cap \mathcal{P}(\mathfrak{p}_2^+,V)$ is decomposed under $\tilde{K}_1$ as

$$\mathcal{H}_\lambda(D,V)_{\tilde{K}} \cap \mathcal{P}(\mathfrak{p}_2^+,V) = \bigoplus_{W\in\hat{K}_1} m(W)W \otimes \chi^{-\lambda},$$

then $\mathcal{H}_\lambda(D,V)$ is decomposed under $\tilde{G}_1$ as the direct sum of Hilbert subspaces

$$\mathcal{H}_\lambda(D,V)|_{\tilde{G}_1} \simeq \sum_{W\in\hat{K}_1}^{\oplus} m(W)\mathcal{H}_\lambda(D_1,W).$$

We note that if λ is sufficiently large, $\mathcal{H}_\lambda(D,V)_{\tilde{K}} = \mathcal{P}(\mathfrak{p}^+,V)$ holds, and hence $(\mathcal{H}_\lambda(D,V)_{\tilde{K}})^{\mathfrak{p}_1^+} \simeq \mathcal{P}(\mathfrak{p}_2^+,V)$ holds. In this case the computation of the decomposition is easier. In such cases we want to construct the $\tilde{G}_1$-intertwining operators

$$\mathcal{H}_\lambda(D,V)|_{\tilde{G}_1} \rightleftarrows \mathcal{H}_\lambda(D_1,W).$$

3. Main results for the simplest cases

In this section, we consider

$$(G,G_1) = (Sp(1,\mathbb{R}) \times Sp(1,\mathbb{R}), \Delta Sp(1,\mathbb{R})), \quad \tilde{G} \curvearrowright \mathcal{H}_\lambda(D_{Sp(1,\mathbb{R})}) \hat{\boxtimes} \mathcal{H}_\mu(D_{Sp(1,\mathbb{R})}),$$

or

$$(G, G_1) = (SU(2,2), Sp(2,\mathbb{R})), \quad \tilde{G} \curvearrowright \mathcal{H}_\lambda(D_{SU(2,2)}),$$

where $\mathcal{H}_\lambda(D_{Sp(s,\mathbb{R})})$, $\mathcal{H}_\lambda(D_{SU(p,q)})$ are the holomorphic discrete series representations of $Sp(s,\mathbb{R})$, $SU(p,q)$ of scalar type respectively. We recall that $\widetilde{Sp}(s,\mathbb{R})$ (resp. $\widetilde{SU}(p,q)$) acts on $\mathcal{O}(D_{Sp(s,\mathbb{R})})$ (resp. $\mathcal{O}(D_{SU(p,q)})$) by

$$\tau_\lambda\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1}\right) f(w) = \det(Cw+D)^{-\lambda} f\left((Aw+B)(Cw+D)^{-1}\right).$$

Then there exists a unitary subrepresentation $\mathcal{H}_\lambda(D) \subset \mathcal{O}(D)$ if and only if

$$\lambda \in \left\{0, \frac{1}{2}, 1, \ldots, \frac{s-1}{2}\right\} \cup \left(\frac{s-1}{2}, \infty\right) \qquad (G = Sp(s,\mathbb{R})),$$
$$\lambda \in \{0, 1, 2, \ldots, \min\{p,q\}-1\} \cup (\min\{p,q\}-1, \infty) \qquad (G = SU(p,q)).$$

The K-finite part $\mathcal{H}_\lambda(D)_{\tilde{K}}$ is equal to the whole polynomial space $\mathcal{P}(\mathrm{Sym}(s,\mathbb{C}))$ (resp. $\mathcal{P}(M(p,q;\mathbb{C}))$) if $\lambda > \frac{s-1}{2}$ (resp. $\lambda > \min\{p,q\}-1$), and $\mathcal{H}_\lambda(D)$ is a holomorphic discrete series if $\lambda > s$ (resp. $\lambda > p+q-1$).

When $(G, G_1) = (Sp(1,\mathbb{R}) \times Sp(1,\mathbb{R}), \Delta Sp(1,\mathbb{R}))$, the spaces $\mathfrak{p}^+$, $\mathfrak{p}_1^+$ and $\mathfrak{p}_2^+$ are given as

$$\mathfrak{p}^+ = \mathrm{Sym}(1,\mathbb{C}) \oplus \mathrm{Sym}(1,\mathbb{C}), \qquad \begin{aligned} \mathfrak{p}_1^+ &= \{(x,x)\} \simeq \mathrm{Sym}(1,\mathbb{C}), \\ \mathfrak{p}_2^+ &= \{(x,-x)\} \simeq \mathrm{Sym}(1,\mathbb{C}). \end{aligned}$$

Then the polynomial space $\mathcal{P}(\mathfrak{p}_2^+)$ is decomposed under $K_1 = \Delta U(1)$ as

$$\mathcal{P}(\mathfrak{p}_2^+) = \bigoplus_{k=0}^{\infty} \mathcal{P}_k(\mathrm{Sym}(1,\mathbb{C})) \simeq \bigoplus_{k=0}^{\infty} \mathbb{C}_{-2k}.$$

Accordingly, when $\lambda, \mu > 0$, $\mathcal{H}_\lambda(D_{Sp(1,\mathbb{R})}) \hat{\boxtimes} \mathcal{H}_\mu(D_{Sp(1,\mathbb{R})})$ is decomposed under $\tilde{G}_1$ as

$$\begin{aligned} \mathcal{H}_\lambda(D_{Sp(1,\mathbb{R})}) \hat{\boxtimes} \mathcal{H}_\mu(D_{Sp(1,\mathbb{R})})|_{\tilde{G}_1} &\simeq \sum_{k=0}^{\infty}{}^{\oplus} \mathcal{H}_{\lambda+\mu}(D_{Sp(1,\mathbb{R})}, \mathbb{C}_{-2k}) \\ &\simeq \sum_{k=0}^{\infty}{}^{\oplus} \mathcal{H}_{\lambda+\mu+2k}(D_{Sp(1,\mathbb{R})}). \end{aligned}$$

Similarly, when $(G, G_1) = (SU(2,2), Sp(2,\mathbb{R}))$, the spaces $\mathfrak{p}^+$, $\mathfrak{p}_1^+$ and $\mathfrak{p}_2^+$ are given as

$$\mathfrak{p}^+ = M(2,\mathbb{C}), \qquad \mathfrak{p}_1^+ = \mathrm{Sym}(2,\mathbb{C}), \qquad \mathfrak{p}_2^+ = \mathrm{Skew}(2,\mathbb{C}).$$

Then the polynomial space $\mathcal{P}(\mathfrak{p}_2^+)$ is decomposed under $K_1 = U(2)$ as

$$\mathcal{P}(\mathfrak{p}_2^+) = \bigoplus_{k=0}^{\infty} \mathcal{P}_k(\mathrm{Skew}(2,\mathbb{C})) \simeq \bigoplus_{k=0}^{\infty} \mathbb{C}_{-k}.$$

Accordingly, when $\lambda > 1$, $\mathcal{H}_\lambda(D_{SU(2,2)})$ is decomposed under $\tilde{G}_1$ as

$$\mathcal{H}_\lambda(D_{SU(2,2)})|_{\tilde{G}_1} \simeq \sum_{k=0}^{\infty}{}^{\oplus} \mathcal{H}_\lambda(D_{Sp(2,\mathbb{R})}, \mathbb{C}_{-k}) \simeq \sum_{k=0}^{\infty}{}^{\oplus} \mathcal{H}_{\lambda+k}(D_{Sp(2,\mathbb{R})}).$$

Now we give the main theorems for these cases.

THEOREM 3.1 (COHEN [**2**], PENG-ZHANG [**22**], KOBAYASHI-PEVZNER [**18**]). *When* $(G, G_1) = (Sp(1,\mathbb{R}) \times Sp(1,\mathbb{R}), \Delta Sp(1,\mathbb{R}))$,

$$\mathcal{F}^*_{\lambda,\mu,k} : \mathcal{H}_\lambda(D_{Sp(1,\mathbb{R})}) \hat{\boxtimes}\, \mathcal{H}_\mu(D_{Sp(1,\mathbb{R})}) \to \mathcal{H}_{\lambda+\mu+2k}(D_{Sp(1,\mathbb{R})}) \quad (\lambda,\mu > 0, k \in \mathbb{Z}_{\geq 0}),$$

$$\mathcal{F}^*_{\lambda,\mu,k} f(y) := \sum_{m=0}^{\infty} \frac{(-k)_m}{(\lambda)_{k-m}(\mu)_m} \frac{1}{m!} \left. \frac{\partial^k}{\partial x_L^{k-m} \partial x_R^m} \right|_{x_L = x_R = y} f(x_L, x_R)$$

intertwines the $\tilde{G}_1$*-action.*

Here $(\lambda)_m := \lambda(\lambda+1)\cdots(\lambda+m-1)$.

THEOREM 3.2. *When* $(G, G_1) = (SU(2,2), Sp(2,\mathbb{R}))$,

$$\mathcal{F}_{\lambda,k} : \mathcal{H}_{\lambda+k}(D_{Sp(2,\mathbb{R})}) \to \mathcal{H}_\lambda(D_{SU(2,2)}) \quad (\lambda > 1, k \in \mathbb{Z}_{\geq 0}),$$

$$\mathcal{F}_{\lambda,k} f \begin{pmatrix} x_{11} & x_{12}+x_2 \\ x_{12}-x_2 & x_{22} \end{pmatrix} = x_2^k \sum_{m=0}^{\infty} \frac{1}{\left(\lambda+k-\frac{1}{2}\right)_m} \frac{(-1)^m}{m!} \times x_2^{2m} \left(\frac{\partial^2}{\partial x_{11} \partial x_{22}} - \frac{1}{4} \frac{\partial^2}{\partial x_{12}^2} \right)^m f \begin{pmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{pmatrix}$$

intertwines the $\tilde{G}_1$*-action.*

Remark 3.3. *The operator* $\mathcal{F}_{\lambda,k} : \mathcal{H}_{\lambda+k}(D_{Sp(2,\mathbb{R})}) \to \mathcal{H}_\lambda(D_{SU(2,2)})$ *gives the adjoint of the* $\tilde{G}_1$*-intertwining operator* $\mathcal{H}_\lambda(D_{SU(2,2)}) \to \mathcal{H}_{\lambda+k}(D_{Sp(2,\mathbb{R})})$ *given by Kobayashi's F-method* [**13, 17, 18**].

4. Proof of main results

We prove Theorems 3.1 and 3.2 in the following steps.

1. Find the $\tilde{G}_1$-invariant kernel function.
2. Write the intertwining operator in the integral expression.
3. Rewrite this in the differential expression.
4. Compute explicitly this by using the series expansion etc.

We work in more general setting until Step 3, namely,

$$(G, G_1) = (Sp(s,\mathbb{R}) \times Sp(s,\mathbb{R}), \Delta Sp(s,\mathbb{R})),$$
$$\mathcal{H}_\lambda(D_{Sp(s,\mathbb{R})}) \hat{\boxtimes} \mathcal{H}_\mu(D_{Sp(s,\mathbb{R})}) \supset \mathcal{H}_{\lambda+\mu}(D_{Sp(s,\mathbb{R})}, W),$$

where $W \subset \mathcal{P}(\mathrm{Sym}(s,\mathbb{C}))$ is a $K_1 = U(s)$-submodule, or

$$(G, G_1) = (SU(s,s), Sp(s,\mathbb{R})),$$
$$\mathcal{H}_\lambda(D_{SU(s,s)}) \supset \mathcal{H}_\lambda(D_{Sp(s,\mathbb{R})}, W),$$

where $W \subset \mathcal{P}(\mathrm{Skew}(s,\mathbb{C}))$ is a $K_1 = U(s)$-submodule.

Step 1: Find the $\tilde{G}_1$-invariant kernel function.

We fix a $K_1 = U(s)$-submodule $W \subset \mathcal{P}(\mathrm{Sym}(s,\mathbb{C}))$ (resp. $W \subset \mathcal{P}(\mathrm{Skew}(s;\mathbb{C}))$). Then there exists uniquely (up to constant multiple) a polynomial $K_W(x,y) \in W \otimes \overline{W} \subset \mathcal{P}(\mathrm{Sym}(s,\mathbb{C}) \times \overline{\mathrm{Sym}(s,\mathbb{C})})$ (resp. $\mathcal{P}(\mathrm{Skew}(s,\mathbb{C}) \times \overline{\mathrm{Skew}(s,\mathbb{C})})$) such that

$$K_W(kx{}^tk, k^{*-1}y\bar{k}^{-1}) = K_W(x,y)$$
$$(x, y \in \mathfrak{p}_2^+ = \mathrm{Sym}(s,\mathbb{C}) \text{ (resp. } \mathrm{Skew}(s,\mathbb{C})),\ k \in K_1^{\mathbb{C}} = GL(s,\mathbb{C}))$$

We define $\hat{K}_W(x_L, x_R; y_1, y_2) \in \mathcal{O}(\mathfrak{p}^+ \times \overline{\mathfrak{p}_1^+ \times \mathfrak{p}_2^+}) = \mathcal{O}(\mathrm{Sym}(s,\mathbb{C}) \times \overline{\mathrm{Sym}(s,\mathbb{C}) \times \mathrm{Sym}(s,\mathbb{C}) \times \mathrm{Sym}(s,\mathbb{C})})$ by

$$\hat{K}_W(x_L, x_R; y_1, y_2)$$
$$:= \det(I - y_1^* x_L)^{-\lambda} \det(I - y_1^* x_R)^{-\mu} K_W\big(x_L(I - y_1^* x_L)^{-1} - x_R(I - y_1^* x_R)^{-1}, y_2\big),$$

or define $\hat{K}'_W(x; y_1, y_2) \in \mathcal{O}(\mathfrak{p}^+ \times \overline{\mathfrak{p}_1^+ \times \mathfrak{p}_2^+}) = \mathcal{O}(M(s,\mathbb{C}) \times \overline{\mathrm{Sym}(s,\mathbb{C}) \times \mathrm{Skew}(s,\mathbb{C})})$ by

$$\hat{K}'_W(x; y_1, y_2) := \det(I - y_1^* x)^{-\lambda} K_W\left(\frac{1}{2}\left(x(I - y_1^* x)^{-1} - {}^t\big(x(I - y_1^* x)^{-1}\big)\right), y_2\right).$$

Then $\hat{K}_W$ and $\hat{K}'_W$ satisfy the $\tilde{G}_1$-invariance in the following sense. For $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(s,\mathbb{R}) \subset U(s,s)$ and $x \in M(s,\mathbb{C})$, we write $gx := (Ax+B)(Cx+D)^{-1}$.

PROPOSITION 4.1. *(1) For $x_L, x_R, y_1, y_2 \in \mathrm{Sym}(s,\mathbb{C})$, $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(s,\mathbb{R})$,*

$$\begin{aligned} &\hat{K}_W\left(gx_L, gx_R; gy_1, {}^t(Cy_1+D)^{-1}y_2(Cy_1+D)^{-1}\right) \\ &\quad = \det(Cx_L+D)^\lambda \det(Cx_R+D)^\mu \hat{K}_W(x_L,x_R;y_1,y_2)\overline{\det(Cy_1+D)}^{\lambda+\mu}. \end{aligned}$$

(2) For $x \in M(s,\mathbb{C})$, $y_1 \in \mathrm{Sym}(s,\mathbb{C})$, $y_2 \in \mathrm{Skew}(s,\mathbb{C})$, $g \in Sp(s,\mathbb{R})$,

$$\begin{aligned} &\hat{K}'_W\left(gx; gy_1, {}^t(Cy_1+D)^{-1}y_2(Cy_1+D)^{-1}\right) \\ &\qquad\qquad = \det(Cx+D)^\lambda \hat{K}'_W(x;y_1,y_2)\overline{\det(Cy_1+D)}^{\lambda}. \end{aligned}$$

Step 2: Write the intertwining operator in the integral expression.

From the above proposition, we easily get the following corollary.

COROLLARY 4.2. *(1) When $(G,G_1) = (Sp(s,\mathbb{R})\times Sp(s,\mathbb{R}), \Delta Sp(s,\mathbb{R}))$, the linear maps*

$$\begin{aligned} \mathcal{F}^*_W &: \mathcal{H}_\lambda(D_{Sp(s,\mathbb{R})}) \hat{\boxtimes} \mathcal{H}_\mu(D_{Sp(s,\mathbb{R})}) \to \mathcal{H}_{\lambda+\mu}(D_{Sp(s,\mathbb{R})}, W), \\ \mathcal{F}^*_W f(y_1,y_2) &:= \langle f, \hat{K}_W(\cdot,\cdot;y_1,y_2)\rangle_{\mathcal{H}_\lambda(D_{Sp(s,\mathbb{R})})\hat{\boxtimes}\mathcal{H}_\mu(D_{Sp(s,\mathbb{R})})}, \\ \mathcal{F}_W &: \mathcal{H}_{\lambda+\mu}(D_{Sp(s,\mathbb{R})}, W) \to \mathcal{H}_\lambda(D_{Sp(s,\mathbb{R})}) \hat{\boxtimes} \mathcal{H}_\mu(D_{Sp(s,\mathbb{R})}), \\ \mathcal{F}_W f(x_L,x_R) &:= \langle f, \overline{\hat{K}_W(x_L,x_R;\cdot,\cdot)}\rangle_{\mathcal{H}_{\lambda+\mu}(D_{Sp(s,\mathbb{R})},W)} \end{aligned}$$

intertwine the $\tilde{G}_1$-action.

(2) When $(G,G_1) = (SU(s,s), Sp(s,\mathbb{R}))$, the linear maps

$$\begin{aligned} \mathcal{F}^*_W &: \mathcal{H}_\lambda(D_{SU(s,s)}) \to \mathcal{H}_\lambda(D_{Sp(s,\mathbb{R})}, W), \\ \mathcal{F}^*_W f(y_1,y_2) &:= \langle f, \hat{K}'_W(\cdot;y_1,y_2)\rangle_{\mathcal{H}_\lambda(D_{SU(s,s)})}, \\ \mathcal{F}_W &: \mathcal{H}_\lambda(D_{Sp(s,\mathbb{R})}, W) \to \mathcal{H}_\lambda(D_{SU(s,s)}), \\ \mathcal{F}_W f(x) &:= \langle f, \overline{\hat{K}'_W(x;\cdot,\cdot)}\rangle_{\mathcal{H}_\lambda(D_{Sp(s,\mathbb{R})},W)} \end{aligned}$$

intertwine the $\tilde{G}_1$-action.

Especially, when $(G,G_1) = (Sp(s,\mathbb{R})\times Sp(s,\mathbb{R}), \Delta Sp(s,\mathbb{R}))$, if $\lambda,\mu > s$, then since the inner product is given by the explicit integral, the intertwining operator

$$\mathcal{F}_W^* : \mathcal{H}_\lambda(D_{Sp(s,\mathbb{R})}) \hat{\boxtimes} \mathcal{H}_\mu(D_{Sp(s,\mathbb{R})}) \to \mathcal{H}_{\lambda+\mu}(D_{Sp(s,\mathbb{R})}, W)$$

is given by the integral operator

$$\begin{aligned}\mathcal{F}_W^* f(y_1, y_2) &= \langle f, \hat{K}_W(\cdot,\cdot; y_1, y_2)\rangle_{\mathcal{H}_\lambda(D_{Sp(s,\mathbb{R})})\hat{\boxtimes}\mathcal{H}_\mu(D_{Sp(s,\mathbb{R})})} \\ &= \iint_{D_{Sp(s,\mathbb{R})}\times D_{Sp(s,\mathbb{R})}} \overline{\hat{K}_W(x_L, x_R; y_1, y_2)} f(x_L, x_R) \\ &\qquad\qquad \times \det(I - x_L x_L^*)^{\lambda-(s+1)} \det(I - x_R x_R^*)^{\mu-(s+1)} dx_L dx_R,\end{aligned}$$

and

$$\mathcal{F}_W : \mathcal{H}_{\lambda+\mu}(D_{Sp(s,\mathbb{R})}, W) \to \mathcal{H}_\lambda(D_{Sp(s,\mathbb{R})}) \hat{\boxtimes} \mathcal{H}_\mu(D_{Sp(s,\mathbb{R})})$$

is written as

$$\begin{aligned}\mathcal{F}_W f(x_L, x_R) &= \langle f, \overline{\hat{K}_W(x_L, x_R; \cdot,\cdot)}\rangle_{\mathcal{H}_{\lambda+\mu}(D_{Sp(s,\mathbb{R})}, W)} \\ &= \iint_{D_{Sp(s,\mathbb{R})}\times \mathrm{Sym}(s,\mathbb{R})} \hat{K}_W(x_L, x_R; y_1, (I - y_1 y_1^*) y_2 (I - y_1^* y_1)) f(y_1, y_2) \\ &\qquad\qquad \times e^{-\operatorname{tr}(y_2 y_2^*)} \det(I - y_1 y_1^*)^{\lambda+\mu-(s+1)} dy_1 dy_2.\end{aligned}$$

Similarly, when $(G, G_1) = (SU(s,s), Sp(s,\mathbb{R}))$ and $\lambda > 2s-1$, the intertwining operators are given by the integral operators.

Step 3: Rewrite the operator in the differential expression.

In the previous steps, we got the intertwining operators in the integral expession. However, the kernel function $\hat{K}_W$ is a bit complicated. Moreover, in general, the intertwining operator $\mathcal{H}|_{G_1} \to \mathcal{H}_1$ from the holomorphic discrete series representation of the larger group to that of the smaller group is given by a differential operator (Kobayashi-Pevzner [**17**]), but we cannot see this fact directly from the integral expression. Therefore we want to rewrite the intertwining operator in different form. In order to do this, we make use of the following fact. Let $\mathfrak{p}^+$ be a complex vector space with the inner product $(\cdot|\cdot)$, and let $\dim \mathfrak{p}^+ = n$. Then for any $f \in \mathcal{P}(\mathfrak{p}^+)$, we have

$$f(x) = \frac{1}{\pi^n} \int_{\mathfrak{p}^+} f(z) e^{(x|z)} e^{-(z|z)} dz.$$

That is, $\mathcal{O} \cap L^2\left(\mathfrak{p}^+, \frac{1}{\pi^n} e^{-(z|z)} dz\right)$ has the reproducing kernel $e^{(x|z)}$.

First we consider $\mathcal{F}_W^* : \mathcal{H}_\lambda(D_{Sp(s,\mathbb{R})}) \hat{\boxtimes} \mathcal{H}_\mu(D_{Sp(s,\mathbb{R})}) \to \mathcal{H}_{\lambda+\mu}(D_{Sp(s,\mathbb{R})}, W)$. By substituting f with the previous equality, we get

$$\begin{aligned}
&\mathcal{F}_W^* f(y_1,y_2) = \langle f, \hat{K}_W(\cdot,\cdot;y_1,y_2)\rangle_{\mathcal{H}_\lambda(D_{Sp(s,\mathbb{R})})\hat{\boxtimes}\mathcal{H}_\mu(D_{Sp(s,\mathbb{R})})} \\
&= \frac{1}{\pi^{s(s+1)}}\iint_{\mathrm{Sym}(s,\mathbb{C})^{\oplus 2}} f(z_L,z_R)\langle e^{\mathrm{tr}(x_Lz_L^*)+\mathrm{tr}(x_Rz_R^*)}, \hat{K}_W(x_L,x_R;y_1,y_2)\rangle_{\mathcal{H}_\lambda\hat{\boxtimes}\mathcal{H}_\mu,(x_L,x_R)} \\
&\qquad\qquad\qquad\qquad \times e^{-\mathrm{tr}(z_Lz_L^*)-\mathrm{tr}(z_Rz_R^*)}dz_Ldz_R \\
&= \frac{1}{\pi^{s(s+1)}}\iint_{\mathrm{Sym}(s,\mathbb{C})^{\oplus 2}} f(z_L,z_R)\mathcal{F}_W^*(e^{\mathrm{tr}(x_Lz_L^*)+\mathrm{tr}(x_Rz_R^*)})_{x_L,x_R}(y_1,y_2) \\
&\qquad\qquad\qquad\qquad \times e^{-\mathrm{tr}(z_Lz_L^*)-\mathrm{tr}(z_Rz_R^*)}dz_Ldz_R.
\end{aligned}$$

Since $\mathcal{F}_W^*$ intertwines the $\tilde{G}_1$-action and $\exp(\mathfrak{p}_1^+)$ acts by translation,

$$\begin{aligned}
&\mathcal{F}_W^*(e^{\mathrm{tr}(x_Lz_L^*)+\mathrm{tr}(x_Rz_R^*)})_{x_L,x_R}(y_1,y_2) = \mathcal{F}_W^*(e^{\mathrm{tr}((x_L+y_1)z_L^*)+\mathrm{tr}((x_R+y_1)z_R^*)})_{x_L,x_R}(0,y_2) \\
&= \langle e^{\mathrm{tr}(x_Lz_L^*)+\mathrm{tr}(x_Rz_R^*)}, \hat{K}_W(x_L,x_R;0,y_2)\rangle_{\mathcal{H}_\lambda\hat{\boxtimes}\mathcal{H}_\mu,(x_L,x_R)} e^{\mathrm{tr}(y_1z_L^*)+\mathrm{tr}(y_1z_R^*)} \\
&= \langle e^{\mathrm{tr}(x_Lz_L^*)+\mathrm{tr}(x_Rz_R^*)}, K_W(x_L-x_R,y_2)\rangle_{\mathcal{H}_\lambda\hat{\boxtimes}\mathcal{H}_\mu,(x_L,x_R)} e^{\mathrm{tr}(y_1z_L^*)+\mathrm{tr}(y_1z_R^*)}.
\end{aligned}$$

Now we define

$$F_W^*(z_L,z_R;y_2) := \langle e^{\mathrm{tr}(x_Lz_L^*)+\mathrm{tr}(x_Rz_R^*)}, K_W(x_L-x_R,y_2)\rangle_{\substack{\mathcal{H}_\lambda(D_{Sp(s,\mathbb{R})})\hat{\boxtimes}\mathcal{H}_\mu(D_{Sp(s,\mathbb{R})}),\\ (x_L,x_R)}}.$$

Then this is a polynomial, and the intertwining operator $\mathcal{F}_W^* : \mathcal{H}_\lambda(D_{Sp(s,\mathbb{R})})\hat{\boxtimes}\mathcal{H}_\mu(D_{Sp(s,\mathbb{R})}) \to \mathcal{H}_{\lambda+\mu}(D_{Sp(s,\mathbb{R})},W)$ is given by

$$\begin{aligned}
&\mathcal{F}_W^* f(y_1,y_2) \\
&= \frac{1}{\pi^{s(s+1)}}\iint_{\mathrm{Sym}(s,\mathbb{C})^{\oplus 2}} f(z_L,z_R)F_W^*(z_L,z_R;y_2)e^{\mathrm{tr}(y_1z_L^*)+\mathrm{tr}(y_1z_R^*)} \\
&\qquad\qquad\qquad\qquad \times e^{-\mathrm{tr}(z_Lz_L^*)-\mathrm{tr}(z_Rz_R^*)}dz_Ldz_R \\
&= \frac{1}{\pi^{s(s+1)}}\iint_{\mathrm{Sym}(s,\mathbb{C})^{\oplus 2}} f(z_L,z_R)\, F_W^*\left(\overline{\frac{\partial}{\partial x_L}},\overline{\frac{\partial}{\partial x_R}};y_2\right) e^{\mathrm{tr}(x_Lz_L^*)+\mathrm{tr}(x_Rz_R^*)}\Big|_{x_L=x_R=y_1} \\
&\qquad\qquad\qquad\qquad \times e^{-\mathrm{tr}(z_Lz_L^*)-\mathrm{tr}(z_Rz_R^*)}dz_Ldz_R \\
&= F_W^*\left(\overline{\frac{\partial}{\partial x_L}},\overline{\frac{\partial}{\partial x_R}};y_2\right)\Big|_{x_L=x_R=y_1} f(x_L,x_R).
\end{aligned}$$

Similarly we define

$$\begin{aligned}
&F_W(x_2;w_1,w_2) := \langle e^{\mathrm{tr}(y_1w_1^*)+\mathrm{tr}(y_2w_2^*)}, \overline{\hat{K}_W(x_2,-x_2;y_1,y_2)}\rangle_{\mathcal{H}_{\lambda+\mu}(D_{Sp(s,\mathbb{R})},W),(y_1,y_2)} \\
&= \langle e^{\mathrm{tr}(y_1w_1^*)+\mathrm{tr}(y_2w_2^*)}, \overline{\det(I-\overline{y_1}x_2)^{-\lambda}\det(I+\overline{y_1}x_2)^{-\mu}} \\
&\qquad\qquad \times \overline{K_W\left(x_2(I-y_1^*x_2)^{-1}+x_2(I+y_1^*x_2)^{-1},y_2\right)}\rangle_{\mathcal{H}_{\lambda+\mu}(D_{Sp(s,\mathbb{R})},W),(y_1,y_2)}
\end{aligned}$$

$$= \langle e^{\mathrm{tr}(y_1 w_1^*)+\mathrm{tr}(y_2 w_2^*)}, \overline{\det(I - y_1^* x_2)^{-\lambda} \det(I + y_1^* x_2)^{-\mu}} \\ \times \overline{K_W\left(2(I - x_2 y_1^*)^{-1} x_2 (I + y_1^* x_2)^{-1}, y_2\right)} \rangle_{\mathcal{H}_{\lambda+\mu}(D_{Sp(s,\mathbb{R})}, W), (y_1, y_2)}.$$

Then $\mathcal{F}_W : \mathcal{H}_{\lambda+\mu}(D_{Sp(s,\mathbb{R})}, W) \to \mathcal{H}_\lambda(D_{Sp(s,\mathbb{R})}) \hat{\boxtimes} \mathcal{H}_\mu(D_{Sp(s,\mathbb{R})})$ is given by

$$\mathcal{F}_W f(x_L, x_R) = F_W \left(x_L - x_R; \overline{\frac{\partial}{\partial y_1}}, \overline{\frac{\partial}{\partial y_2}} \right)\Bigg|_{\substack{y_1 = x_L + x_R, \\ y_2 = 0}} f(y_1, y_2).$$

This is an infinite-order differential operator, but we can show this is well-defined on $\mathcal{O}(D_{Sp(s,\mathbb{R})}, W)$.

Similarly, when $(G, G_1) = (SU(s,s), Sp(s,\mathbb{R}))$, we define

$$\begin{aligned} F'^*_W(z; y_2) &:= \left\langle e^{\mathrm{tr}(xz^*)}, \hat{K}'_W(x; 0, y_2) \right\rangle_{\mathcal{H}_\lambda(D_{SU(s,s)}), x} \\ &= \left\langle e^{\mathrm{tr}(xz^*)}, K'_W(x_2, y_2) \right\rangle_{\mathcal{H}_\lambda(D_{SU(s,s)}), x}, \end{aligned}$$

$$F'_W(x_2; w_1, w_2) := \left\langle e^{\mathrm{tr}(yw^*)}, \overline{\hat{K}'_W(x_2; y_1, y_2)} \right\rangle_{\mathcal{H}_\lambda(D_{Sp(s,\mathbb{R})}, W), y}$$

$$= \left\langle e^{\mathrm{tr}(yw^*)}, \overline{\det(I - y_1^* x_2 y_1^* x_2)^{-\lambda/2} K'_W\big(x_2 (I - y_1^* x_2 y_1^* x_2)^{-1}, y_2\big)} \right\rangle_{\mathcal{H}_\lambda(D_{Sp(s,\mathbb{R})}, W), y}$$

(where we write $M(s,\mathbb{C}) \ni x = x_1 + x_2 \in \mathrm{Sym}(s,\mathbb{C}) \oplus \mathrm{Skew}(s,\mathbb{C})$ etc.). Then the intertwining operators $\mathcal{H}_\lambda(D_{SU(s,s)}) \underset{\mathcal{F}_W}{\overset{\mathcal{F}^*_W}{\rightleftarrows}} \mathcal{H}_\lambda(D_{Sp(s,\mathbb{R})}, W)$ are given by

$$\begin{aligned} \mathcal{F}^*_W f(y_1, y_2) &:= F'^*_W \left(\overline{\frac{\partial}{\partial x}}; y_2 \right)\Bigg|_{x_1 = y_1, x_2 = 0} f(x), \\ \mathcal{F}_W f(x) &:= F'_W \left(x_2; \overline{\frac{\partial}{\partial y_1}}, \overline{\frac{\partial}{\partial y_2}} \right)\Bigg|_{y_1 = x_1, y_2 = 0} f(y_1, y_2). \end{aligned}$$

Thus we want to compute explicitly

$$F^*_W(z_L, z_R; y_2),\ F_W(x_2; w_1, w_2),\ F'^*_W(z; y_2),\ F'_W(x_2; w_1, w_2).$$

In fact, $F^*_W(z_L, z_R; y_2)$ and $F'_W(x_2; w_1, w_2)$ are explicitly computable for some $W \subset \mathcal{P}(\mathrm{Sym}(s,\mathbb{C}))$ and $W \subset \mathcal{P}(\mathrm{Skew}(s,\mathbb{C}))$, respectively.

Step 4: Compute the differential expression explicitly.

Now we compute the differential expression of the intertwining operators in the simplest cases. First we let $s = 1$ and consider $(G, G_1) = (Sp(1,\mathbb{R}) \times Sp(1,\mathbb{R}), \Delta Sp(1,\mathbb{R}))$. If $\lambda, \mu > 1$, $F^*_W(z_L, z_R; y_2)$ is given by

$$F^*_W(z_L, z_R; y_2) = \langle e^{x_L \overline{z_L} + x_R \overline{z_R}}, K_W(x_L - x_R, y_2) \rangle_{\mathcal{H}_\lambda(D_{Sp(1,\mathbb{R})}) \hat{\boxtimes} \mathcal{H}_\mu(D_{Sp(1,\mathbb{R})}), (x_L, x_R)}$$

$$= \iint_{\substack{\{|x_L|<1\} \\ \times\{|x_R|<1\}}} e^{x_L\overline{z_L}+x_R\overline{z_R}}\overline{K_W(x_L-x_R,y_2)}(1-|x_L|^2)^{\lambda-2}(1-|x_R|^2)^{\mu-2}dx_Ldx_R.$$

Let $W=\mathcal{P}_k(\mathrm{Sym}(1,\mathbb{C}))$. Then we have $K_W(x,y)=x^k\bar{y}^k$, and

$$\begin{aligned}
&F_W^*(z_L,z_R;y_2) \\
&= y_2^k \iint_{\substack{\{|x_L|<1\} \\ \times\{|x_R|<1\}}} e^{x_L\overline{z_L}+x_R\overline{z_R}}\overline{(x_L-x_R)^k}(1-|x_L|^2)^{\lambda-2}(1-|x_R|^2)^{\mu-2}dx_Ldx_R \\
&= y_2^k\sum_{m=0}^k\frac{(-k)_m}{m!}\int_{|x_L|<1} e^{x_L\overline{z_L}}\overline{x_L^{k-m}}(1-|x_L|^2)^{\lambda-2}dx_L \\
&\qquad\qquad\qquad\times\int_{|x_R|<1} e^{x_R\overline{z_R}}\overline{x_R^m}(1-|x_R|^2)^{\mu-2}dx_R.
\end{aligned}$$

Since we have

$$\begin{aligned}
&\int_{|x|<1} e^{x\bar{z}}\overline{x^m}(1-|x|^2)^{\mu-2}dx \\
&= \sum_{j=0}^\infty\frac{\overline{z^j}}{j!}\int_0^1\int_0^{2\pi}(re^{\sqrt{-1}\theta})^j(re^{-\sqrt{-1}\theta})^m(1-r^2)^{\mu-2}rd\theta dr \\
&= 2\pi\frac{\overline{z^m}}{m!}\int_0^1 r^{2m}(1-r^2)^{\mu-2}rdr = \pi\frac{\overline{z^m}}{m!}\int_0^1 s^m(1-s)^{\mu-2}ds \\
&= \pi\frac{\overline{z^m}}{m!}B(m+1,\mu-1) = \pi\frac{\overline{z^m}}{m!}\frac{\Gamma(m+1)\Gamma(\mu-1)}{\Gamma(\mu+m)} = \frac{\pi}{\mu-1}\frac{\overline{z^m}}{(\mu)_m},
\end{aligned}$$

it follows that

$$F_W^*(z_L,z_R;y_2) = \frac{\pi^2}{(\lambda-1)(\mu-1)}y_2^k\sum_{m=0}^k\frac{(-k)_m}{(\lambda)_{k-m}(\mu)_m}\frac{1}{m!}\overline{z_L^{k-m}z_R^m}.$$

Hence $\mathcal{F}_W^*:\mathcal{H}_\lambda(D_{Sp(1,\mathbb{R})})\mathbin{\hat{\boxtimes}}\mathcal{H}_\mu(D_{Sp(1,\mathbb{R})})\to\mathcal{H}_{\lambda+\mu}(D_{Sp(1,\mathbb{R})},\mathcal{P}_k(\mathrm{Sym}(1,\mathbb{C})))$ is given by

$$\mathcal{F}_W^*f(y_1,y_2) = y_2^k\sum_{m=0}^k\frac{(-k)_m}{(\lambda)_{k-m}(\mu)_m}\frac{1}{m!}\left.\frac{\partial^k}{\partial x_L^{k-m}\partial x_R^m}\right|_{x_L=x_R=y_1} f(x_L,x_R).$$

Finally, since $\mathcal{H}_{\lambda+\mu}(D_{Sp(1,\mathbb{R})},\mathcal{P}_k(\mathrm{Sym}(1,\mathbb{C})))\simeq\mathcal{H}_{\lambda+\mu+2k}(D_{Sp(1,\mathbb{R})})$ via $y_2^kf(y_1)\mapsto f(y)$, $\mathcal{F}_{\lambda,\mu,k}^*:\mathcal{H}_\lambda(D_{Sp(1,\mathbb{R})})\mathbin{\hat{\boxtimes}}\mathcal{H}_\mu(D_{Sp(1,\mathbb{R})})\to\mathcal{H}_{\lambda+\mu+2k}(D_{Sp(1,\mathbb{R})})$ is given by

$$\mathcal{F}_{\lambda,\mu,k}^*f(y) = \sum_{m=0}^k\frac{(-k)_m}{(\lambda)_{k-m}(\mu)_m}\frac{1}{m!}\left.\frac{\partial^k}{\partial x_L^{k-m}\partial x_R^m}\right|_{x_L=x_R=y} f(x_L,x_R).$$

Next we consider $(G,G_1)=(SU(2,2),Sp(2,\mathbb{R}))$. If we write $y_1=\begin{pmatrix} y_{11} & y_{12} \\ y_{12} & y_{22}\end{pmatrix}\in$ $\mathrm{Sym}(2,\mathbb{C})$, $y_2=\begin{pmatrix} 0 & y_2 \\ -y_2 & 0\end{pmatrix}\in\mathrm{Skew}(2,\mathbb{C})$, then F'_W is given as

$$
\begin{aligned}
&F'_W(x_2;w_1,w_2)\\
&=\Big\langle e^{\mathrm{tr}(y_1w_1^*)+2y_2\overline{w_2}},\overline{\det(I-y_1^*x_2y_1^*x_2)^{-\lambda/2}}\\
&\qquad\qquad\times\overline{K'_W\big(x_2(I-y_1^*x_2y_1^*x_2)^{-1},y_2\big)}\Big\rangle_{\mathcal{H}_\lambda(D_{Sp(2,\mathbb{R})},W),y}\\
&=\left\langle e^{\mathrm{tr}(y_1w_1^*)+2y_2\overline{w_2}},\overline{\det\left(I-\begin{pmatrix} y_{11} & y_{12} \\ y_{12} & y_{22}\end{pmatrix}^*\begin{pmatrix} 0 & x_2 \\ -x_2 & 0\end{pmatrix}\begin{pmatrix} y_{11} & y_{12} \\ y_{12} & y_{22}\end{pmatrix}^*\begin{pmatrix} 0 & x_2 \\ -x_2 & 0\end{pmatrix}\right)^{-\lambda/2}}\right.\\
&\quad\overline{\times K'_W\left(\begin{pmatrix} 0 & x_2 \\ -x_2 & 0\end{pmatrix}\left(I-\begin{pmatrix} y_{11} & y_{12} \\ y_{12} & y_{22}\end{pmatrix}^*\begin{pmatrix} 0 & x_2 \\ -x_2 & 0\end{pmatrix}\begin{pmatrix} y_{11} & y_{12} \\ y_{12} & y_{22}\end{pmatrix}^*\begin{pmatrix} 0 & x_2 \\ -x_2 & 0\end{pmatrix}\right)^{-1},\right.}\\
&\qquad\qquad\qquad\qquad\left.\left.\overline{\begin{pmatrix} 0 & y_2 \\ -y_2 & 0\end{pmatrix}}\right)\right\rangle_{\mathcal{H}_\lambda(D_{Sp(2,\mathbb{R})},W),y}\\
&=\Big\langle e^{\mathrm{tr}(y_1w_1^*)+2y_2\overline{w_2}},\overline{(1+x_2^2(\overline{y_{11}y_{22}-y_{12}^2}))^{-\lambda}}\\
&\qquad\left.\overline{\times K'_W\left(x_2(1+x_2^2(\overline{y_{11}y_{22}-y_{12}^2}))^{-1}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix},\begin{pmatrix} 0 & y_2 \\ -y_2 & 0\end{pmatrix}\right)}\right\rangle_{\mathcal{H}_\lambda(D_{Sp(2,\mathbb{R})},W),y}.
\end{aligned}
$$

Let $W=\mathcal{P}_k(\mathrm{Skew}(2,\mathbb{C}))\simeq\mathbb{C}_{-k}$. Then we have $K_W\left(\begin{pmatrix} 0 & x_2 \\ -x_2 & 0\end{pmatrix},\begin{pmatrix} 0 & y_2 \\ -y_2 & 0\end{pmatrix}\right)=$ $(x_2\overline{y_2})^k$, and

$$
F'_W(x_2;w_1,w_2)=\Big\langle e^{\mathrm{tr}(y_1w_1^*)+2y_2\overline{w_2}},\overline{(1+x_2^2\overline{\det(y_1)})^{-\lambda-k}(x_2\overline{y_2})^k}\Big\rangle_{\mathcal{H}_\lambda(D_{Sp(2,\mathbb{R})},\mathcal{P}_k),y}.
$$

Since $\mathcal{H}_\lambda(D_{Sp(2,\mathbb{R})},\mathcal{P}_k)\simeq\mathcal{H}_{\lambda+k}(D_{Sp(2,\mathbb{R})})$ via $y_2^kf(y_1)\mapsto f(y_1)$, we have

$$
\begin{aligned}
&F'_W(x_2;w_1,w_2)=(x_2\overline{w_2})^k\Big\langle e^{\mathrm{tr}(y_1w_1^*)},\overline{(1+x_2^2\overline{\det(y_1)})^{-\lambda-k}}\Big\rangle_{\mathcal{H}_{\lambda+k}(D_{Sp(2,\mathbb{R})}),y}\\
&=(x_2\overline{w_2})^k\int_{D_{Sp(2,\mathbb{R})}}e^{\mathrm{tr}(y_1w_1^*)}(1+x_2^2\overline{\det(y_1)})^{-\lambda-k}\det(I-y_1y_1^*)^{\lambda-k-3}dy_1\\
&=(x_2\overline{w_2})^k\sum_{m=0}^{\infty}\frac{(-1)^m(\lambda+k)_m}{m!}x_2^{2m}\int_{D_{Sp(2,\mathbb{R})}}e^{\mathrm{tr}(y_1w_1^*)}\overline{\det(y_1)^m}\det(I-y_1y_1^*)^{\lambda-k-3}dy_1.
\end{aligned}
$$

Then by Faraut-Korányi's result [**3**], it holds that

$$\int_{D_{Sp(2,\mathbb{R})}} e^{\mathrm{tr}(y_1 w_1^*)}\overline{\det(y_1)^m}\det(I-y_1y_1^*)^{\lambda-k-3}dy_1 = \frac{C}{(\lambda+k)_m\left(\lambda+k-\frac{1}{2}\right)_m}\overline{\det(w_1)^m},$$

where C is a constant which does not depend on m. Therefore we have

$$\begin{aligned}
&F'_W(x_2;w_1,w_2)\\
&= (x_2\overline{w_2})^k\sum_{m=0}^{\infty}\frac{(-1)^m(\lambda+k)_m}{m!}x_2^{2m}\frac{C}{(\lambda+k)_m\left(\lambda+k-\frac{1}{2}\right)_m}\overline{\det(w_1)^m}\\
&= C(x_2\overline{w_2})^k\sum_{m=0}^{\infty}\frac{1}{\left(\lambda+k-\frac{1}{2}\right)_m}\frac{(-1)^m}{m!}x_2^{2m}(\overline{w_{11}w_{22}-w_{12}^2})^m.
\end{aligned}$$

Therefore the intertwining operator $\mathcal{F}_W : \mathcal{H}_\lambda(D_{Sp(2,\mathbb{R})},\mathcal{P}_k) \to \mathcal{H}_\lambda(D_{SU(2,2)})$ is given by

$$\begin{aligned}
\mathcal{F}_W f\begin{pmatrix} x_{11} & x_{12}+x_2\\ x_{12}-x_2 & x_{22}\end{pmatrix} &= \left(x_2\frac{\partial}{\partial y_2}\right)^k\sum_{m=0}^{\infty}\frac{1}{\left(\lambda+k-\frac{1}{2}\right)_m}\frac{(-1)^m}{m!}\\
&\quad\times x_2^{2m}\left(\frac{\partial^2}{\partial y_{11}\partial y_{22}}-\frac{1}{4}\frac{\partial^2}{\partial y_{22}^2}\right)^m\bigg|_{y_1=x_1,y_2=0} f\left(\begin{pmatrix} y_{11} & y_{12}\\ y_{12} & y_{22}\end{pmatrix},\begin{pmatrix} 0 & y_2\\ -y_2 & 0\end{pmatrix}\right),
\end{aligned}$$

and $\mathcal{F}_{\lambda,k} : \mathcal{H}_{\lambda+k}(D_{Sp(2,\mathbb{R})}) \to \mathcal{H}_\lambda(D_{SU(2,2)})$ is given by

$$\begin{aligned}
\mathcal{F}_{\lambda,k} f\begin{pmatrix} x_{11} & x_{12}+x_2\\ x_{12}-x_2 & x_{22}\end{pmatrix} &= x_2^k\sum_{m=0}^{\infty}\frac{1}{\left(\lambda+k-\frac{1}{2}\right)_m}\frac{(-1)^m}{m!}\\
&\quad\times x_2^{2m}\left(\frac{\partial^2}{\partial x_{11}\partial x_{22}}-\frac{1}{4}\frac{\partial^2}{\partial x_{22}^2}\right)^m f\begin{pmatrix} x_{11} & x_{12}\\ x_{12} & x_{22}\end{pmatrix}.
\end{aligned}$$

5. Results for groups of higher rank

In this section we give the results on $\tilde{G}_1$-intertwining operators for $(G,G_1) = (SU(s,s),Sp(s,\mathbb{R}))$ or $(SU(s,s),SO^*(2s))$ for general s. For simplicity, we only consider the case both representations are of scalar type. Then for $\lambda > s-1$, we have

$$\mathcal{H}_\lambda(D_{SU(s,s)}) \supset \mathcal{H}_\mu(D_{Sp(s,\mathbb{R})}) \quad \text{if and only if} \quad \mu = \lambda+k, \begin{cases} k\in\mathbb{Z}_{\geq 0} & (s:\text{even}),\\ k=0 & (s:\text{odd}),\end{cases}$$

$$\mathcal{H}_\lambda(D_{SU(s,s)}) \supset \mathcal{H}_\mu(D_{SO^*(2s)}) \quad \text{if and only if} \quad \mu = 2\lambda+4k,\ k\in\mathbb{Z}_{\geq 0}.$$

In order to state the main result, we prepare some notations. We set $r := \left\lfloor \frac{s}{2} \right\rfloor$, and let

$$\mathbb{Z}^r_{++} := \{(m_1, \dots, m_r) \in \mathbb{Z}^r : m_1 \geq m_2 \geq \cdots \geq m_r \geq 0\}.$$

For $\mathbf{m} \in \mathbb{Z}^r_{++}$, $d \in \mathbb{N}$ and $\lambda \in \mathbb{C}$, let

$$(\lambda)_{\mathbf{m},d} := \prod_{j=1}^{r} \left(\lambda - \frac{d}{2}(j-1) \right)_{m_j}.$$

Also, for $\mathbf{m} \in \mathbb{Z}^r_{++}$, let $\tilde{\Phi}_{\mathbf{m}}(t_1, \dots, t_r)$ be the Schur polynomial

$$\tilde{\Phi}_{\mathbf{m}}(t_1, \dots, t_r) = \frac{\prod_{i<j}(m_i - m_j - i + j)}{\prod_{i=1}^{r}(m_i + r - i)!} \frac{\det\left((t_i^{m_j + r - j})_{i,j} \right)}{\det\left((t_i^{r-j})_{i,j} \right)},$$

normalized such that

$$\sum_{\mathbf{m} \in \mathbb{Z}^r_{++}} \tilde{\Phi}_{\mathbf{m}}(t_1, \dots, t_r) = e^{t_1 + \cdots + t_r},$$

$$\sum_{\mathbf{m} \in \mathbb{Z}^r_{++}} (\lambda)_{\mathbf{m},2} \tilde{\Phi}_{\mathbf{m}}(t_1, \dots, t_r) = \prod_{j=1}^{r} (1 - t_j)^{-\lambda}.$$

For $z \in \mathrm{Sym}(s, \mathbb{C})$, $w \in \mathrm{Skew}(s, \mathbb{C})$, if the eigenvalues of zw are $t_1, -t_1, t_2, -t_2, \dots, t_r, -t_r(,0)$, then we write

$$\tilde{\Phi}'_{\mathbf{m}}(zwzw) = \tilde{\Phi}'_{\mathbf{m}}(wzwz) := \tilde{\Phi}_{\mathbf{m}}(t_1^2, \dots, t_r^2).$$

This becomes a polynomial on $\mathrm{Sym}(s, \mathbb{C}) \oplus \mathrm{Skew}(s, \mathbb{C})$. Then the intertwining operators are given as follows.

THEOREM 5.1. *(1) When $(G, G_1) = (SU(s,s), Sp(s, \mathbb{R}))$,*

$$\mathcal{F}_{\lambda,k} : \mathcal{H}_{\lambda+k}(D_{Sp(s,\mathbb{R})}) \to \mathcal{H}_\lambda(D_{SU(s,s)}) \qquad \left(\lambda > s-1, \begin{cases} k \in \mathbb{Z}_{\geq 0} & s : \textit{even}, \\ k = 0 & s : \textit{odd} \end{cases} \right),$$

$$\mathcal{F}_{\lambda,k} f(x_1 + x_2) = \mathrm{Pf}(x_2)^k \sum_{\mathbf{m} \in \mathbb{Z}^r_{++}} \frac{1}{\left(\lambda + k - \frac{1}{2}\right)_{\mathbf{m},2}} \tilde{\Phi}'_{\mathbf{m}} \left(x_2 \frac{\partial}{\partial x_1} x_2 \frac{\partial}{\partial x_1} \right) f(x_1)$$

$(x_1 \in \mathrm{Sym}(s, \mathbb{C})$*,* $x_2 \in \mathrm{Skew}(s, \mathbb{C}))$ *intertwines the* $\tilde{G}_1$*-action.*

(2) When $(G, G_1) = (SU(s,s), SO^*(2s))$,

$$\mathcal{F}_{\lambda,k} : \mathcal{H}_{2\lambda+4k}(D_{SO^*(2s)}) \to \mathcal{H}_\lambda(D_{SU(s,s)}) \quad (\lambda > s-1,\ k \in \mathbb{Z}_{\geq 0}),$$

$$\mathcal{F}_{\lambda,k}f(x_1+x_2)=\det(x_2)^k\sum_{\mathbf{m}\in\mathbb{Z}^r_{++}}\frac{1}{\left(\lambda+2k+\frac{1}{2}\right)_{\mathbf{m},2}}\tilde{\Phi}'_{\mathbf{m}}\left(x_2\frac{\partial}{\partial x_1}x_2\frac{\partial}{\partial x_1}\right)f(x_1)$$

$(x_1\in\mathrm{Skew}(s,\mathbb{C}),\ x_2\in\mathrm{Sym}(s,\mathbb{C}))$ *intertwines the* $\tilde{G}_1$*-action.*

For the proof, we use the expansion

$$\det(I-zwzw)^{-\mu/2}=\sum_{\mathbf{m}\in\mathbb{Z}^r_{++}}(\mu)_{\mathbf{m},2}\tilde{\Phi}'_{\mathbf{m}}(zwzw),$$

and

$$\left\langle e^{\mathrm{tr}(y_1w_1^*)},\overline{\tilde{\Phi}'_{\mathbf{m}}(x_2y_1^*x_2y_1^*)}\right\rangle_{\mathcal{H}_\mu(D_{Sp(s,\mathbb{R})}),y_1}=\frac{C}{(\mu)_{\mathbf{m}^2,1}}\tilde{\Phi}'_{\mathbf{m}}(x_2w_1^*x_2w_1^*)$$

where $\mathbf{m}^2=(m_1,m_1,m_2,m_2,\ldots,m_r,m_r)\in\mathbb{Z}^{2r}_{++}$,

$$\left\langle e^{\mathrm{tr}(y_1w_1^*)},\overline{\tilde{\Phi}'_{\mathbf{m}}(x_2y_1^*x_2y_1^*)}\right\rangle_{\mathcal{H}_{2\mu}(D_{SO^*(2s)}),y_1}=\frac{2^{2|\mathbf{m}|}C}{(2\mu)_{2\mathbf{m},4}}\tilde{\Phi}'_{\mathbf{m}}(x_2w_1^*x_2w_1^*)$$

where $2\mathbf{m}=(2m_1,2m_2,\ldots,2m_r)\in\mathbb{Z}^r_{++}$, which follow from Faraut-Korányi's results plus some observation. Then by

$$\frac{(\mu)_{\mathbf{m},2}}{(\mu)_{\mathbf{m}^2,1}}=\frac{1}{\left(\mu-\frac{1}{2}\right)_{\mathbf{m},2}},\qquad\frac{2^{2|\mathbf{m}|}(\mu)_{\mathbf{m},2}}{(2\mu)_{2\mathbf{m},4}}=\frac{1}{\left(\mu+\frac{1}{2}\right)_{\mathbf{m},2}},$$

the theorem holds.

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