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Whittaker functions on $GL(2,\mathbb{C})$ and the local zeta integrals

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Abstract. This is an announcement of a new result which is a generalization of Popa's result in [**Po**]. Popa gives explicit formulas of Whittaker functions on $GL(2,\mathbb{C})$ at the minimal K-types, and shows that the local zeta integrals for $GL(2,\mathbb{C})$ defined from some Whittaker functions are equal to the associated L-factors. In this article, we will give explicit formulas of Whittaker functions on $GL(2,\mathbb{C})$ at all K-types, and show that the local zeta integrals for $GL(2,\mathbb{C}) \times GL(2,\mathbb{C})$ defined from some Whittaker functions are equal to the associated L-factors. Proofs will appear elsewhere.

1. Notation

Let $G = GL(2,\mathbb{C})$ be the complex general linear group of degree 2. In this article, we regard G as a real reductive Lie group. We fix an Iwasawa decomposition $G = NAK$ of G with

$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \middle| \, x \in \mathbb{C} \right\}, \quad A = \left\{ \begin{pmatrix} y_1 y_2 & 0 \\ 0 & y_2 \end{pmatrix} \middle| \, y_1, y_2 \in \mathbb{R}_{>0} \right\}, \quad K = U(2).$$

Let $\mathfrak{g}$ be the associated Lie algebra of G. We denote by $\mathfrak{g}_{\mathbb{C}}$ the complexification $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ of $\mathfrak{g}$. It is known that $\mathfrak{g}_{\mathbb{C}}$ is isomorphic to $\mathfrak{gl}(2,\mathbb{C}) \oplus \mathfrak{gl}(2,\mathbb{C})$ via

$$\mathfrak{g}_{\mathbb{C}} \ni X \otimes t \mapsto (tX, t\overline{X}) \in \mathfrak{gl}(2,\mathbb{C}) \oplus \mathfrak{gl}(2,\mathbb{C}).$$

Here $\overline{X}$ means the complex conjugate of X. We denote by $U(\mathfrak{g}_{\mathbb{C}})$ the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$, and by $Z(\mathfrak{g}_{\mathbb{C}})$ the center of $U(\mathfrak{g}_{\mathbb{C}})$. We note that

$$U(\mathfrak{g}_{\mathbb{C}}) \simeq U(\mathfrak{gl}(2,\mathbb{C}) \oplus \mathfrak{gl}(2,\mathbb{C})) \simeq U(\mathfrak{gl}(2,\mathbb{C})) \otimes_{\mathbb{C}} U(\mathfrak{gl}(2,\mathbb{C})).$$

Here the second isomorphism above is induced from

$$\mathfrak{gl}(2,\mathbb{C}) \oplus \mathfrak{gl}(2,\mathbb{C}) \ni (X_1, X_2) \mapsto X_1 \otimes 1 + 1 \otimes X_2 \in U(\mathfrak{gl}(2,\mathbb{C})) \otimes_{\mathbb{C}} U(\mathfrak{gl}(2,\mathbb{C})).$$

Let $\Lambda = \{(\lambda_1, \lambda_2) \in \mathbb{Z}^2 \mid \lambda_1 \geq \lambda_2\}$. For $\lambda = (\lambda_1, \lambda_2) \in \Lambda$, let V_λ be the $\mathbb{C}$-vector space of degree $\lambda_1 - \lambda_2$ homogeneous polynomials in two variables z_1, z_2, on which

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K acts by

$$\tau_\lambda(k)p(z_1,z_2)=\det(k)^{\lambda_2}p((z_1,z_2)k) \qquad (k\in K,\ p(z_1,z_2)\in V_\lambda).$$

We define $\{v_q^\lambda \mid 0\le q\le \lambda_1-\lambda_2\}$ as a basis of V_λ by $v_q^\lambda = z_1^{\lambda_1-\lambda_2-q}z_2^q$. It is known that the equivalence classes of irreducible representations of K is exhausted by $\{(\tau_\lambda,V_\lambda)\mid \lambda\in\Lambda\}$.

2. Whittaker functions on $GL(2,\mathbb{C})$

For $c\in\mathbb{C}^\times$, we define an additive character $\psi_c\colon \mathbb{C}\to\mathbb{C}^\times$ by

$$\psi_c(x)=\exp(2\pi\sqrt{-1}(cx+\overline{cx})) \qquad (x\in\mathbb{C}).$$

Let $C^\infty(N\backslash G;\psi_c)$ be the space of smooth functions f on G satisfying

$$f\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}g\right)=\psi_c(x)f(g) \qquad (x\in\mathbb{C},\ g\in G),$$

on which G acts by the right translation. For an irreducible admissible representation (π,H_π) of G, let $\mathcal{I}_{\pi,\psi_c}$ be the subspace of $\mathrm{Hom}_{\mathfrak{g}_\mathbb{C},K}(H_{\pi,K},C^\infty(N\backslash G;\psi_c)_K)$ consisting of all homomorphisms Φ such that $\Phi(f)$ is of moderate growth for any $f\in H_{\pi,K}$. Here we denote by $H_{\pi,K}$ and $C^\infty(N\backslash G;\psi_c)_K$ the subspaces of H_π and $C^\infty(N\backslash G;\psi_c)$ consisting of all K-finite vectors, respectively.

The multiplicity one theorem (*cf.* [**Sha**], [**Wa**]) tells that the intertwining space $\mathcal{I}_{\pi,\psi_c}$ is at most one dimensional. If $\mathcal{I}_{\pi,\psi_c}\neq\{0\}$, we say that π is generic, and define the Whittaker model $\mathcal{W}(\pi,\psi_c)$ of π by

$$\mathcal{W}(\pi,\psi_c)=\{\Phi(f)\mid f\in H_{\pi,K},\ \Phi\in\mathcal{I}_{\pi,\psi_c}\},$$

and functions in $\mathcal{W}(\pi,\psi_c)$ are called Whittaker functions for π.

For an irreducible admissible representation (π,H_π) of G, it is known that π is generic if and only if π is infinitesimally equivalent to an irreducible principal series representation ([**Ja2**, Lemma 2.5]). We define principal series representations of G in the next section.

3. Principal series representations of $GL(2,\mathbb{C})$

Let $\nu=(\nu_1,\nu_2)\in\mathbb{C}^2$ and $d=(d_1,d_2)\in\mathbb{Z}^2$. Let $H^0_{(\nu,d)}$ be the space of continuous functions f on G satisfying

$$f\left(\begin{pmatrix}m_1 & x\\ 0 & m_2\end{pmatrix}g\right)=m_1^{d_1}m_2^{d_2}|m_1|^{2\nu_1-d_1+1}|m_2|^{2\nu_2-d_2-1}f(g)$$
$$(m_1,m_2\in\mathbb{C}^\times,\ x\in\mathbb{C},\ g\in G).$$

The group G acts on $H^0_{(\nu,d)}$ by the right translation

$$(\pi_{(\nu,d)}(g)f)(h) = f(hg) \qquad (g,h \in G,\ f \in H^0_{(\nu,d)}).$$

Let $(\pi_{(\nu,d)}, H_{(\nu,d)})$ be the Hilbert representation of G, which is the completion of $(\pi_{(\nu,d)}, H^0_{(\nu,d)})$ relative to the inner product

$$(f_1,f_2) = \int_K f_1(k)\overline{f_2(k)}\,dk \qquad (f_1,f_2 \in H^0_{(\nu,d)}).$$

We call $(\pi_{(\nu,d)}, H_{(\nu,d)})$ a principal series representation of G. It is known that a principal series representation $\pi_{(\nu,d)}$ is irreducible for generic parameter ν. In this article, we only consider irreducible principal series representations of G. We may assume $d=(d_1,d_2)\in\Lambda$ without loss of generalities, since

$$H_{((\nu_1,\nu_2),(d_1,d_2)),K} \simeq H_{((\nu_2,\nu_1),(d_2,d_1)),K} \tag{1}$$

as $(\mathfrak{g}_{\mathbb{C}},K)$-modules if $\pi_{((\nu_1,\nu_2),(d_1,d_2))}$ is irreducible (*cf.* [**SV**, Corollary 2.8]).

By the Frobenius reciprocity law, the K-types of principal series representations of G are given as follows.

LEMMA 3.1. *For $\nu=(\nu_1,\nu_2)\in\mathbb{C}^2$ and $d=(d_1,d_2)\in\Lambda$, it holds that*

$$H_{(\nu,d),K} \simeq \bigoplus_{m=0}^{\infty} V_{(d_1+m,d_2-m)}$$

as K-modules.

4. Explicit formulas of Whittaker functions

In this section, we give explicit formulas of Whittaker functions for an irreducible principal series representation $(\pi_{(\nu,d)}, H_{(\nu,d)})$ of G with $\nu=(\nu_1,\nu_2)\in\mathbb{C}^2$, $d=(d_1,d_2)\in\Lambda$. Let $c\in\mathbb{C}^\times$ and $m\in\mathbb{Z}_{\geq 0}$. We set $\lambda=(\lambda_1,\lambda_2)=(d_1+m,d_2-m)$, and take a K-embedding $\phi\colon V_\lambda \to \mathcal{W}(\pi_{(\nu,d)},\psi_c)$. Then, for $g\in G$ with the Iwasawa decomposition

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y_1y_2 & 0 \\ 0 & y_2 \end{pmatrix} k \qquad (x\in\mathbb{C},\ y_1,y_2\in\mathbb{R}_{>0},\ k\in K),$$

we have

$$\phi(v)(g) = \psi_c(x)y_2^{2\nu_1+2\nu_2}\phi(\tau_\lambda(k)v)\left(\begin{pmatrix} y_1 & 0 \\ 0 & 1 \end{pmatrix}\right).$$

Hence, the functions φ_q $(0 \le q \le \lambda_1 - \lambda_2)$ on $\mathbb{R}_{>0}$ defined by

$$\varphi_q(y_1) = \phi(v_q^\lambda)\left(\begin{pmatrix} y_1 & 0 \\ 0 & 1 \end{pmatrix}\right) \qquad (y_1 \in \mathbb{R}_{>0})$$

characterize the K-embedding ϕ.

We denote by $\mathcal{C}$ the Casimir element of $\mathfrak{sl}(2,\mathbb{C})$. From the actions of two elements of $Z(\mathfrak{g}_\mathbb{C})$ corresponding to $\mathcal{C}\otimes 1$ and $1\otimes\mathcal{C}$, we obtain the following equalities:

$$\begin{aligned}&\{(\vartheta - 2\nu_1 + m - q - 1)(\vartheta - 2\nu_2 - m + \lambda_1 - \lambda_2 - q - 1) - (4\pi c y_1)^2\}\varphi_q \\ &= -(8\pi c\sqrt{-1}y_1)(\lambda_1 - \lambda_2 - q)\varphi_{q+1},\end{aligned} \tag{2}$$

$$\begin{aligned}&\{(\vartheta - 2\nu_1 - m + q - 1)(\vartheta - 2\nu_2 + m - \lambda_1 + \lambda_2 + q - 1) - (4\pi c y_1)^2\}\varphi_q \\ &= (8\pi c\sqrt{-1}y_1) q\varphi_{q-1}\end{aligned} \tag{3}$$

for $0 \le q \le \lambda_1 - \lambda_2$. Here $\vartheta = y_1 \dfrac{d}{dy_1}$, and $\varphi_q = 0$ if $q < 0$ or $q > \lambda_1 - \lambda_2$.

The system of differential equations (2), (3) has a unique moderate growth solution. Solving this system, we obtain the following theorem.

THEOREM 4.1. *Let $(\pi_{(\nu,d)}, H_{(\nu,d)})$ be an irreducible principal series representation of G with $\nu = (\nu_1,\nu_2) \in \mathbb{C}^2$ and $d = (d_1,d_2) \in \Lambda$. Let $c \in \mathbb{C}^\times$, $m \in \mathbb{Z}_{\ge 0}$, and set $\lambda = (\lambda_1,\lambda_2) = (d_1 + m, d_2 - m)$. Then there exists a K-embedding*

$$\phi^{(c)}_{[\nu,d;m]} \colon V_\lambda \to \mathcal{W}(\pi_{(\nu,d)}, \psi_c)$$

which is characterized by

$$\begin{aligned}\phi^{(c)}_{[\nu,d;m]}(v_q^\lambda)\left(\begin{pmatrix} y_1 & 0 \\ 0 & 1 \end{pmatrix}\right) &= \left(\frac{c\sqrt{-1}}{|c|}\right)^{\lambda_1 - q} \sum_{i=0}^{q} \binom{q}{i} \frac{(-m)_i\left(-\nu_1 + \nu_2 - \frac{\lambda_1 - \lambda_2}{2}\right)_i}{(-\lambda_1 + \lambda_2)_i (2\pi|c|y_1)^i} \\ &\times \frac{y_1}{4\pi\sqrt{-1}} \int_{\alpha - \sqrt{-1}\infty}^{\alpha + \sqrt{-1}\infty} \Gamma_\mathbb{C}\left(s + \nu_1 + \frac{q + m - i}{2}\right) \\ &\times \Gamma_\mathbb{C}\left(s + \nu_2 + \frac{\lambda_1 - \lambda_2 - q - m + i}{2}\right)(|c|y_1)^{-2s} ds\end{aligned}$$

for $y_1 \in \mathbb{R}_{>0}$ and $0 \le q \le \lambda_1 - \lambda_2$. Here $(s)_i = \Gamma(s+i)/\Gamma(s)$, $\Gamma_\mathbb{C}(s) = 2(2\pi)^{-s}\Gamma(s)$ $(s \in \mathbb{C},\ i \in \mathbb{Z}_{\ge 0})$ as usual, and α is a sufficiently large real number.

Remark 4.2. *When $m = 0$, we note that the formula in Theorem 4.1 is simplified as follows:*

$$\phi^{(c)}_{[\nu,d;0]}(v_q^d)\left(\begin{pmatrix} y_1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \left(\frac{c\sqrt{-1}}{|c|}\right)^{d_1 - q} \frac{y_1}{4\pi\sqrt{-1}} \int_{\alpha - \sqrt{-1}\infty}^{\alpha + \sqrt{-1}\infty} \Gamma_\mathbb{C}\left(s + \nu_1 + \frac{q}{2}\right)$$

$$\times \Gamma_{\mathbb{C}}\left(s+\nu_2+\frac{d_1-d_2-q}{2}\right)(|c|y_1)^{-2s}ds$$

for $y_1\in\mathbb{R}_{>0}$ *and* $0\le q\le d_1-d_2$. *This formula is given by Popa in* [**Po**].

5. The local zeta integrals for $GL(2,\mathbb{C})\times GL(2,\mathbb{C})$

Let $\mathcal{S}(\mathbb{C}^2)$ be the space of Schwartz functions on $\mathbb{C}^2$. For $f\in\mathcal{S}(\mathbb{C}^2)$, we define the Fourier transform $\mathcal{F}(f)$ of f with respect to ψ_1 by

$$\mathcal{F}(f)(w_1,w_2)=\int_{\mathbb{C}^2} f(z_1,z_2)\psi_1(z_1w_1+z_2w_2)\,dz_1dz_2 \qquad (w_1,w_2\in\mathbb{C}).$$

Here dz_i is the self dual Haar measure on $\mathbb{C}$ with respect to ψ_1, that is, dz_i is the twice Lebesgue measure on $\mathbb{C}\simeq\mathbb{R}^2$ for $i=1,2$. Then we note that

$$\mathcal{F}(\mathcal{F}(f))(z_1,z_2)=f(-z_1,-z_2) \qquad (f\in\mathcal{S}(\mathbb{C}^2),\ z_1,z_2\in\mathbb{C}).$$

Let $\mathcal{S}(\mathbb{C}^2)^{\mathrm{std}}$ be the subspace of $\mathcal{S}(\mathbb{C}^2)$ consisting of all functions f of the form

$$f(z_1,z_2)=p(z_1,z_2,\overline{z_1},\overline{z_2})\exp\left(-2\pi(|z_1|^2+|z_2|^2)\right) \qquad (z_1,z_2\in\mathbb{C}) \tag{4}$$

with polynomial functions p on $\mathbb{C}^4$. We call functions in $\mathcal{S}(\mathbb{C}^2)^{\mathrm{std}}$ standard Schwartz functions on $\mathbb{C}^2$.

Let $(\pi_{(\nu,d)},H_{(\nu,d)})$ and $(\pi_{(\nu',d')},H_{(\nu',d')})$ be irreducible principal series representations of G with $\nu=(\nu_1,\nu_2)\in\mathbb{C}^2$, $d=(d_1,d_2)\in\mathbb{Z}^2$, $\nu'=(\nu'_1,\nu'_2)\in\mathbb{C}^2$ and $d'=(d'_1,d'_2)\in\mathbb{Z}^2$. From the Langlands parameters of $\pi_{(\nu,d)}$ and $\pi_{(\nu',d')}$, we define the local L-factor $L(s,\pi_{(\nu,d)}\times\pi_{(\nu',d')})$ by

$$L(s,\pi_{(\nu,d)}\times\pi_{(\nu',d')})=\prod_{1\le i,j\le 2}\Gamma_{\mathbb{C}}\left(s+\nu_i+\nu'_j+\frac{|d_i+d'_j|}{2}\right).$$

Let $\varepsilon\in\{\pm1\}$. For $W\in\mathcal{W}(\pi_{(\nu,d)},\psi_\varepsilon)$, $W'\in\mathcal{W}(\pi_{(\nu',d')},\psi_{-\varepsilon})$ and $f\in\mathcal{S}(\mathbb{C}^2)$, we define the local zeta integral $Z(s,W,W',f)$ by

$$Z(s,W,W',f)=\int_{N\backslash G} W(g)W'(g)f((0,1)g)|\det g|^{2s}\,d\dot{g},$$

where $d\dot{g}$ is the right invariant measure on $N\backslash G$. The local zeta integrals converge for $\mathrm{Re}(s)\gg 0$. Jacquet proves the following theorems ([**Ja1**], [**Ja2**]).

THEOREM 5.1 (JACQUET). *Retain the notation. Let* $W\in\mathcal{W}(\pi_{(\nu,d)},\psi_1)$, $W'\in\mathcal{W}(\pi_{(\nu',d')},\psi_{-1})$ *and* $f\in\mathcal{S}(\mathbb{C}^2)$. *Then the ratio* $Z(s,W,W',f)/L(s,\pi_{(\nu,d)}\times\pi_{(\nu',d')})$

extends to an entire function of s, and satisfies the local functional equation:

$$\frac{Z(1-s,\widetilde{W},\widetilde{W}',\mathcal{F}(f))}{L(1-s,\pi_{(-\nu,-d)}\times\pi_{(-\nu',-d')})}=(\sqrt{-1})^{\sum_{i=1}^{2}(|d_1+d_i'|-|d_2+d_i'|)}\frac{Z(s,W,W',f)}{L(s,\pi_{(\nu,d)}\times\pi_{(\nu',d')})}$$

where $\widetilde{W}(g)=W\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}{}^tg^{-1}\right)$ *and* $\widetilde{W}'(g)=W'\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}{}^tg^{-1}\right)$ $(g\in G)$.

THEOREM 5.2 (JACQUET). *Retain the notation. There exist* $m\in\mathbb{Z}_{>0}$ *and* $(W_i,W_i',f_i)\in\mathcal{W}(\pi_{(\nu,d)},\psi_1)\times\mathcal{W}(\pi_{(\nu',d')},\psi_{-1})\times\mathcal{S}(\mathbb{C}^2)^{\mathrm{std}}$ $(i=1,\cdots,m)$ *such that*

$$\sum_{i=1}^{m}Z(s,W_i,W_i',f_i)=L(s,\pi_{(\nu,d)}\times\pi_{(\nu',d')}).$$

By the calculation based on the explicit formulas in the previous section, we can improve Theorem 5.2 as follows:

THEOREM 5.3. *Retain the notation. Then there exist* $W_0\in\mathcal{W}(\pi_{(\nu,d)},\psi_1)$, $W_0'\in\mathcal{W}(\pi_{(\nu',d')},\psi_{-1})$ *and* $f_0\in\mathcal{S}(\mathbb{C}^2)^{\mathrm{std}}$ *such that*

$$Z(s,W_0,W_0',f_0)=L(s,\pi_{(\nu,d)}\times\pi_{(\nu',d')}). \tag{5}$$

In the proof of this theorem, appropriate choices of Whittaker functions play important roles. We regard $\mathcal{S}(\mathbb{C}^2)^{\mathrm{std}}$ as a K-module via

$$(\tau(k)f)(z_1,z_2)=f((z_1,z_2)k)\qquad (k\in K,\ f\in\mathcal{S}(\mathbb{C}^2)^{\mathrm{std}}).$$

For $p,q\in\mathbb{Z}_{\geq 0}$, let $\mathcal{S}(\mathbb{C}^2)^{\mathrm{std}}_{p,q}$ be the subspace of $\mathcal{S}(\mathbb{C}^2)^{\mathrm{std}}$ consisting of all functions of the form (4) with polynomial functions $p(w_1,w_2,w_3,w_4)$ which are degree p homogeneous with respect to w_1,w_2, and degree q homogeneous with respect to w_3,w_4. Then we have

$$\mathcal{S}(\mathbb{C}^2)^{\mathrm{std}}=\bigoplus_{p,q\geq 0}\mathcal{S}(\mathbb{C}^2)^{\mathrm{std}}_{p,q},\qquad \mathcal{S}(\mathbb{C}^2)^{\mathrm{std}}_{p,q}\simeq V_{(p,0)}\otimes_{\mathbb{C}}V_{(0,-q)}.$$

Because of the isomorphism (1), we may assume $d\in\Lambda$ and $d'\in\Lambda$. Let $m,m'\in\mathbb{Z}_{\geq 0}$, and we set $\lambda=(d_1+m,d_2-m)$, $\lambda'=(d_1'+m',d_2'-m')$. For each $s\in\mathbb{C}$, we note that

$$V_\lambda\otimes_{\mathbb{C}}V_{\lambda'}\otimes_{\mathbb{C}}\mathcal{S}(\mathbb{C}^2)^{\mathrm{std}}_{p,q}\ni v\otimes v'\otimes f\mapsto Z\big(s,\phi^{(1)}_{[\nu,d;m]}(v),\phi^{(-1)}_{[\nu',d';m']}(v'),f\big)\in V_{(0,0)}$$

is a K-homomorphism, and this homomorphism vanishes if

$$\mathrm{Hom}_K\big(V_\lambda\otimes_{\mathbb{C}}V_{\lambda'}\otimes_{\mathbb{C}}\mathcal{S}(\mathbb{C}^2)^{\mathrm{std}}_{p,q},V_{(0,0)}\big)=\{0\}.$$

Because of the local functional equations, we may assume $d_1+d_2+d_1'+d_2'\geq 0$. Moreover, interchanging $\pi_{(\nu,d)}$ and $\pi_{(\nu',d')}$ if necessary, we may assume $d_1+d_2'\geq 0$. Under these assumptions, if we set $m=0$ and $p=0$, then the smallest integers $q=q_0$ and $m'=m_0'$ satisfying $\mathrm{Hom}_K\big(V_d\otimes_{\mathbb{C}}V_{\lambda'}\otimes_{\mathbb{C}}\mathcal{S}(\mathbb{C}^2)^{\mathrm{std}}_{0,q},V_{(0,0)}\big)\neq\{0\}$ are given by

$$q_0=d_1+d_2+d_1'+d_2',\qquad\qquad m_0'=\max\{d_2+d_2',0,-d_2-d_1'\},$$

and the space

$$\mathrm{Hom}_K\big(V_d\otimes_{\mathbb{C}}V_{(d_1'+m_0',d_2'-m_0')}\otimes_{\mathbb{C}}\mathcal{S}(\mathbb{C}^2)^{\mathrm{std}}_{0,q_0},V_{(0,0)}\big)\tag{6}$$

is one dimensional. Let ι be a unique nonzero element of the space (6), and take $v_0\in V_d$, $v_0'\in V_{(d_1'+m_0',d_2'-m_0')}$ and $f_1\in\mathcal{S}(\mathbb{C}^2)^{\mathrm{std}}_{0,q_0}$ so that $\iota(v_0\otimes v_0'\otimes f_1)\neq 0$. We can show that there exists some nonzero constant C such that

$$Z\big(s,\phi^{(1)}_{[\nu,d;0]}(v_0),\phi^{(-1)}_{[\nu',d';m_0']}(v_0'),f_1\big)=C\,L(s,\pi_{(\nu,d)}\times\pi_{(\nu',d')})\qquad(s\in\mathbb{C}),$$

by the calculation using Barnes' lemma ([**Ba1**, §1.7]):

$$\begin{aligned}&\frac{1}{2\pi\sqrt{-1}}\int_{\alpha-\sqrt{-1}\infty}^{\alpha+\sqrt{-1}\infty}\Gamma(a+s)\Gamma(b+s)\Gamma(c-s)\Gamma(d-s)\,ds\\&=\frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}\end{aligned}$$

for $a,b,c,d\in\mathbb{C}$ and $\alpha\in\mathbb{R}$ such that

$$-\min\{\mathrm{Re}(a),\mathrm{Re}(b)\}<\alpha<\min\{\mathrm{Re}(c),\mathrm{Re}(d)\}.$$

Hence, $W_0=\phi^{(1)}_{[\nu,d;0]}(v_0)$, $W_0'=\phi^{(-1)}_{[\nu',d';m_0']}(v_0')$ and $f_0=C^{-1}f_1$ satisfy (5).

References

[Ba1] W. N. Bailey. *Generalized hypergeometric series.* Cambridge Tracts in Mathematics and Mathematical Physics, No. 32. Stechert-Hafner, Inc., New York, 1964.

[Ja1] Hervé Jacquet. *Automorphic forms on* GL(2)*. Part II.* Lecture Notes in Mathematics, Vol. 278. Springer-Verlag, Berlin-New York, 1972.

[Ja2] Hervé Jacquet. Archimedean Rankin-Selberg integrals. In *Automorphic forms and L-functions II. Local aspects*, volume 489 of *Contemp. Math.*, pages 57–172. Amer. Math. Soc., Providence, RI, 2009.

[Po] Alexandru A. Popa. Whittaker newforms for Archimedean representations. *J. Number Theory*, 128(6):1637–1645, 2008.

[Sha] J. A. Shalika. The multiplicity one theorem for GL_n. *Ann. of Math. (2)*, 100:171–193, 1974.

[SV] Birgit Speh and David A. Vogan, Jr. Reducibility of generalized principal series representations. *Acta Math.*, 145(3-4):227–299, 1980.

[Wa] Nolan R. Wallach. Asymptotic expansions of generalized matrix entries of representations of real reductive groups. In *Lie group representations, I (College Park, Md., 1982/1983)*, volume 1024 of *Lecture Notes in Math.*, pages 287–369. Springer, Berlin, 1983.

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