Whittaker functions on $GL(2, \mathbb{C})$ and the local zeta integrals

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Abstract. This is an announcement of a new result which is a generalization of Popa’s result in [Po]. Popa gives explicit formulas of Whittaker functions on $GL(2, \mathbb{C})$ at the minimal $K$-types, and shows that the local zeta integrals for $GL(2, \mathbb{C})$ defined from some Whittaker functions are equal to the associated $L$-factors. In this article, we will give explicit formulas of Whittaker functions on $GL(2, \mathbb{C})$ at all $K$-types, and show that the local zeta integrals for $GL(2, \mathbb{C}) \times GL(2, \mathbb{C})$ defined from some Whittaker functions are equal to the associated $L$-factors. Proofs will appear elsewhere.

1. Notation

Let $G = GL(2, \mathbb{C})$ be the complex general linear group of degree 2. In this article, we regard $G$ as a real reductive Lie group. We fix an Iwasawa decomposition $G = NAK$ of $G$ with

$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \Big| x \in \mathbb{C} \right\}, \quad A = \left\{ \begin{pmatrix} y_1 y_2 & 0 \\ 0 & y_2 \end{pmatrix} \Big| y_1, y_2 \in \mathbb{R}^> \right\}, \quad K = U(2).$$

Let $\mathfrak{g}$ be the associated Lie algebra of $G$. We denote by $\mathfrak{g}_\mathbb{C}$ the complexification $\mathfrak{g} \otimes \mathbb{C}$ of $\mathfrak{g}$. It is known that $\mathfrak{g}_\mathbb{C}$ is isomorphic to $\mathfrak{gl}(2, \mathbb{C}) \oplus \mathfrak{gl}(2, \mathbb{C})$ via

$$\mathfrak{g}_\mathbb{C} \ni X \otimes t \mapsto (tX, t\overline{X}) \in \mathfrak{gl}(2, \mathbb{C}) \oplus \mathfrak{gl}(2, \mathbb{C}).$$

Here $\overline{X}$ means the complex conjugate of $X$. We denote by $U(\mathfrak{g}_\mathbb{C})$ the universal enveloping algebra of $\mathfrak{g}_\mathbb{C}$, and by $Z(\mathfrak{g}_\mathbb{C})$ the center of $U(\mathfrak{g}_\mathbb{C})$. We note that

$$U(\mathfrak{g}_\mathbb{C}) \simeq U(\mathfrak{gl}(2, \mathbb{C}) \oplus \mathfrak{gl}(2, \mathbb{C})) \simeq U(\mathfrak{gl}(2, \mathbb{C})) \otimes_{\mathbb{C}} U(\mathfrak{gl}(2, \mathbb{C})).$$

Here the second isomorphism above is induced from

$$\mathfrak{gl}(2, \mathbb{C}) \oplus \mathfrak{gl}(2, \mathbb{C}) \ni (X_1, X_2) \mapsto X_1 \otimes 1 + 1 \otimes X_2 \in U(\mathfrak{gl}(2, \mathbb{C})) \otimes_{\mathbb{C}} U(\mathfrak{gl}(2, \mathbb{C})).$$

Let $\Lambda = \{ (\lambda_1, \lambda_2) \in \mathbb{Z}^2 \mid \lambda_1 \geq \lambda_2 \}$. For $\lambda = (\lambda_1, \lambda_2) \in \Lambda$, let $V_\lambda$ be the $\mathbb{C}$-vector space of degree $\lambda_1 - \lambda_2$ homogeneous polynomials in two variables $z_1, z_2$, on which

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$K$ acts by
\[
\tau_\lambda(k)p(z_1, z_2) = \det(k)^\lambda p((z_1, z_2)k) \quad (k \in K, \ p(z_1, z_2) \in V_\lambda).
\]
We define \( \{v^\lambda_q \mid 0 \leq q \leq \lambda_1 - \lambda_2 \} \) as a basis of \( V_\lambda \) by \( v^\lambda_q = z_1^{\lambda_1 - q}z_2^q \). It is known that the equivalence classes of irreducible representations of \( K \) is exhausted by \( \{(\tau_\lambda, V_\lambda) \mid \lambda \in \Lambda\} \).

2. Whittaker functions on \( GL(2, \mathbb{C}) \)

For \( c \in \mathbb{C}^\times \), we define an additive character \( \psi_c : \mathbb{C} \to \mathbb{C}^\times \) by
\[
\psi_c(x) = \exp(2\pi \sqrt{-1}(cx + \overline{cx})) \quad (x \in \mathbb{C}).
\]
Let \( C^\infty(N\backslash G; \psi_c) \) be the space of smooth functions \( f \) on \( G \) satisfying
\[
f\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) = \psi_c(x)f(g) \quad (x \in \mathbb{C}, \ g \in G),
\]
on which \( G \) acts by the right translation. For an irreducible admissible representation \((\pi, H_\pi)\) of \( G \), let \( I_{\pi, \psi_c} \) be the subspace of \( \text{Hom}_{\mathbb{C}, K}(H_\pi, K, C^\infty(N\backslash G; \psi_c)_K) \) consisting of all homomorphisms \( \Phi \) such that \( \Phi(f) \) is of moderate growth for any \( f \in H_\pi \). Here we denote by \( H_\pi, K \) and \( C^\infty(N\backslash G; \psi_c)_K \) the subspaces of \( H_\pi \) and \( C^\infty(N\backslash G; \psi_c) \) consisting of all \( K \)-finite vectors, respectively.

The multiplicity one theorem (cf. [Sha], [Wa]) tells that the intertwining space \( I_{\pi, \psi_c} \) is at most one dimensional. If \( I_{\pi, \psi_c} \neq \{0\} \), we say that \( \pi \) is generic, and define the Whittaker model \( \mathcal{W}(\pi, \psi_c) \) of \( \pi \) by
\[
\mathcal{W}(\pi, \psi_c) = \{\Phi(f) \mid f \in H_\pi, K, \Phi \in I_{\pi, \psi_c}\},
\]
and functions in \( \mathcal{W}(\pi, \psi_c) \) are called Whittaker functions for \( \pi \).

For an irreducible admissible representation \((\pi, H_\pi)\) of \( G \), it is known that \( \pi \) is generic if and only if \( \pi \) is infinitesimally equivalent to an irreducible principal series representation ([Ja2, Lemma 2.5]). We define principal series representations of \( G \) in the next section.

3. Principal series representations of \( GL(2, \mathbb{C}) \)

Let \( \nu = (\nu_1, \nu_2) \in \mathbb{C}^2 \) and \( d = (d_1, d_2) \in \mathbb{Z}^2 \). Let \( H^0_{(\nu, d)} \) be the space of continuous functions \( f \) on \( G \) satisfying
\[
f\left(\begin{pmatrix} m_1 & x \\ 0 & m_2 \end{pmatrix} g \right) = m_1^{d_1}m_2^{d_2}|m_1|^{2\nu_1-d_1+1}|m_2|^{2\nu_2-d_2-1}f(g) \quad (m_1, m_2 \in \mathbb{C}^\times, \ x \in \mathbb{C}, \ g \in G).
\]
The group $G$ acts on $H^0_{(\nu,d)}$ by the right translation
\[(\pi_{(\nu,d)}(g)f)(h) = f(hg) \quad (g, h \in G, \ f \in H^0_{(\nu,d)}).\]

Let $(\pi_{(\nu,d)}, H_{(\nu,d)})$ be the Hilbert representation of $G$, which is the completion of $(\pi_{(\nu,d)}, H^0_{(\nu,d)})$ relative to the inner product
\[(f_1, f_2) = \int_K f_1(k) \overline{f_2(k)} \, dk \quad (f_1, f_2 \in H^0_{(\nu,d)}).\]

We call $(\pi_{(\nu,d)}, H_{(\nu,d)})$ a principal series representation of $G$. It is known that a principal series representation $\pi_{(\nu,d)}$ is irreducible for generic parameter $\nu$. In this article, we only consider irreducible principal series representations of $G$. We may assume $d = (d_1, d_2) \in \Lambda$ without loss of generalities, since $H_{((\nu_1, \nu_2), (d_1, d_2))} \simeq H_{((\nu_2, \nu_1), (d_2, d_1))}$ as $(g_{\mathbb{C}}, K)$-modules if $\pi_{((\nu_1, \nu_2), (d_1, d_2))}$ is irreducible (cf. [SV, Corollary 2.8]).

By the Frobenius reciprocity law, the $K$-types of principal series representations of $G$ are given as follows.

**Lemma 3.1.** For $\nu = (\nu_1, \nu_2) \in \mathbb{C}^2$ and $d = (d_1, d_2) \in \Lambda$, it holds that
\[H_{(\nu, d), K} \simeq \bigoplus_{m=0}^{\infty} V_{(d_1+m,d_2-m)}\]
as $K$-modules.

4. Explicit formulas of Whittaker functions

In this section, we give explicit formulas of Whittaker functions for an irreducible principal series representation $(\pi_{(\nu,d)}, H_{(\nu,d)})$ of $G$ with $\nu = (\nu_1, \nu_2) \in \mathbb{C}^2$, $d = (d_1, d_2) \in \Lambda$. Let $c \in \mathbb{C}^\times$ and $m \in \mathbb{Z}_{\geq 0}$. We set $\lambda = (\lambda_1, \lambda_2) = (d_1 + m, d_2 - m)$, and take a $K$-embedding $\phi: V_{\lambda} \to W(\pi_{(\nu,d)}, \psi_c)$. Then, for $g \in G$ with the Iwasawa decomposition
\[g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 & y_2 \\ 0 & y_2 \end{pmatrix} k \quad (x \in \mathbb{C}, \ y_1, y_2 \in \mathbb{R}_{>0}, \ k \in K),\]
we have
\[\phi(v)(g) = \psi_c(x) y_2^{2\nu_1 + 2\nu_2} \phi(\tau_{\lambda}(k)v) \left( \begin{pmatrix} y_1 & 0 \\ 0 & 1 \end{pmatrix} \right).\]
Hence, the functions $\varphi_q (0 \leq q \leq \lambda_1 - \lambda_2)$ on $\mathbb{R}_{>0}$ defined by

$$\varphi_q(y_1) = \phi(r_q^\lambda)\begin{pmatrix} y_1 & 0 \\ 0 & 1 \end{pmatrix}$$

characterize the $K$-embedding $\phi$.

We denote by $C$ the Casimir element of $\mathfrak{sl}(2, \mathbb{C})$. From the actions of two elements of $Z(g_C)$ corresponding to $C \otimes 1$ and $1 \otimes C$, we obtain the following equalities:

$$\{(\vartheta - 2\nu_1 + m - q - 1)(\vartheta - 2\nu_2 - m + \lambda_1 - \lambda_2 - q - 1) - (4\pi cy_1)^2\} \varphi_q = -(8\pi c\sqrt{-1}y_1)(\lambda_1 - \lambda_2 - q)\varphi_{q+1},$$

$$\{(\vartheta - 2\nu_1 + m - q - 1)(\vartheta - 2\nu_2 + m - \lambda_1 + \lambda_2 + q - 1) - (4\pi cy_1)^2\} \varphi_q = (8\pi c\sqrt{-1}y_1)q\varphi_{q-1}$$

for $0 \leq q \leq \lambda_1 - \lambda_2$. Here $\vartheta = y_1 \frac{d}{dy_1}$, and $\varphi_q = 0$ if $q < 0$ or $q > \lambda_1 - \lambda_2$.

The system of differential equations (2), (3) has a unique moderate growth solution. Solving this system, we obtain the following theorem.

**Theorem 4.1.** Let $(\pi(\nu,d), H(\nu,d))$ be an irreducible principal series representation of $G$ with $\nu = (\nu_1, \nu_2) \in \mathbb{C}^2$ and $d = (d_1, d_2) \in \Lambda$. Let $c \in \mathbb{C}^\times$, $m \in \mathbb{Z}_{\geq 0}$, and set $\lambda = (\lambda_1, \lambda_2) = (d_1 + m, d_2 - m)$. Then there exists a $K$-embedding

$$\phi^{(c)}_{[\nu,d;m]} : V_{\lambda} \rightarrow \mathcal{W}(\pi(\nu,d), \psi_c)$$

which is characterized by

$$\phi^{(c)}_{[\nu,d;m]}(r_q^\lambda)\begin{pmatrix} y_1 & 0 \\ 0 & 1 \end{pmatrix} = \left(\frac{c\sqrt{-1}}{|c|}\right)^{\lambda_1-q} \sum_{i=0}^{q} \binom{q}{i} \frac{(-m)_i(-\nu_1 + \nu_2 - \frac{\lambda_1 - \lambda_2}{2})_i}{(-\lambda_1 + \lambda_2)_i(2\pi |c|y_1)^i} \times \frac{y_1}{4\pi \sqrt{-1}} \int_{\alpha - \sqrt{-1}\infty}^{\alpha + \sqrt{-1}\infty} \Gamma_C(s + \nu_1 + \frac{q + m - i}{2}) \times \Gamma_C(s + \nu_2 + \frac{\lambda_1 - \lambda_2 - q - m + i}{2})(|c|y_1)^{-2s} ds$$

for $y_1 \in \mathbb{R}_{>0}$ and $0 \leq q \leq \lambda_1 - \lambda_2$. Here $(s)_i = \Gamma(s + i)/\Gamma(s)$, $\Gamma_C(s) = 2(2\pi)^{-s}\Gamma(s)$ ($s \in \mathbb{C}$, $i \in \mathbb{Z}_{\geq 0}$) as usual, and $\alpha$ is a sufficiently large real number.

**Remark 4.2.** When $m = 0$, we note that the formula in Theorem 4.1 is simplified as follows:

$$\phi^{(c)}_{[\nu,d;0]}(r_q^\lambda)\begin{pmatrix} y_1 & 0 \\ 0 & 1 \end{pmatrix} = \left(\frac{c\sqrt{-1}}{|c|}\right)^{d_1-q} \frac{y_1}{4\pi \sqrt{-1}} \int_{\alpha - \sqrt{-1}\infty}^{\alpha + \sqrt{-1}\infty} \Gamma_C(s + \nu_1 + \frac{q}{2})$$
\[
\times \Gamma_C \left( s + \nu_2 + \frac{d_1 - d_2 - q}{2} \right) \left( |c|y_1 \right)^{-2s} ds
\]
for \( y_1 \in \mathbb{R}_{>0} \) and \( 0 \leq q \leq d_1 - d_2 \). This formula is given by Popa in [Po].

5. The local zeta integrals for \( GL(2, \mathbb{C}) \times GL(2, \mathbb{C}) \)

Let \( S(\mathbb{C}^2) \) be the space of Schwartz functions on \( \mathbb{C}^2 \). For \( f \in S(\mathbb{C}^2) \), we define the Fourier transform \( \mathcal{F}(f) \) of \( f \) with respect to \( \psi_1 \) by

\[
\mathcal{F}(f)(w_1, w_2) = \int_{\mathbb{C}^2} f(z_1, z_2) \psi_1(z_1w_1 + z_2w_2) dz_1 dz_2 \quad (w_1, w_2 \in \mathbb{C}).
\]

Here \( dz_i \) is the self-dual Haar measure on \( \mathbb{C} \) with respect to \( \psi_1 \), that is, \( dz_i \) is the twice Lebesgue measure on \( \mathbb{C} \simeq \mathbb{R}^2 \) for \( i = 1, 2 \). Then we note that

\[
\mathcal{F}(\mathcal{F}(f))(z_1, z_2) = f(-z_1, -z_2) \quad (f \in S(\mathbb{C}^2), \ z_1, z_2 \in \mathbb{C}).
\]

Let \( S(\mathbb{C}^2)^{std} \) be the subspace of \( S(\mathbb{C}^2) \) consisting of all functions \( f \) of the form

\[
f(z_1, z_2) = p(z_1, z_2, \overline{z_1}, \overline{z_2}) \exp(-2\pi(|z_1|^2 + |z_2|^2)) \quad (z_1, z_2 \in \mathbb{C}) \tag{4}
\]
with polynomial functions \( p \) on \( \mathbb{C}^4 \). We call functions in \( S(\mathbb{C}^2)^{std} \) standard Schwartz functions on \( \mathbb{C}^2 \).

Let \( (\pi_{(\nu, d)}, H_{(\nu, d)}) \) and \( (\pi_{(\nu', d')}, H_{(\nu', d')}) \) be irreducible principal series representations of \( G \) with \( \nu = (\nu_1, \nu_2) \in \mathbb{C}^2, \ d = (d_1, d_2) \in \mathbb{Z}^2, \ \nu' = (\nu'_1, \nu'_2) \in \mathbb{C}^2 \) and \( d' = (d'_1, d'_2) \in \mathbb{Z}^2 \). From the Langlands parameters of \( \pi_{(\nu, d)} \) and \( \pi_{(\nu', d')} \), we define the local \( L \)-factor \( L(s, \pi_{(\nu, d)} \times \pi_{(\nu', d')}) \) by

\[
L(s, \pi_{(\nu, d)} \times \pi_{(\nu', d')}) = \prod_{1 \leq i, j \leq 2} \Gamma_C \left( s + \nu_i + \nu'_j + \frac{|d_i + d'_j|}{2} \right).
\]

Let \( \varepsilon \in \{\pm 1\} \). For \( W \in \mathcal{W}(\pi_{(\nu, d)}, \psi_{\varepsilon}), \ W' \in \mathcal{W}(\pi_{(\nu', d')}, \psi_{-\varepsilon}) \) and \( f \in S(\mathbb{C}^2) \), we define the local zeta integral \( Z(s, W, W', f) \) by

\[
Z(s, W, W', f) = \int_{N \backslash G} W(g)W'(g) f((0, 1)g) |\det g|^{2s} d\dot{g},
\]
where \( d\dot{g} \) is the right invariant measure on \( N \backslash G \). The local zeta integrals converge for \( \text{Re}(s) \gg 0 \). Jacquet proves the following theorems ([Ja1], [Ja2]).

**Theorem 5.1 (Jacquet).** Retain the notation. Let \( W \in \mathcal{W}(\pi_{(\nu, d)}, \psi_1), \ W' \in \mathcal{W}(\pi_{(\nu', d')}, \psi_{-1}) \) and \( f \in S(\mathbb{C}^2) \). Then the ratio \( Z(s, W, W', f)/L(s, \pi_{(\nu, d)} \times \pi_{(\nu', d')}) \)
can improve Theorem 5.2 as follows:

\[ L(1 - s, \mathcal{L}(\phi, \tau, \nu), \mathfrak{h}) = (1 - \mathfrak{h})^{-1} \]

where \( \mathfrak{h} \) denotes an entire function of \( s \) and satisfies the local functional equation:

\[ L(1, \mathfrak{h}) = \mathfrak{h}(s) L(1 - s, \mathfrak{h}) \]

extends to an entire function of \( s \) and satisfies the local functional equation:

By the calculation based on the explicit formulas in the previous section, we can improve Theorem 5.2 as follows:

\[ Z(s, W, W', f) = \sum_{i=1}^{m} \mathcal{Z}(a, b, c) \choose \end{multline}

where \( \mathcal{Z}(a, b, c) \) is a \( K \)-homomorphism, and this homomorphism vanishes if

\[ V_{0}(f) \mathcal{Z}(a, b, c) = (0). \]
Because of the local functional equations, we may assume \( d_1 + d_2 + d'_1 + d'_2 \geq 0 \). Moreover, interchanging \( \pi(\nu, d) \) and \( \pi(\nu', d') \) if necessary, we may assume \( d_1 + d'_2 \geq 0 \).

Under these assumptions, if we set \( m = 0 \) and \( p = 0 \), then the smallest integers \( q = q_0 \) and \( m' = m'_0 \) satisfying \( \text{Hom}_K(V_\nu \otimes_C V_\nu' \otimes_C S(\mathbb{C}^2)^{\text{std}}, V(0,0)) \neq \{0\} \) are given by

\[
q_0 = d_1 + d_2 + d'_1 + d'_2, \quad m'_0 = \max\{d_2 + d'_2, 0, -d_2 - d'_1\},
\]

and the space

\[
\text{Hom}_K(V_\nu \otimes_C V_{d_1 + m'_0, d_2 - m'_0} \otimes_C S(\mathbb{C}^2)^{\text{std}}, V(0,0)) \tag{6}
\]
is one dimensional. Let \( \iota \) be a unique nonzero element of the space (6), and take \( v_0 \in V_\nu, v'_0 \in V_{d_1 + m'_0, d_2 - m'_0} \) and \( f_1 \in S(\mathbb{C}^2)^{\text{std}} \) so that \( \iota(v_0 \otimes v'_0 \otimes f_1) \neq 0 \). We can show that there exists some nonzero constant \( C \) such that

\[
Z(s, \phi_{[\nu, d; 0]}^{(1)}(v_0), \phi_{[\nu', d'; m'_0]}^{(-1)}(v'_0), f_1) = C L(s, \pi(\nu, d) \times \pi(\nu', d')) \quad (s \in \mathbb{C}),
\]

by the calculation using Barnes' lemma ([Ba1, §1.7]):

\[
\frac{1}{2\pi \sqrt{-1}} \int_{a - \sqrt{-1} \infty}^{a + \sqrt{-1} \infty} \frac{\Gamma(a + s)\Gamma(b + s)\Gamma(c - s)\Gamma(d - s)}{\Gamma(a + d)\Gamma(b + c)\Gamma(b + d)} \, ds = \frac{\Gamma(a + c)\Gamma(a + d)\Gamma(b + c)\Gamma(b + d)}{\Gamma(a + b + c + d)}
\]

for \( a, b, c, d \in \mathbb{C} \) and \( \alpha \in \mathbb{R} \) such that

\[-\min\{\text{Re}(a), \text{Re}(b)\} < \alpha < \min\{\text{Re}(c), \text{Re}(d)\}.
\]

Hence, \( W_0 = \phi_{[\nu, d; 0]}^{(1)}(v_0), W'_0 = \phi_{[\nu', d'; m'_0]}^{(-1)}(v'_0) \) and \( f_0 = C^{-1}f_1 \) satisfy (5).

References


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