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Multivariate circular Jacobi polynomials

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Abstract. We introduce a new family of multivariate orthogonal polynomials. They are a two-parameter deformation of the spherical polynomials by harmonic analysis on symmetric cones. They are regarded as a multivariate analogue of the circular Jacobi polynomials. Furthermore, the weight function of their orthogonality relation coincides with the circular Jacobi ensemble defined by Bourgade et al. We obtain their main properties : generating function, pseudo differential equation and determinant expression.

1. Introduction

The one variable circular Jacobi (orthogonal) polynomials are named by M. Ismail (see [**10**, p. 229]). They are defined by the Gaussian hypergeometric representation as

$$\phi_m^{(\alpha)}(x) := \frac{(\alpha)_m}{m!}\,{}_2F_1\left(\begin{matrix}-m, \frac{\alpha+1}{2}\\ \alpha\end{matrix}; 1-x\right) = \frac{\left(\frac{\alpha-1}{2}\right)_m}{m!}\,{}_2F_1\left(\begin{matrix}-m, \frac{\alpha+1}{2}\\ -m-\frac{\alpha-3}{2}\end{matrix}; x\right)$$
$$= \frac{(\alpha)_m}{m!}\sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{\left(\frac{\alpha+1}{2}\right)_k}{(\alpha)_k}(1-x)^k. \qquad (1)$$

Here, $(\alpha)_m := \frac{\Gamma(\alpha+m)}{\Gamma(\alpha)} = \alpha(\alpha+1)\cdots(\alpha+m-1)$ and $\binom{m}{k} := (-1)^k\frac{(-m)_k}{k!}$. For $\alpha > 0$, these polynomials $\phi_m^{(\alpha)}$ satisfy with the following orthogonality which was given by R. Askey [**1**] in 1982 (he proved more general results).

$$\frac{1}{2\pi}\int_0^{2\pi}\phi_m^{(\alpha)}(e^{i\theta})\overline{\phi_n^{(\alpha)}(e^{i\theta})}|(1-e^{i\theta})^{\frac{\alpha-1}{2}}|^2\,d\theta = \frac{\Gamma(\alpha+m)}{m!}\frac{1}{\Gamma(\frac{\alpha+1}{2})^2}\delta_{mn}. \qquad (2)$$

Li-Chien Shen [**14**] provided a useful framework for introducing the circular Jacobi polynomials. Specifically, Shen's picture considers the following function spaces and their complete orthogonal bases:
(1)

$$L^2_\alpha(\mathbb{R}_{>0}) := \{\psi : \mathbb{R}_{>0} \longrightarrow \mathbb{C} \mid \|\psi\|^2_{\alpha,\mathbb{R}_{>0}} < \infty\},$$
$$\|\psi\|^2_{\alpha,\mathbb{R}_{>0}} := \frac{2^\alpha}{\Gamma(\alpha)}\int_0^\infty |\psi(u)|^2 u^{\alpha-1}\,du,$$

$$\psi_m^{(\alpha)}(u) := e^{-u} L_m^{(\alpha-1)}(2u) = \frac{(\alpha)_m}{m!} e^{-u} \sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{1}{(\alpha)_k} (2u)^k$$

; exponential multiplied by Laguerre polynomials.

(2) Let H be the upper half plane $\{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$.

$$H_\alpha^2(\mathbb{R}) := \Big\{ \Psi : \mathbb{R} \longrightarrow \mathbb{C} \mid \|\Psi\|_{\alpha,\mathbb{R}}^2 < \infty \text{ and } \Psi \text{ is continued analytically to } H \text{ as a holomorphic function which satisfies } \sup_{0<y<\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} |\Psi(x+iy)|^2 \, dx < \infty \Big\},$$

$$\|\Psi\|_{\alpha,\mathbb{R}}^2 := \frac{\Gamma\left(\frac{\alpha+1}{2}\right)^2}{2\pi} \frac{2^\alpha}{\Gamma(\alpha)} \int_{-\infty}^{\infty} |\Psi(t)|^2 \, dt,$$

$$\Psi_m^{(\alpha)}(t) := (1-it)^{-\frac{\alpha+1}{2}} \frac{(\alpha)_m}{m!} \sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{\left(\frac{\alpha+1}{2}\right)_k}{(\alpha)_k} \left(\frac{2}{1-it}\right)^k$$

; modified Fourier transform of the Laguerre polynomials.

(3) Let $\mathcal{D}$ be the open unit disc $\{z \in \mathbb{C} \mid |z| < 1\}$, and $\Sigma := \{z \in \mathbb{C} \mid |z| = 1\}$.

$$H_\alpha^2(\Sigma) := \{\phi : \Sigma \longrightarrow \mathbb{C} \mid \phi \text{ is continued analytically to } \mathcal{D} \text{ as a holomorphic function and } \|\phi\|_{\alpha,\Sigma}^2 < \infty\},$$

$$\|\phi\|_{\alpha,\Sigma}^2 := \frac{\Gamma\left(\frac{\alpha+1}{2}\right)^2}{2\pi i} \frac{1}{\Gamma(\alpha)} \int_\Sigma |\phi(\sigma)|^2 |(1-\sigma)^{\frac{\alpha-1}{2}}|^2 \, d\mu(\sigma),$$

$$\phi_m^{(\alpha)}(\sigma) := \frac{(\alpha)_m}{m!} \sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{\left(\frac{\alpha+1}{2}\right)_k}{(\alpha)_k} (1-\sigma)^k$$

; circular Jacobi polynomials.

Here, $d\mu(\sigma) := \frac{d\sigma}{\sigma}$.

We remark that

$$\|\psi_m^{(\alpha)}\|_{\alpha,\mathbb{R}_{>0}}^2 = \|\Psi_m^{(\alpha)}\|_{\alpha,\mathbb{R}}^2 = \|\phi_m^{(\alpha)}\|_{\alpha,\Sigma}^2 = \frac{(\alpha)_m}{m!}.$$

Furthermore, the following unitary isomorphisms are known.
<u>Modified (inverse) Fourier transform</u>

$$\mathcal{F}_\alpha^{-1} : L_\alpha^2(\mathbb{R}_{>0}) \xrightarrow{\simeq} H_\alpha^2(\mathbb{R}), \ \ (\mathcal{F}_\alpha^{-1}\psi)(t) := \frac{1}{\Gamma\left(\frac{\alpha+1}{2}\right)} \int_0^\infty e^{itu} u^{\frac{\alpha-1}{2}} \psi(u) \, du.$$

Modified Cayley transform

$$\mathcal{C}_\alpha^{-1} : H_\alpha^2(\mathbb{R}) \xrightarrow{\simeq} H_\alpha^2(\Sigma), \;\; (\mathcal{C}_\alpha^{-1}\Psi)(\sigma) := \left(\frac{1-\sigma}{2}\right)^{-\frac{\alpha+1}{2}} \Psi\left(i\frac{1+\sigma}{1-\sigma}\right).$$

These two results of Shen can be described as follows.

PROPOSITION 1.1 ([**14**]).

$$\begin{array}{ccccc} L_\alpha^2(\mathbb{R}_{\geq 0}) & \xrightarrow[\mathcal{F}_\alpha^{-1}]{\simeq} & H_\alpha^2(\mathbb{R}) & \xrightarrow[\mathcal{C}_\alpha^{-1}]{\simeq} & H_\alpha^2(\Sigma), \;\; \text{(unitary)}. \\ \Cup & & \Cup & & \Cup \\ \psi_m^{(\alpha)} & \longmapsto & \Psi_m^{(\alpha)} & \longmapsto & \phi_m^{(\alpha)} \\ (\mathbf{1}) & & (\mathbf{2}) & & (\mathbf{3}) \end{array}$$

This setting is suitable for introducing the above orthogonal systems, but studying their fundamental properties (orthogonality, generating functions, differential equations).

The purpose of this article is to provide a multivariate analogue of the results obtained by Shen. Namely, we consider a modified Fourier transform of $L_\alpha^2(\Omega)^K$ and multivariate Laguerre polynomials. Using this unitary isomorphism and the modified Cayley transform, we introduce new multivariate special orthogonal polynomials, which we call them *multivariate circular Jacobi (MCJ) polynomials*. They are a generalization of the spherical (zonal) polynomials as well. Our generalization is different from the Jack or Macdonald polynomials that are well known as an extension of the spherical polynomials. We also remark that the weight function of their orthogonality relation coincides with the circular Jacobi ensemble defined by Bourgade et al. [**3**]. Furthermore, we provide a generating function for the MCJ polynomials and a differential equation that is satisfied by the modified Cayley transform of the MCJ polynomials. In case of the multiplicity $d = 2$, we establish a determinant formula for the MCJ polynomials.

Let us now describe the content in this paper. The basic definitions and fundamental properties of Jordan algebras and symmetric cones, and lemmas for analysis on symmetric cones are presented in the first subsection of Section 2. Section 3 which is the main part of this paper provides a generalization of Proposition 1.1. Using the generalization, we obtain the MCJ polynomials and their fundamental properties. Finally, in Section 4, we present a conjecture and some problems for a further generalization of the MCJ polynomials.

2. Preliminaries

Throughout this paper, we denote the ring of rational integers by $\mathbb{Z}$, the field of real numbers by $\mathbb{R}$, the field of complex numbers by $\mathbb{C}$. We fix a positive integer

r and denote the partition set of length r by

$$\mathscr{P} := \{\mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{Z}^r_{\geq 0} \mid m_1 \geq \cdots \geq m_r\}. \tag{3}$$

For any vector $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{C}^r$, we put

$$\Re\mathbf{s} := (\Re s_1, \ldots, \Re s_r), \tag{4}$$
$$|\mathbf{s}| := s_1 + \cdots + s_r, \tag{5}$$
$$\|\mathbf{s}\| := (|s_1|, \ldots, |s_r|). \tag{6}$$

Moreover, for $\mathbf{m} \in \mathscr{P}$ let

$$\mathbf{m}! := m_1! \cdots m_r!$$

and we set $\delta := (r-1, r-2, \ldots, 1, 0)$. Refer to Faraut and Koranyi [**7**] for further details in this section.

2.1. Analysis on symmetric cones

Let Ω be an irreducible symmetric cone in V which is a finite dimensional simple Euclidean Jordan algebra of dimension n as a real vector space and rank r. The classification of irreducible symmetric cones is well-known: there are four families of classical irreducible symmetric cones $\Pi_r(\mathbb{R}), \Pi_r(\mathbb{C}), \Pi_r(\mathbb{H})$, the cones of all $r \times r$ positive definite matrices over $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, the Lorentz cones Λ_r and an exceptional cone $\Pi_3(\mathbb{O})$ (see [**7**, p. 97]). Let $V^{\mathbb{C}} := V + iV$ be the complexification of V. Furthermore, we put $T_\Omega := \Omega + iV$ and $H_\Omega := V + i\Omega$. For $w, z \in V^{\mathbb{C}}$, we define

$$\begin{aligned}
L(w)z &:= wz, \\
w\square z &:= L(wz) + [L(w), L(z)], \\
P(w,z) &:= L(w)L(z) + L(z)L(w) - L(wz), \\
P(w) &:= P(w,w) = 2L(w)^2 - L(w^2).
\end{aligned}$$

We denote the Jordan trace and determinant of the complex Jordan algebra $V^{\mathbb{C}}$ by tr x and by $\Delta(x)$ respectively.

Fix a Jordan frame $\{c_1, \ldots, c_r\}$ that is a complete system of orthogonal primitive idempotents in V and define the following subspaces:

$$\begin{aligned}
V_j &:= \{x \in V \mid L(c_j)x = x\}, \\
V_{jk} &:= \left\{x \in V \,\middle|\, L(c_j)x = \frac{1}{2}x \text{ and } L(c_k)x = \frac{1}{2}x\right\}.
\end{aligned}$$

Then, $V_j = \mathbb{R}c_j$ for $j = 1, \ldots, r$ are one-dimensional subalgebras of V, while the subspaces V_{jk} for $j, k = 1, \ldots, r$ with $j < k$ all have a common dimension

$d = \dim_{\mathbb{R}} V_{jk}$. The space V has the Peirce decomposition

$$V = \left(\bigoplus_{j=1}^{r} V_j \right) \oplus \left(\bigoplus_{j<k} V_{jk} \right),$$

which is the orthogonal direct sum. It follows that $n = r + \frac{d}{2}r(r-1)$. Let $G(\Omega)$ denote the automorphism group of Ω and let G be the identity component in $G(\Omega)$. Then G acts transitively on Ω and $\Omega \cong G/K$ where $K \subset G$ is the isotropy subgroup of the unit element $e \in V$. K is also the identity component in Aut (V).

For any $x \in V$, there exist $k \in K$ and $\lambda_1, \dots, \lambda_r \in \mathbb{R}$ such that

$$x = k \sum_{j=1}^{r} \lambda_j c_j \quad (\lambda_1 \geq \cdots \geq \lambda_r).$$

From this polar decomposition, we obtain the following integral formula (see [**7**, Theorem VI.2.3]).

Lemma 2.1. *Let f be an integrable function on V. Then*

$$\int_V f(x)\, dx = \widetilde{c_0} \int_{K \times \mathbb{R}^r} f(k\lambda) \prod_{1 \leq p < q \leq r} |\lambda_p - \lambda_q|^d \, dk d\lambda_1 \cdots d\lambda_r. \tag{7}$$

Here, dx is the Euclidean measure associated with the Euclidean structure on V given by $(u|v) = \mathrm{tr}(uv)$; dk is the normalized Haar measure on the compact group K ; $\lambda = \sum_{j=1}^{r} \lambda_j c_j$; and $\widetilde{c_0}$ is defined by

$$\widetilde{c_0} := (2\pi)^{\frac{n-r}{2}} \prod_{j=1}^{r} \frac{\Gamma\left(\frac{d}{2}+1\right)}{\Gamma\left(\frac{d}{2}j+1\right)} = \frac{(2\pi)^{\frac{n-r}{2}}}{r!} \prod_{j=1}^{r} \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2}j\right)}. \tag{8}$$

In particular, for $f \in L^1(V)^K$

$$\int_V f(x)\, dx = \widetilde{c_0} \int_{\mathbb{R}^r} f(\lambda_1, \dots, \lambda_r) \prod_{1 \leq p < q \leq r} |\lambda_p - \lambda_q|^d \, d\lambda_1 \cdots d\lambda_r. \tag{9}$$

As in the case of V, we have the following spectral decomposition for $V^{\mathbb{C}}$. Every z in $V^{\mathbb{C}}$ can be written

$$z = u \sum_{j=1}^{r} \lambda_j c_j$$

with $u \in U$ as the identity component of $Str(V^{\mathbb{C}}) \cap U(V^{\mathbb{C}})$, $\lambda_1 \geq \cdots \geq \lambda_r \geq 0$. Here $Str(V^{\mathbb{C}})$ and $U(V^{\mathbb{C}})$ are the structure and unitary groups of V, respectively (for details, see [**7**, Chapter VIII]). Moreover, we define the spectral norm of $z \in V^{\mathbb{C}}$

by $|z| = \lambda_1$ and introduce the open unit ball $\mathcal{D} \in V^{\mathbb{C}}$ by

$$\mathcal{D} = \{z \in V^{\mathbb{C}} \mid |z| < 1\}.$$

We define Σ as the set of invertible elements in $V^{\mathbb{C}}$ such that $z^{-1} = \overline{z}$. This coincides with the Shilov boundary of $\mathcal{D}$. For Σ, the following result is well known (see [**7**, Proposition X.2.3]).

Lemma 2.2. *For $z \in V^{\mathbb{C}}$, the following properties are equivalent:*
(i) $z \in \Sigma$,
(ii) $z = e^{i\theta} = \sum_{j=1}^{r} e^{i\theta_j} c_j$ *with* $\theta = \sum_{j=1}^{r} \theta_j c_j \in V$,
(iii) $z \in \overline{c^{-1}(V)}$,
where $c^{-1}(t) := (t - ie)(t + ie)^{-1} = e - 2i(t + ie)^{-1}$ *is called the inverse Cayley transform.*

We will later need the following integral formula on Σ to describe the orthogonal relation of the MCJ polynomials (Theorem 3.1 (2)).

Lemma 2.3. *Let μ denote the measure associated with the Riemannian structure on Σ induced by the Euclidean structure of $V^{\mathbb{C}}$.*
(1) *If ϕ is an integrable function on Σ, then*

$$\int_{\Sigma} \phi(\sigma)\, d\mu(\sigma) = 2^n \int_V \phi(c^{-1}(t)) |\Delta(e - it)^{-\frac{n}{r}}|^2\, dt. \tag{10}$$

(2) *If Ψ is an integrable function on V, then*

$$\int_V \Psi(t)\, dt = 2^n \int_{\Sigma} \Psi(c(\sigma)) |\Delta(e - \sigma)^{-\frac{n}{r}}|^2\, d\mu(\sigma). \tag{11}$$

Here, c is a Cayley transform defined by $c(\sigma) := i(e + \sigma)(e - \sigma)^{-1} = -ie + 2i(e - \sigma)^{-1}$.
(3) *If Ψ is an integrable function on V and a K-invariant, then*

$$\int_{\Sigma} \phi(\sigma)\, d\mu(\sigma) = \widetilde{c_0} \int_{\mathbb{T}^r} \phi(e^{i\theta}) \prod_{1 \le p < q \le r} |e^{i\theta_p} - e^{i\theta_q}|^d\, d\theta_1 \cdots d\theta_r. \tag{12}$$

Here $\mathbb{T}^r$ is the direct product of r circles.

Proof. (1) is [**7**, Proposition X.2.4] itself. (2) is proved similarly. Hence we only prove (3).

Let $\phi \in L^1(\Sigma)^K$. First, we remark for any $k \in K$

$$c^{-1}(kt) = (k(t - ie))(k(t + ie))^{-1} = k((t + ie)(t - ie)^{-1}) = kc^{-1}(t).$$

From Lemma 2.3 (1) and Lemma 2.1, we have

$$\begin{aligned}\int_\Sigma \phi(\sigma)\,d\mu(\sigma) &= 2^n \int_V \phi(c^{-1}(t))\Delta(e+t^2)^{-\frac{n}{r}}\,dt \\ &= 2^n \widetilde{c_0} \int_{K\times\mathbb{R}^r} \phi(c^{-1}(k\lambda))\Delta(e+(k\lambda)^2)^{-\frac{n}{r}} \\ &\quad \cdot \prod_{1\le p<q\le r} |\lambda_p-\lambda_q|^d\, dk d\lambda_1\cdots d\lambda_r \\ &= 2^n \widetilde{c_0} \int_{\mathbb{R}^r} \phi(c^{-1}(\lambda))\Delta(e+\lambda^2)^{-\frac{n}{r}} \prod_{1\le p<q\le r} |\lambda_p-\lambda_q|^d\, d\lambda_1\cdots d\lambda_r.\end{aligned}$$

If we put $\lambda_j = -\cot\left(\frac{\theta_j}{2}\right)$, then

$$\lambda = i\sum_{j=1}^r \frac{1+e^{i\theta_j}}{1-e^{i\theta_j}} c_j = i\left(\sum_{j=1}^r (1+e^{i\theta_j})c_j\right)\left(\sum_{l=1}^r (1-e^{i\theta_l})c_l\right)^{-1} = c(e^{i\theta}).$$

Therefore,

$$\begin{aligned}\int_\Sigma \phi(\sigma)\,d\mu(\sigma) &= 2^{n-r}\widetilde{c_0}\int_{\mathbb{T}^r} \phi(e^{i\theta}) \prod_{j=1}^r \sin\left(\frac{\theta_j}{2}\right)^{2\left(\frac{n}{r}-1\right)} \\ &\quad \cdot \prod_{1\le p<q\le r} \left| \frac{\sin\left(\frac{1}{2}(\theta_p-\theta_q)\right)}{\sin\left(\frac{\theta_p}{2}\right)\sin\left(\frac{\theta_q}{2}\right)} \right|^d d\theta_1\cdots d\theta_r \\ &= \widetilde{c_0}\int_{\mathbb{T}^r} \phi(e^{i\theta}) \prod_{1\le p<q\le r} |e^{i\theta_p}-e^{i\theta_q}|^d\, d\theta_1\cdots d\theta_r.\end{aligned}$$

□

For $j = 1,\dots,r$, let $e_j := c_1+\cdots+c_j$, and set

$$V^{(j)} := \{x\in V \mid L(e_j)x = x\}.$$

Let us denote the orthogonal projection of V onto the subalgebra $V^{(j)}$ by P_j, and define

$$\Delta_j(x) := \delta_j(P_j x)$$

for $x \in V$, where δ_j denotes the Koecher norm function for $V^{(j)}$. In particular, $\delta_r = \Delta$. Then Δ_j is a polynomial on V that is homogeneous of degree j. Let

$\mathbf{s} := (s_1, \dots, s_r) \in \mathbb{C}^r$ and define the function $\Delta_{\mathbf{s}}$ on V by

$$\Delta_{\mathbf{s}}(x) := \Delta(x)^{s_r} \prod_{j=1}^{r-1} \Delta_j(x)^{s_j - s_{j+1}}. \tag{13}$$

That is the generalized power function on V. We remark that for $\mathbf{m} \in \mathscr{P}$, $\Delta_{\mathbf{m}}$ becomes a polynomial function on V that is homogeneous of degree $|\mathbf{m}|$. Furthermore, $\Delta_{\mathbf{s}}$ can be extended to a function on $V^{\mathbb{C}}$ by analytic continuation.

The gamma function Γ_Ω for the symmetric cone Ω is defined, for $\mathbf{s} \in \mathbb{C}^r$ and $\Re s_j > \frac{d}{2}(j-1)$ $(j = 1, \dots, r)$ by

$$\Gamma_\Omega(\mathbf{s}) := \int_\Omega e^{-\mathrm{tr}(x)} \Delta_{\mathbf{s}}(x) \Delta(x)^{-\frac{n}{r}}\, dx. \tag{14}$$

Its evaluation gives

$$\Gamma_\Omega(\mathbf{s}) = (2\pi)^{\frac{n-r}{2}} \prod_{j=1}^{r} \Gamma\left(s_j - \frac{d}{2}(j-1)\right). \tag{15}$$

Hence, Γ_Ω extends analytically as a meromorphic function on $\mathbb{C}^r$.

For $\mathbf{s} \in \mathbb{C}^r$ and $\mathbf{m} \in \mathscr{P}$, we define the generalized shifted factorial by

$$(\mathbf{s})_{\mathbf{m}} := \frac{\Gamma_\Omega(\mathbf{s} + \mathbf{m})}{\Gamma_\Omega(\mathbf{s})}. \tag{16}$$

It follows from (15) that

$$(\mathbf{s})_{\mathbf{m}} = \prod_{j=1}^{r} \left(s_j - \frac{d}{2}(j-1)\right)_{m_j}. \tag{17}$$

The following lemma is useful to estimate the approximation of spherical Taylor expansions (see [**15**, Lemma 2.1]).

Lemma 2.4. *If* $\mathbf{s} \in \mathbb{C}^r$, $\mathbf{m}, \mathbf{k} \in \mathscr{P}$ *and* $\mathbf{m} \supset \mathbf{k}$, *then*

$$\left| \frac{(\mathbf{s})_{\mathbf{m}}}{(\mathbf{s})_{\mathbf{k}}} \right| \le \frac{(\|\mathbf{s}\| + d(r-1))_{\mathbf{m}}}{(\|\mathbf{s}\| + d(r-1))_{\mathbf{k}}}. \tag{18}$$

The space $\mathcal{P}(V)$ of the polynomial ring on V has the following decomposition:

$$\mathcal{P}(V) = \bigoplus_{\mathbf{m} \in \mathscr{P}} \mathcal{P}_{\mathbf{m}}.$$

Here $\mathcal{P}_{\mathbf{m}}$ are mutually inequivalent as finite dimensional irreducible G-modules. Their dimensions are denoted by $d_{\mathbf{m}}$. The following formula is known for $d_{\mathbf{m}}$(see, [**16**, Lemma 2.6] or [**7**, p. 315]).

Lemma 2.5. *For any* $\mathbf{m} \in \mathscr{P}$,

$$d_{\mathbf{m}} = \frac{c(-\rho)}{c(\rho - \mathbf{m})c(\mathbf{m} - \rho)} \tag{19}$$

$$= \prod_{1 \le p < q \le r} \frac{m_p - m_q + \frac{d}{2}(q-p)}{\frac{d}{2}(q-p)} \frac{B\left(m_p - m_q, \frac{d}{2}(q-p-1)+1\right)}{B\left(m_p - m_q, \frac{d}{2}(q-p+1)\right)} \tag{20}$$

$$= \prod_{j=1}^{r} \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2}j\right)\Gamma\left(\frac{d}{2}(j-1)+1\right)}$$

$$\cdot \prod_{1 \le p < q \le r} \left(m_p - m_q + \frac{d}{2}(q-p)\right) \frac{\Gamma\left(m_p - m_q + \frac{d}{2}(q-p+1)\right)}{\Gamma\left(m_p - m_q + \frac{d}{2}(q-p-1)+1\right)}. \tag{21}$$

Here $\rho = (\rho_1, \ldots, \rho_r)$, $\rho_j := \frac{d}{4}(2j - r - 1)$, *and* c *is the Harish-Chandra function:*

$$c(\mathbf{s}) = \prod_{1 \le p < q \le r} \frac{B\left(s_q - s_p, \frac{d}{2}\right)}{B\left(\frac{d}{2}(q-p), \frac{d}{2}\right)}.$$

In particular, for $d = 2$

$$d_{\mathbf{m}} = \prod_{1 \le p < q \le r} \left(\frac{m_p - m_q + q - p}{q - p}\right)^2 = s_{\mathbf{m}}(1, \ldots, 1)^2. \tag{22}$$

Here $s_{\mathbf{m}}$ *is the Schur polynomial corresponding to* $\mathbf{m} \in \mathscr{P}$ *defined by*

$$s_{\mathbf{m}}(\lambda_1, \ldots, \lambda_r) := \frac{\det\left(\lambda_j^{m_k + r - k}\right)}{\det\left(\lambda_j^{r-k}\right)}.$$

The following lemma is necessary to evaluate the Fourier transform of the multivariate Laguerre polynomial.

Lemma 2.6 ([**7**, Theorem XI.2.3]). *For* $p \in \mathcal{P}_{\mathbf{m}}$, $\Re\alpha > (r-1)\frac{d}{2}$ *and* $z \in T_\Omega$,

$$\int_\Omega e^{-(z|x)} p(x) \Delta(x)^{\alpha - \frac{n}{r}}\, dx = \Gamma_\Omega(\mathbf{m} + \alpha)\Delta(z)^{-\alpha} p(z^{-1}). \tag{23}$$

Here α *is regarded as* $(\alpha, \ldots, \alpha) \in \mathbb{C}^r$.

For each $\mathbf{m} \in \mathscr{P}$, the spherical polynomial of weight $\mathbf{m}$ on Ω is defined by

$$\Phi_{\mathbf{m}}^{(d)}(x) := \int_K \Delta_{\mathbf{m}}(kx)\, dk. \tag{24}$$

We will omit the multiplicity d and simply write $\Phi_{\mathbf{m}}$. The algebra of all K-invariant polynomials on V, denoted by $\mathcal{P}(V)^K$, decomposes as

$$\mathcal{P}(V)^K = \bigoplus_{\mathbf{m}\in\mathscr{P}} \mathbb{C}\Phi_{\mathbf{m}}.$$

By analytic continuation to the complexification $V^{\mathbb{C}}$ of V, we can extend tr, Δ and $\Phi_{\mathbf{m}}$ to polynomial functions on $V^{\mathbb{C}}$.

Remark 2.7. (1) *Since* $\Phi_{\mathbf{m}} \in \mathcal{P}_{\mathbf{m}}^K$, *for* $x = k\sum_{j=1}^r \lambda_j c_j$ $\Phi_{\mathbf{m}}(x)$ *can be expressed by*

$$\Phi_{\mathbf{m}}(\lambda_1,\dots,\lambda_r) := \Phi_{\mathbf{m}}\left(\sum_{j=1}^r \lambda_j c_j\right) (= \Phi_{\mathbf{m}}(x)).$$

$\Phi_{\mathbf{m}}(x)$ *has the following expression (see* [**6**]*) as well:*

$$\Phi_{\mathbf{m}}^{(d)}(\lambda_1,\dots,\lambda_r) = \frac{P_{\mathbf{m}}^{(\frac{2}{d})}(\lambda_1,\dots,\lambda_r)}{P_{\mathbf{m}}^{(\frac{2}{d})}(1,\dots,1)}. \tag{25}$$

Here $P_{\mathbf{m}}^{(\frac{2}{d})}(\lambda_1,\dots,\lambda_r)$ *is an* r*-variable Jack polynomial (see* [**11**, Chapter. VI. 10]*). When* $P_{\mathbf{m}}^{(1)}(\lambda_1,\dots,\lambda_r) = s_{\mathbf{m}}(\lambda_1,\dots,\lambda_r)$, $\Phi_{\mathbf{m}}^{(2)}$ *becomes the Schur polynomial:*

$$\Phi_{\mathbf{m}}^{(2)}(\lambda_1,\dots,\lambda_r) = \frac{s_{\mathbf{m}}(\lambda_1,\dots,\lambda_r)}{s_{\mathbf{m}}(1,\dots,1)} = \prod_{1\le p<q\le r} \frac{q-p}{m_p - m_q + q - p} s_{\mathbf{m}}(\lambda_1,\dots,\lambda_r). \tag{26}$$

(2) *When* $r = 2$, $\Phi_{\mathbf{m}}^{(d)}$ *has the following hypergeometric expression (see* [**13**]*).*

$$\begin{aligned}\Phi_{m_1,m_2}^{(d)}(\lambda_1,\lambda_2) &= \lambda_1^{m_1}\lambda_2^{m_2}\,{}_2F_1\left(\begin{matrix}-(m_1-m_2),\frac{d}{2}\\ d\end{matrix};\frac{\lambda_1-\lambda_2}{\lambda_1}\right)\\ &= \lambda_1^{m_1}\lambda_2^{m_2}\frac{\left(\frac{d}{2}\right)_{m_1-m_2}}{(d)_{m_1-m_2}}\,{}_2F_1\left(\begin{matrix}-(m_1-m_2),\frac{d}{2}\\ -(m_1-m_2)-\frac{d}{2}+1\end{matrix};\frac{\lambda_2}{\lambda_1}\right).\end{aligned}$$

Note that the function $\Phi_{\mathbf{m}}(e+x)$ is a K-invariant polynomial of degree $|\mathbf{m}|$. We define the generalized binomial coefficients $\binom{\mathbf{m}}{\mathbf{k}}_{\frac{d}{2}}$ by using the following expansion:

$$\Phi_{\mathbf{m}}^{(d)}(e+x) = \sum_{|\mathbf{k}|\le|\mathbf{m}|} \binom{\mathbf{m}}{\mathbf{k}}_{\frac{d}{2}} \Phi_{\mathbf{k}}^{(d)}(x). \tag{27}$$

For $\binom{\mathbf{m}}{\mathbf{k}}_{\frac{d}{2}}$, we also often omit $\frac{d}{2}$. It is well-known that $\binom{\mathbf{m}}{\mathbf{k}} = 0$ when $\mathbf{k} \not\subset \mathbf{m}$.

Hence, we have

$$\Phi_{\mathbf{m}}(e+x) = \sum_{\mathbf{k}\subset\mathbf{m}} \binom{\mathbf{m}}{\mathbf{k}} \Phi_{\mathbf{k}}(x). \tag{28}$$

We recall two Lemmas in [**7**] for the spherical polynomials.

Lemma 2.8 ([**7**, Theorem XII.1.1 (i)]). *For $z = u\sum_{j=1}^{r}\lambda_j c_j$ with $u \in U$, $\lambda_1 \geq \cdots \geq \lambda_r \geq 0$ and $\mathbf{m} \in \mathscr{P}$, we have*

$$|\Phi_{\mathbf{m}}(z)| \leq \lambda_1^{m_1}\cdots\lambda_r^{m_r} \leq \lambda_1^{|\mathbf{m}|} = \Phi_{\mathbf{m}}(\lambda_1). \tag{29}$$

Lemma 2.9 ([**7**, Chapter XV. Exercises 3. (a)]). *For any $\alpha \in \mathbb{C}$, $z \in \overline{\mathcal{D}}$, $w \in \mathcal{D}$, we have*

$$\sum_{\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \Phi_{\mathbf{m}}(z)\Phi_{\mathbf{m}}(w) = \Delta(w)^{-\alpha}\int_K \Delta(kw^{-1}-z)^{-\alpha}\,dk. \tag{30}$$

The spherical function $\varphi_{\mathbf{s}}$ on Ω for $\mathbf{s} \in \mathbb{C}^r$ is defined by

$$\varphi_{\mathbf{s}}(x) := \int_K \Delta_{\mathbf{s}+\rho}(kx)\,dk. \tag{31}$$

We remark that for $x \in \Omega$

$$\varphi_{\mathbf{s}}(x^{-1}) = \varphi_{-\mathbf{s}}(x), \tag{32}$$

and for $x \in \Omega, \mathbf{m} \in \mathscr{P}$

$$\Phi_{\mathbf{m}}(x) = \varphi_{\mathbf{m}-\mathbf{s}}(x). \tag{33}$$

Let $\mathbb{D}(\Omega)$ denote the algebra of G-invariant differential operators on Ω; let $\mathcal{P}(V)^K$ denote the space of K-invariant polynomials on V; and let $\mathcal{P}(V\times V)^G$ denote the space of polynomials on $V\times V$ that are invariant as follows

$$p(gx,\xi) = p(x,g^*\xi) \quad (g \in G).$$

Here we write g^* for the adjoint of an element g; i.e., $(gx|y) = (x|g^*y)$ for all $x, y \in V$. The spherical function $\varphi_{\mathbf{s}}$ is an eigenfunction of every $D \in \mathbb{D}(\Omega)$. We denote its eigenvalues by $\gamma(D)(\mathbf{s})$, so $D\varphi_{\mathbf{s}} = \gamma(D)(\mathbf{s})\varphi_{\mathbf{s}}$.

Let σ_D denote the symbol of a partial differential operator D that acts on the variable $x \in V$ by

$$De^{(x|\xi)} = \sigma_D(x,\xi)e^{(x|\xi)} \quad (x,\xi \in V).$$

A differential operator D on Ω is invariant under G if and only if its symbol

σ_D belongs to $\mathcal{P}(V\times V)^G$. In addition, the map $D\mapsto\sigma_D$ establishes a linear isomorphism from $\mathbb{D}(\Omega)$ onto $\mathcal{P}(V\times V)^G$. The map $D\mapsto\sigma_D(e,u)$ is a vector space isomorphism from $\mathbb{D}(\Omega)$ onto $\mathcal{P}(V)^K$. In particular, for $\mathbf{k}\in\mathscr{P}$, $\mathbf{s}\in\mathbb{C}^r$, we put

$$\gamma_{\mathbf{k}}(\mathbf{s}) := \gamma(\Phi_{\mathbf{k}}(\partial_x))(\mathbf{s}) = \Phi_{\mathbf{k}}(\partial_x)\varphi_{\mathbf{s}}(x)|_{x=e}. \tag{34}$$

Here $\Phi_{\mathbf{k}}(\partial_x)$ is a unique G-invariant differential operator satisfying

$$\sigma_{\Phi_{\mathbf{k}}(\partial_x)}(e,\xi) = \Phi_{\mathbf{k}}(\xi)\in\mathcal{P}(V)^K,\quad \text{i.e., } \Phi_{\mathbf{k}}(\partial_x)e^{(x|\xi)}|_{x=e} = \Phi_{\mathbf{k}}(\xi)e^{\operatorname{tr}\xi}.$$

We remark that $\Phi_k(\partial_x)=\partial_x^k$ and $\gamma_k(s)=s(s-1)\cdots(s-k+1)$ in the case $r=1$. For any $\alpha\in\mathbb{C}$, $\mathbf{k}\in\mathscr{P}$, we have

$$\gamma_{\mathbf{k}}(\alpha-\rho) = (-1)^{|\mathbf{k}|}(-\alpha)_{\mathbf{k}}. \tag{35}$$

The function γ_D is an r variable symmetric polynomial and the map $D\mapsto\gamma_D$ is an algebra isomorphism from $\mathbb{D}(\Omega)$ onto the algebra $\mathcal{P}(\mathbb{R}^r)^{\mathfrak{S}_r}$, which is a special case of the Harish-Chandra isomorphism.

If a K-invariant function ψ is analytic in the neighborhood of e, it admits a spherical Taylor expansion near e:

$$\psi(e+x) = \sum_{\mathbf{k}\in\mathscr{P}} d_{\mathbf{k}}\frac{1}{\left(\frac{n}{r}\right)_{\mathbf{k}}}\{\Phi_{\mathbf{k}}(\partial_x)\psi(x)|_{x=e}\}\Phi_{\mathbf{k}}(x).$$

By the definition of $\gamma_{\mathbf{k}}$, we have

$$\varphi_{\mathbf{s}}(e+x) = \sum_{\mathbf{k}\in\mathscr{P}} d_{\mathbf{k}}\frac{1}{\left(\frac{n}{r}\right)_{\mathbf{k}}}\gamma_{\mathbf{k}}(\mathbf{s})\Phi_{\mathbf{k}}(x).$$

Since $\Phi_{\mathbf{m}}=\varphi_{\mathbf{m}-\rho}$,

$$\binom{\mathbf{m}}{\mathbf{k}} = d_{\mathbf{k}}\frac{1}{\left(\frac{n}{r}\right)_{\mathbf{k}}}\gamma_{\mathbf{k}}(\mathbf{m}-\rho).$$

For a complex number α, we define the following differential operator on Ω:

$$D_\alpha = \Delta(x)^{1+\alpha}\Delta(\partial_x)\Delta(x)^{-\alpha}.$$

For this operator, we have

$$\gamma(D_\alpha)(\mathbf{s}) = \prod_{j=1}^{r}\left(s_j-\alpha+\frac{d}{4}(r-1)\right). \tag{36}$$

The operators $D_{j\frac{d}{2}}$, $j=0,\dots,r-1$ generate the algebra $\mathbb{D}(\Omega)$.

For $\gamma_{\mathbf{k}}$, the following two lemmas are known.

Lemma 2.10 ([**15**, Lemma 2.8]). *For all* $\mathbf{k} \in \mathscr{P}$*, there exist some constant* $C > 0$ *and integer* N *such that for any* $\mathbf{s} \in \mathbb{C}^r$

$$|\gamma_{\mathbf{k}}(\mathbf{s})| \leq C \prod_{l=1}^{r} \left(|s_l| + \frac{d}{4}(r-1) \right)^N . \tag{37}$$

Lemma 2.11 ([**15**, Lemma 2.9]). *For all* $\mathbf{m}, \mathbf{k} \in \mathscr{P}$*, we have*

$$\gamma_{\mathbf{k}}(\mathbf{m} - \rho) \geq 0. \tag{38}$$

Theorem 2.12 ([**15**, Theorem 2.13]). (1) *For* $w \in \mathcal{D}$, $\mathbf{k} \in \mathscr{P}$, $\alpha \in \mathbb{C}$, *we have*

$$(\alpha)_{\mathbf{k}} \Delta(e - w)^{-\alpha} \Phi_{\mathbf{k}}(w(e-w)^{-1}) = \sum_{\mathbf{x} \in \mathscr{P}} d_{\mathbf{x}} \frac{(\alpha)_{\mathbf{x}}}{\left(\frac{n}{r}\right)_{\mathbf{x}}} \gamma_{\mathbf{k}}(\mathbf{x} - \rho) \Phi_{\mathbf{x}}(w). \tag{39}$$

We choose the branch of $\Delta(e - w)^{-\alpha}$ *that takes the value* 1 *at* $w = 0$.
(2) *For* $w \in V^{\mathbb{C}}$, $\mathbf{k} \in \mathscr{P}$, *a* K*-invariant analytic function* $e^{\operatorname{tr} w} \Phi_{\mathbf{k}}(w)$ *has the following expansion:*

$$e^{\operatorname{tr} w} \Phi_{\mathbf{k}}(w) = \sum_{\mathbf{x} \in \mathscr{P}} d_{\mathbf{x}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{x}}} \gamma_{\mathbf{k}}(\mathbf{x} - \rho) \Phi_{\mathbf{x}}(w). \tag{40}$$

Now we consider the gradient for a $\mathbb{C}$-valued function f on simple Euclidean Jordan algebra V; see [**5**] for details. For a scalar or vector valued differentiable function f, we define the gradient $\nabla f(x)$ by

$$(\nabla f(x)|u) = D_u f(x) = \frac{d}{dt} f(x + tu) \bigg|_{t=0} .$$

For a $\mathbb{C}$-valued function $f = f_1 + i f_2$, we define $\nabla f = \nabla f_1 + i \nabla f_2$. For $z = x + iy \in V^{\mathbb{C}}$, we define $D_z = D_x + i D_y$. If $\{e_1, \ldots, e_n\}$ is an orthonormal basis of V and $x = \sum_{j=1}^{n} x_j e_j \in V^{\mathbb{C}}$, then

$$\nabla f(x) = \sum_{j=1}^{n} \frac{\partial f(x)}{\partial x_j} e_j.$$

This expression is independent of the choice of an orthonormal basis of V.

For a V-valued function $f : V \to V$ expressed by $f(x) = \sum_{j=1}^{r} f_j(x) e_j$, we define ∇f by

$$\nabla f(x) = \sum_{j,l=1}^{n} \frac{\partial f_j(x)}{\partial x_l} e_j e_l.$$

Let us present some derivation formulas.

LEMMA 2.13. (1) *The product rule of differentiation: For V-valued functions f and h, we have*

$$\mathrm{tr}\ (\nabla(f(x)h(x))) = \mathrm{tr}\ (\nabla f(x))h(x) + f(x)\,\mathrm{tr}\ (\nabla h(x)). \tag{41}$$

For $\mathbb{C}$-valued functions f and h,

$$\nabla(f(x)h(x)) = (\nabla f(x))h(x) + f(x)(\nabla h(x)). \tag{42}$$

(2) *For $x \in V$,*

$$\nabla x = \frac{n}{r}e. \tag{43}$$

(3) *For any invertible element $x \in V^{\mathbb{C}}$,*

$$\mathrm{tr}\ (x\nabla)x^{-1} := \mathrm{tr}\ (x(\nabla x^{-1})) = -\frac{n}{r}\,\mathrm{tr}\ x^{-1}. \tag{44}$$

(4) *For $\beta \in \mathbb{C}$ and an invertible element $x \in V^{\mathbb{C}}$,*

$$\nabla(\Delta(x)^{\beta}) = \beta\Delta(x)^{\beta}x^{-1}. \tag{45}$$

(1), (2), and (4) are well known (see [**7, 5, 8**]). (3) follows from (1), (2), and $\nabla(xx^{-1}) = \nabla(e) = 0$.

We provide a Plancherel theorem, as it is useful to investigate the MCJ polynomials.

LEMMA 2.14. *Put*

$$\begin{aligned}
L^2(\Omega) &:= \{\psi : \Omega \longrightarrow \mathbb{C} \mid \|\psi\|_{\Omega}^2 < \infty\},\\
L^2(V) &:= \{\Psi : V \longrightarrow \mathbb{C} \mid \|\Psi\|_{V}^2 < \infty\},\\
H^2(H_\Omega) &:= \{\widetilde{\Psi} : H_\Omega := V + i\Omega \longrightarrow \mathbb{C} \mid \widetilde{\Psi} \textit{ is analytic in } H_\Omega \textit{ and } \|\widetilde{\Psi}\|_{H_\Omega}^2 < \infty\},\\
H^2(V) &:= H^2(H_\Omega)|_V \cap L^2(V).
\end{aligned}$$

Here

$$\begin{aligned}
\|\psi\|_{\Omega}^2 &:= \int_{\Omega} |\psi(u)|^2\, du,\\
\|\Psi\|_{V}^2 &:= \frac{1}{(2\pi)^n}\int_V |\Psi(t)|^2\, dt,\\
\|\widetilde{\Psi}\|_{H_\Omega}^2 &:= \sup_{y\in\Omega}\frac{1}{(2\pi)^n}\int_V |\widetilde{\Psi}(x+iy)|^2\, dx.
\end{aligned}$$

The (inverse) Fourier transform of an integrable function ψ *on* Ω *is defined as*

$$(F^{-1}\psi)(t) := \int_{\Omega} e^{i(t|u)}\psi(u)\,du. \tag{46}$$

We have

$$F^{-1} : L^2(\Omega) \xrightarrow{\simeq} H^2(V) \ \ \text{(unitary)}. \tag{47}$$

In particular,

$$F^{-1} : L^2(\Omega)^K \xrightarrow{\simeq} H^2(V)^K \ \ \text{(unitary)}. \tag{48}$$

Proof. From [**8**, Theorem IX.4.1], we have

$$\widetilde{F}^{-1} : L^2(\Omega) \xrightarrow{\simeq} H^2(H_{\Omega}) \ \ \text{(unitary)},$$

where

$$\widetilde{F}^{-1}(\psi)(z) := \int_{\Omega} e^{i(z|u)}\psi(u)\,du.$$

For the function $\widetilde{\Psi} \in H^2(H_{\Omega})$, $y \in \Omega$, we write $\widetilde{\Psi}_y(x) := \widetilde{\Psi}(x+iy)$. By [**7**, Corollary IX.4.2],

$$\lim_{y\to 0, y\in\Omega} \widetilde{\Psi}_y = \widetilde{\Psi}_0, \quad \widetilde{\Psi}_0(t) := \int_{\Omega} e^{i(t|u)}\psi(u)\,du = F^{-1}(\psi)(t),$$

exists in $L^2(V)$, and the map $\widetilde{\Psi} \mapsto \widetilde{\Psi}_0$ is an isometric embedding of $H^2(H_{\Omega})$ into $L^2(V)$. Hence, from the definition of $H^2(V)$, we obtain (47).

Finally, since the inverse Fourier transform F^{-1} and the action of K are commutative, the above unitary isomorphism also holds for the K-invariant spaces. □

2.2. Multivariate Laguerre polynomials and their unitary picture

In this subsection, we recall the multivariate Laguerre polynomials and provide two fundamental lemmas based on [**7, 8**].

Let $\alpha > \frac{n}{r} - 1 = \frac{d}{2}(r-1)$, $\mathbf{m} \in \mathscr{P}$.

(1) $\psi_{\mathbf{m}}^{(\alpha)}$; Multivariate Laguerre polynomials (up to exponential factor)

$$L^2_{\alpha}(\Omega)^K := \{\psi : \Omega \longrightarrow \mathbb{C} \mid \psi \text{ is } K\text{-invariant and } \|\psi\|^2_{\alpha,\Omega} < \infty\},$$

$$\|\psi\|^2_{\alpha,\Omega} := \frac{2^{r\alpha}}{\Gamma_{\Omega}(\alpha)} \int_{\Omega} |\psi(u)|^2 \Delta(u)^{\alpha-\frac{n}{r}}\,du,$$

$$\psi_{\mathbf{m}}^{(\alpha)}(u) := e^{-\operatorname{tr} u} L_{\mathbf{m}}^{(\alpha-\frac{n}{r})}(2u)$$

; multivariate Laguerre polynomials (up to exponential factor). (49)

Here $L_{\mathbf{m}}^{(\alpha-\frac{n}{r})}(u)$ is the multivariate Laguerre polynomial defined by

$$\begin{aligned} L_{\mathbf{m}}^{(\alpha-\frac{n}{r})}(u) &:= d_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \sum_{\mathbf{k}\subset\mathbf{m}} (-1)^{|\mathbf{k}|} \binom{\mathbf{m}}{\mathbf{k}} \frac{1}{(\alpha)_{\mathbf{k}}} \Phi_{\mathbf{k}}(u) \\ &= d_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \sum_{\mathbf{k}\subset\mathbf{m}} (-1)^{|\mathbf{k}|} d_{\mathbf{k}} \frac{\gamma_{\mathbf{k}}(\mathbf{m}-\rho)}{\left(\frac{n}{r}\right)_{\mathbf{k}} (\alpha)_{\mathbf{k}}} \Phi_{\mathbf{k}}(u). \end{aligned}$$

We remark that $\{\psi_{\mathbf{m}}^{(\alpha)}\}_{\mathbf{m}\in\mathscr{P}}$ form complete orthogonal basis of $L_{\alpha}^2(\Omega)^K$, and

$$\|\psi_{\mathbf{m}}^{(\alpha)}\|_{\alpha,\Omega}^2 = d_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}.$$

The multivariate Laguerre polynomials have the following generating function.

LEMMA 2.15 ([**4**, Proposition 2.8], [**15**, Lemma 2.13]). *For any $\alpha \in \mathbb{C}$, $u \in \Omega$ and $z \in \mathcal{D}$, we have*

$$\sum_{\mathbf{m}\in\mathscr{P}} L_{\mathbf{m}}^{(\alpha-\frac{n}{r})}(u)\Phi_{\mathbf{m}}(z) = \Delta(e-z)^{-\alpha} \int_K e^{-(ku|z(e-z)^{-1})}\, dk. \tag{50}$$

The multivariate Laguerre polynomials also satisfy the following differential equation.

LEMMA 2.16 ([**5**, Theorem 4.3]). *Let us consider the operator $D_{\alpha}^{(1)}$:*

$$D_{\alpha}^{(1)} = \mathrm{tr}\,(-u\nabla_u^2 - \alpha\nabla_u + u - \alpha e). \tag{51}$$

We have

$$D_{\alpha}^{(1)} \psi_{\mathbf{m}}^{(\alpha)}(u) = 2|\mathbf{m}|\psi_{\mathbf{m}}^{(\alpha)}(u). \tag{52}$$

3. Multivariate circular Jacobi polynomials

This section is the main part of the article. In § 3.1, we introduce new multivariate orthogonal polynomials $\phi_{\mathbf{m}}^{(d)}(\sigma;\alpha,\nu) = \phi_{\mathbf{m}}^{(\alpha,\nu)}(\sigma)$, which are a two-parameter deformation of the spherical polynomials. They are also regarded as a multivariate analogue of the circular Jacobi polynomials and we call this polynomial the *multivariate circular Jacobi (MCJ) polynomials*. In the case of one variable, our MCJ degenerate to a one-parameter deformation of the usual circular Jacobi polynomials $\phi_m^{(\alpha)}(e^{i\theta})$. We remark the weight function of its orthogonality relation coincides with the circular Jacobi ensemble defined by Bourgade et al. [**3**].

We derive a generating function of $\phi_{\mathbf{m}}^{(\alpha,\nu)}$ in § 3.2 and a pseudo-differential equation for $\Psi_{\mathbf{m}}^{(\alpha,\nu)}$ in § 3.3. In case of the multiplicity $d = 2$, we give a determinant formula for the MCJ polynomials in § 3.4. We study the one variable case in more detail in § 3.5 as well. Finally, we describe future work for $\phi_{\mathbf{m}}^{(\alpha,\nu)}$.

In this section, we assume $\alpha > \frac{n}{r} - 1 = \frac{d}{2}(r-1)$ and $\nu \in \mathbb{R}$ unless otherwise specified.

3.1. Definitions and orthogonality

Following § 2, we introduce these function spaces and their complete orthogonal bases.

(2)

$$
\begin{aligned}
H^2_{\alpha,\nu}(V)^K &:= \{\Psi : V \longrightarrow \mathbb{C} \mid \Psi \in H^2(V) \text{ is } K\text{-invariant and } \|\Psi\|^2_{\alpha,\nu,V} < \infty\},\\
\|\Psi\|^2_{\alpha,\nu,V} &:= \frac{2^{r\alpha}}{\Gamma_\Omega(\alpha)} \left|\Gamma_\Omega\left(\frac{1}{2}\left(\alpha+\frac{n}{r}\right)+i\nu\right)\right|^2 \|\Psi\|^2_V\\
&= \frac{\widetilde{c_0}}{(2\pi)^n}\frac{2^{r\alpha}}{\Gamma_\Omega(\alpha)} \left|\Gamma_\Omega\left(\frac{1}{2}\left(\alpha+\frac{n}{r}\right)+i\nu\right)\right|^2\\
&\quad \cdot \int_{\mathbb{R}^r} |\Psi(\lambda)|^2 \prod_{1\le p<q\le r} |\lambda_p-\lambda_q|^d \, d\lambda_1\cdots d\lambda_r,\\
\Psi^{(\alpha,\nu)}_{\mathbf{m}}(t) &:= \Delta(e-it)^{-\frac{1}{2}\left(\alpha+\frac{n}{r}\right)-i\nu}\,\widetilde{\Psi^{(\alpha,\nu)}_{\mathbf{m}}}(t),\\
\widetilde{\Psi^{(\alpha,\nu)}_{\mathbf{m}}}(t) &:= d_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \sum_{\mathbf{k}\subset\mathbf{m}} (-1)^{|\mathbf{k}|} \binom{\mathbf{m}}{\mathbf{k}} \frac{\left(\frac{1}{2}\left(\alpha+\frac{n}{r}\right)+i\nu\right)_{\mathbf{k}}}{(\alpha)_{\mathbf{k}}} \Phi_{\mathbf{k}}(2(e-it)^{-1})\\
&\qquad ;\text{Modified Fourier transform of } \psi^{(\alpha)}_{\mathbf{m}}.
\end{aligned}
$$

Here $\lambda = \sum_{j=1}^r \lambda_j c_j$ and we choose the branch of $\Delta(e-it)^{-\frac{1}{2}\left(\alpha+\frac{n}{r}\right)-i\nu}$ that takes the value 1 at $t=0$.

(3)

$$
\begin{aligned}
H^2_{\alpha,\nu}(\Sigma)^K &:= \{\phi : \Sigma \longrightarrow \mathbb{C} \mid \phi \text{ is } K\text{-invariant and continued analytically to } \mathcal{D}\\
&\qquad\qquad \text{as a holomorphic function which satisfies}\\
&\qquad\qquad \|\phi\|^2_{\alpha,\nu,\Sigma} < \infty\},\\
\|\phi\|^2_{\alpha,\nu,\Sigma} &:= \frac{1}{(2\pi)^n}\frac{1}{\Gamma_\Omega(\alpha)} \left|\Gamma_\Omega\left(\frac{1}{2}\left(\alpha+\frac{n}{r}\right)+i\nu\right)\right|^2\\
&\quad \cdot \int_\Sigma |\phi(\sigma)|^2 |\Delta(e-\sigma)^{\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu}|^2 \, d\mu(\sigma)\\
&= \frac{\widetilde{c_0}}{(2\pi)^n}\frac{1}{\Gamma_\Omega(\alpha)} \left|\Gamma_\Omega\left(\frac{1}{2}\left(\alpha+\frac{n}{r}\right)+i\nu\right)\right|^2\\
&\quad \cdot \int_{\mathbb{T}^r} |\phi(e^{i\theta})|^2 \prod_{j=1}^r |(1-e^{i\theta_j})^{\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu}|^2 \prod_{1\le p<q\le r} |e^{i\theta_p}-e^{i\theta_q}|^d \, d\theta_1\cdots d\theta_r.
\end{aligned}
$$

We define the multivariate circular Jacobi polynomial by

$$\phi^{(d)}_{\mathbf{m}}(\sigma;\alpha,\nu)=\phi^{(\alpha,\nu)}_{\mathbf{m}}(\sigma):=d_{\mathbf{m}}\frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}\sum_{\mathbf{k}\subset\mathbf{m}}(-1)^{|\mathbf{k}|}\binom{\mathbf{m}}{\mathbf{k}}\frac{\left(\frac{1}{2}\left(\alpha+\frac{n}{r}\right)+i\nu\right)_{\mathbf{k}}}{(\alpha)_{\mathbf{k}}}\Phi_{\mathbf{k}}(e-\sigma). \tag{53}$$

The main purpose of this subsection is to show that these polynomials form a complete orthogonal basis of $H^2_{\alpha,\nu}(\Sigma)^K$ and to write their orthogonal relations explicitly. To achieve that purpose, we introduce a modified Fourier transform $\mathcal{F}^{-1}_{\alpha}$ for a function ψ on Ω and the second inverse modified Cayley transform $\mathcal{C}^{-1}_{\alpha,\nu}$ as follows:

$$(\mathcal{F}^{-1}_{\alpha}\psi)(t):=\frac{1}{\Gamma_\Omega\left(\frac{1}{2}\left(\alpha+\frac{n}{r}\right)\right)}(F^{-1}(\Delta(u)^{\frac{1}{2}(\alpha-\frac{n}{r})}\psi))(t) \tag{54}$$

$$=\frac{1}{\Gamma_\Omega\left(\frac{1}{2}\left(\alpha+\frac{n}{r}\right)\right)}\int_\Omega e^{i(t|u)}\psi(u)\Delta(u)^{\frac{1}{2}(\alpha-\frac{n}{r})}\,du, \tag{55}$$

$$(\mathcal{C}^{-1}_{\alpha,\nu}\Psi)(\sigma):=\Delta(e-ic(\sigma))^{\frac{1}{2}(\alpha+\frac{n}{r})+i\nu}\Psi(c(\sigma))=\Delta\left(\frac{e-\sigma}{2}\right)^{-\frac{1}{2}(\alpha+\frac{n}{r})-i\nu}\Psi(c(\sigma)). \tag{56}$$

These formulas give the following unitary isomorphisms which is regarded as a multivariate analogue of Proposition 1.1.

Theorem 3.1. (1)

$$\begin{array}{ccccc}\mathcal{F}^{-1}_{\alpha,\nu}:=\mathcal{F}^{-1}_{\alpha+2i\nu}: & L^2_\alpha(\Omega)^K & \xrightarrow{\simeq} & H^2_{\alpha,\nu}(V)^K & \text{(unitary)}.\\ & \Cup & & \Cup & \\ & \psi^{(\alpha)}_{\mathbf{m}} & \longmapsto & \Psi^{(\alpha,\nu)}_{\mathbf{m}} & \end{array}$$

In particular, $\{\Psi^{(\alpha,\nu)}_{\mathbf{m}}\}_{\mathbf{m}\in\mathscr{P}}$ *form a complete orthogonal basis of* $H^2_{\alpha,\nu}(V)^K$ *and for all* $\mathbf{m},\mathbf{n}\in\mathscr{P}$,

$$\frac{1}{(2\pi)^n}\int_V\Psi^{(\alpha,\nu)}_{\mathbf{m}}(t)\overline{\Psi^{(\alpha,\nu)}_{\mathbf{n}}(t)}\,dt=d_{\mathbf{m}}\frac{\Gamma_\Omega(\alpha+\mathbf{m})}{\left(\frac{n}{r}\right)_{\mathbf{m}}}\frac{1}{\left|\Gamma_\Omega\left(\frac{1}{2}\left(\alpha+\frac{n}{r}\right)+i\nu\right)\right|^2}\delta_{\mathbf{mn}}. \tag{57}$$

(2)

$$\begin{array}{ccccc}\mathcal{C}^{-1}_{\alpha,\nu}: & H^2_{\alpha,\nu}(V)^K & \xrightarrow{\simeq} & H^2_{\alpha,\nu}(\Sigma)^K & \text{(unitary)}.\\ & \Cup & & \Cup & \\ & \Psi^{(\alpha,\nu)}_{\mathbf{m}} & \longmapsto & \phi^{(\alpha,\nu)}_{\mathbf{m}} & \end{array}$$

The MCJ polynomials form a complete orthogonal basis of $H^2_{\alpha,\nu}(\Sigma)^K$, *and for all*

$\mathbf{m}, \mathbf{n} \in \mathscr{P}$,

$$\begin{aligned}
&\frac{1}{(2\pi)^n} \int_{\Sigma} \phi_{\mathbf{m}}^{(\alpha,\nu)}(\sigma) \overline{\phi_{\mathbf{n}}^{(\alpha,\nu)}(\sigma)} |\Delta(e-\sigma)^{\frac{1}{2}(\alpha-\frac{n}{r})+i\nu}|^2 \, d\mu(\sigma) \\
&= \frac{\widetilde{c_0}}{(2\pi)^n} \int_{\mathbb{T}^r} \phi_{\mathbf{m}}^{(\alpha,\nu)}(e^{i\theta}) \overline{\phi_{\mathbf{n}}^{(\alpha,\nu)}(e^{i\theta})} \\
&\quad \cdot \prod_{j=1}^{r} |(1-e^{i\theta_j})^{\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu}|^2 \prod_{1\le k<l\le r} |e^{i\theta_k} - e^{i\theta_l}|^d \, d\theta_1 \cdots d\theta_r \\
&= d_{\mathbf{m}} \frac{\Gamma_\Omega(\alpha+\mathbf{m})}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \frac{1}{\left|\Gamma_\Omega\left(\frac{1}{2}\left(\alpha+\frac{n}{r}\right)+i\nu\right)\right|^2} \delta_{\mathbf{mn}}. \qquad (58)
\end{aligned}$$

Proof. (1) Observe that

$$\begin{array}{ccc}
L^2_\alpha(\Omega)^K & \xrightarrow{\simeq} & L^2(\Omega)^K \qquad \text{(unitary)}, \\
\cup & & \cup \\
\psi & \longmapsto & \left(\frac{2^{r\alpha}}{\Gamma_\Omega(\alpha)}\right)^{\frac{1}{2}} \Delta(u)^{\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu}\psi
\end{array}$$

$$\begin{array}{ccc}
H^2_{\alpha,\nu}(V)^K & \xrightarrow{\simeq} & H^2(V)^K \qquad \text{(unitary)}. \\
\cup & & \cup \\
\Psi & \longmapsto & \frac{2^{\frac{r\alpha}{2}}}{\Gamma_\Omega(\alpha)^{\frac{1}{2}}} \Gamma_\Omega\left(\frac{1}{2}\left(\alpha+\frac{n}{r}\right)+i\nu\right)\Psi
\end{array}$$

Using Lemma 2.14 immediately gives the following unitary isomorphism $\mathcal{F}^{-1}_{\alpha,\nu}$.

$$\begin{array}{ccccccc}
L^2_\alpha(\Omega)^K & \xrightarrow{\simeq} & L^2(\Omega)^K & \xrightarrow{\simeq} & H^2(V)^K & \xrightarrow{\simeq} & H^2_{\alpha,\nu}(V)^K. \\
\cup & & \cup & & \cup & & \cup \\
\psi & \mapsto & \frac{2^{\frac{r\alpha}{2}}}{\Gamma_\Omega(\alpha)^{\frac{1}{2}}}\Delta(u)^{\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu}\psi & \mapsto & F^{-1}\left(\frac{2^{\frac{r\alpha}{2}}}{\Gamma_\Omega(\alpha)^{\frac{1}{2}}}\Delta(u)^{\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu}\psi\right) & \mapsto & \mathcal{F}^{-1}_{\alpha,\nu}(\psi)
\end{array}$$

Next, we evaluate the modified Fourier transform of $\psi_{\mathbf{m}}^{(\alpha)}$ which forms the complete orthogonal basis for $L^2_\alpha(\Omega)^K$. From Lemma 2.6, we obtain

$$\mathcal{F}^{-1}_{\alpha,\nu}(e^{-\operatorname{tr} u}\Phi_{\mathbf{k}})(t) = \left(\frac{1}{2}\left(\alpha+\frac{n}{r}\right)+i\nu\right)_{\mathbf{k}} \Delta(e-it)^{-\frac{1}{2}\left(\alpha+\frac{n}{r}\right)-i\nu}\Phi_{\mathbf{k}}((e-it)^{-1}).$$

Hence,

$$\begin{aligned}
\mathcal{F}^{-1}_{\alpha,\nu}(\psi_{\mathbf{m}}^{(\alpha)})(t) &= \frac{1}{\Gamma\left(\frac{1}{2}\left(\alpha+\frac{n}{r}\right)+i\nu\right)} F^{-1}(\Delta(u)^{\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu}\psi_{\mathbf{m}}^{(\alpha)})(t) \\
&= d_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \sum_{\mathbf{k}\subset\mathbf{m}} (-2)^{|\mathbf{k}|} \binom{\mathbf{m}}{\mathbf{k}} \frac{1}{(\alpha)_{\mathbf{k}}}
\end{aligned}$$

$$\cdot \frac{1}{\Gamma\left(\frac{1}{2}\left(\alpha+\frac{n}{r}\right)+i\nu\right)} \int_\Omega e^{i(t|u)} \Delta(u)^{\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu} e^{-\operatorname{tr} u} \Phi_{\mathbf{k}}(u)\, du$$
$$= \Psi_{\mathbf{m}}^{(\alpha,\nu)}(t).$$

The formula (57) follows from

$$\begin{aligned} d_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \delta_{\mathbf{mn}} &= (\psi_{\mathbf{m}}^{(\alpha)}, \psi_{\mathbf{n}}^{(\alpha)})_{\alpha,\Omega} \\ &= (\mathcal{F}_{\alpha,\nu}^{-1}(\psi_{\mathbf{m}}^{(\alpha)}), \mathcal{F}_{\alpha,\nu}^{-1}(\psi_{\mathbf{n}}^{(\alpha)}))_{\alpha,\nu,V} \\ &= \frac{2^{r\alpha}}{\Gamma_\Omega(\alpha)} \frac{\left|\Gamma_\Omega\left(\frac{1}{2}\left(\alpha+\frac{n}{r}\right)\right)+i\nu\right|^2}{(2\pi)^n} \int_V \Psi_{\mathbf{m}}^{(\alpha,\nu)}(t) \overline{\Psi_{\mathbf{n}}^{(\alpha,\nu)}(t)}\, dt. \end{aligned}$$

(2) The inverse Cayley transform c^{-1} is a holomorphic bijection of H_Ω onto $\mathcal{D}$ and the inverse map of $\mathcal{C}_{\alpha,\nu}^{-1}$ is given by

$$(\mathcal{C}_{\alpha,\nu}\phi)(t) := \Delta\left(\frac{e-c^{-1}(t)}{2}\right)^{\frac{1}{2}\left(\alpha+\frac{n}{r}\right)+i\nu} \phi(c^{-1}(t)) = \Delta(e-it)^{-\frac{1}{2}\left(\alpha+\frac{n}{r}\right)-i\nu} \phi(c^{-1}(t)). \tag{59}$$

Since $\{\Psi_{\mathbf{m}}^{(\alpha,\nu)}\}_{\mathbf{m}\in\mathscr{P}}$ form a complete orthogonal basis of $H^2_{\alpha,\nu}(V)^K$, it is sufficient to show the statement for $\{\Psi_{\mathbf{m}}^{(\alpha,\nu)}\}_{\mathbf{m}\in\mathscr{P}}$ and $\{\phi_{\mathbf{m}}^{(\alpha,\nu)}\}_{\mathbf{m}\in\mathscr{P}}$.

By the proof of [**7**, Theorem IX.4.1],

$$\begin{aligned} \|\Psi_{\mathbf{m}}^{(\alpha,\nu)}\|^2_{H_\Omega} &= \sup_{y\in\Omega} \frac{1}{(2\pi)^n} \int_V |\Psi_{\mathbf{m}}^{(\alpha,\nu)}(x+iy)|^2\, dx \\ &= \frac{\Gamma_\Omega(\alpha)}{2^{r\alpha}\left|\Gamma_\Omega\left(\frac{1}{2}\left(\alpha+\frac{n}{r}\right)\right)+i\nu\right|^2} \|\psi_{\mathbf{m}}^{(\alpha)}\|^2_\Omega < \infty. \end{aligned}$$

By the definitions of $\mathcal{C}_{\alpha,\nu}^{-1}$ and $\phi_{\mathbf{m}}^{(\alpha,\nu)}$, we have

$$\begin{aligned} (\mathcal{C}_{\alpha,\nu}^{-1}\Psi_{\mathbf{m}}^{(\alpha,\nu)})(\sigma) &= \Delta\left(\frac{e-\sigma}{2}\right)^{-\frac{1}{2}\left(\alpha+\frac{n}{r}\right)-i\nu} \Delta(e-ic(\sigma))^{-\frac{1}{2}\left(\alpha+\frac{n}{r}\right)-i\nu} \widetilde{\Psi_{\mathbf{m}}^{(\alpha,\nu)}}(c(\sigma)) \\ &= \phi_{\mathbf{m}}^{(\alpha,\nu)}(\sigma), \end{aligned}$$

and from (11) in Lemma 2.3, we have

$$\begin{aligned} \int_V |\Psi_{\mathbf{m}}^{(\alpha,\nu)}(t)|^2\, dt &= 2^n \int_\Sigma |\Delta(e-\sigma)^{-\frac{n}{r}}|^2 |\Psi_{\mathbf{m}}^{(\alpha,\nu)}(c(\sigma))|^2\, d\mu(\sigma) \\ &= 2^n \int_\Sigma |\Delta(e-\sigma)^{-\frac{n}{r}}|^2 \left|\Delta\left(\frac{e-\sigma}{2}\right)^{\frac{1}{2}\left(\alpha+\frac{n}{r}\right)+i\nu}\right|^2 \\ &\quad \cdot |(\mathcal{C}_{\alpha,\nu}^{-1}\Psi_{\mathbf{m}}^{(\alpha,\nu)})(\sigma)|^2\, d\mu(\sigma) \end{aligned}$$

$$= 2^{-r\alpha} \int_{\Sigma} |\phi_{\mathbf{m}}^{(\alpha,\nu)}(\sigma)|^2 |\Delta(e-\sigma)^{\frac{1}{2}(\alpha-\frac{n}{r})+i\nu}|^2 \, d\mu(\sigma).$$

Therefore we obtain

$$(\phi_{\mathbf{m}}^{(\alpha,\nu)}, \phi_{\mathbf{n}}^{(\alpha,\nu)})_{\alpha,\nu,\Sigma} = (\mathcal{C}_{\alpha,\nu}^{-1}(\Psi_{\mathbf{m}}^{(\alpha,\nu)}), \mathcal{C}_{\alpha,\nu}^{-1}(\Psi_{\mathbf{n}}^{(\alpha,\nu)}))_{\alpha,\nu,\Sigma} = d_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \delta_{\mathbf{mn}}.$$

The formula (58) follows from (12) of Lemma 2.3 immediately. □

Remark 3.2. (1) *Our multivariate orthogonal polynomials* $\phi_{\mathbf{m}}^{(\alpha,\nu)}$ *are not the BC-type multivariate Jacobi polynomials. The weight function of the left hand side for (58)*

$$\prod_{j=1}^{r} (1-e^{i\theta_j})^{\frac{1}{2}(\alpha-\frac{n}{r})+i\nu} (1-e^{-i\theta_j})^{\frac{1}{2}(\alpha-\frac{n}{r})-i\nu} \prod_{1\le p<q\le r} |e^{i\theta_p} - e^{i\theta_q}|^d$$

coincides with the circular Jacobi ensemble defined by [**3**], *which is not the BC-type (Weyl group invariant) weight function.*
(2) *When* $\alpha = \frac{n}{r}$, $\nu = 0$, *(53) and (58) degenerate to*

$$\phi_{\mathbf{m}}^{(\frac{n}{r},0)}(e^{i\theta}) = d_{\mathbf{m}} \sum_{\mathbf{k}\subset\mathbf{m}} (-1)^{|\mathbf{k}|} \binom{\mathbf{m}}{\mathbf{k}} \Phi_{\mathbf{k}}(e-e^{i\theta}) = d_{\mathbf{m}} \Phi_{\mathbf{m}}(e^{i\theta}), \tag{60}$$

$$\begin{aligned} &\frac{1}{(2\pi)^n} \int_{\Sigma} \phi_{\mathbf{m}}^{(\frac{n}{r},0)}(\sigma) \overline{\phi_{\mathbf{n}}^{(\frac{n}{r},0)}(\sigma)} \, d\mu(\sigma) \\ &= \frac{\widetilde{c_0}}{(2\pi)^n} \int_{\mathbb{T}^r} \phi_{\mathbf{m}}^{(\frac{n}{r},0)}(e^{i\theta}) \overline{\phi_{\mathbf{n}}^{(\frac{n}{r},0)}(e^{i\theta})} \prod_{1\le p<q\le r} |e^{i\theta_p} - e^{i\theta_q}|^d \, d\theta_1 \cdots d\theta_r = d_{\mathbf{m}} \frac{1}{\Gamma_\Omega\left(\frac{n}{r}\right)} \delta_{\mathbf{mn}}. \end{aligned} \tag{61}$$

Therefore, $\phi_{\mathbf{m}}^{(\alpha,\nu)}(e^{i\theta})$ *is regarded as a two-parameter deformation of the spherical polynomials.*

As a generalization of the spherical polynomials $\Phi_{\mathbf{m}}^{(d)}$, *the Jack polynomial* $P_{\mathbf{m}}^{(\frac{2}{d})}$ *are well known (see* [**11**, Chapter VI]*). This multivariate special orthogonal system is a generalization of the spherical polynomials* $\Phi_{\mathbf{m}}^{(d)}$. *It is defined by the simultaneous eigenfunctions of some commuting differential operators. On the other hand, using the unitary picture, we obtain another extension* $\phi_{m}^{(\alpha,\nu)}$. *Our* $\phi_{m}^{(\alpha,\nu)}$ *are different from the Jack polynomials. We consider deformations for real two-parameters* α *and* ν *instead of the multiplicity* d.
(3) *The Multivariate Meixner-Pollaczek polynomials* $P_{\mathbf{m}}^{(\alpha)}(\mathbf{s};\theta)$ *are introduced by Faraut-Wakayama* [**8**]*:*

$$P_{\mathbf{m}}^{(\alpha)}(\mathbf{s};\theta) := e^{i|\mathbf{m}|\theta} d_{\mathbf{m}} \frac{(2\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \sum_{\mathbf{k}\subset\mathbf{m}} \binom{\mathbf{m}}{\mathbf{k}} \frac{\gamma_{\mathbf{k}}(-i\mathbf{s}-\alpha)}{(2\alpha)_{\mathbf{k}}} (1-e^{-2i\theta})^{|\mathbf{k}|}$$

$$= e^{i|\mathbf{m}|\theta} d_{\mathbf{m}} \frac{(2\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \sum_{\mathbf{k}\subset\mathbf{m}} d_{\mathbf{k}} \frac{\gamma_{\mathbf{k}}(\mathbf{m}-\rho)\gamma_{\mathbf{k}}(-i\mathbf{s}-\alpha)}{\left(\frac{n}{r}\right)_{\mathbf{k}}(2\alpha)_{\mathbf{k}}}(1-e^{-2i\theta})^{|\mathbf{k}|}.$$

For $\alpha > \frac{n}{r}-1$, $0<\theta<2\pi$, this system has the following orthogonality relations.

$$\frac{1}{(2\pi)^r}\int_{\mathbb{R}^r} q_{\mathbf{m}}^{(\alpha,\theta)}(\mathbf{s})\overline{q_{\mathbf{n}}^{(\alpha,\theta)}(\mathbf{s})}e^{(2\theta-\pi)|\mathbf{s}|}\left|\Gamma_\Omega\left(i\mathbf{s}+\frac{\alpha}{2}+\rho\right)\right|^2 \frac{m(d\mathbf{s})}{|c(i\mathbf{s})|^2}$$
$$= d_{\mathbf{m}} \frac{\Gamma_\Omega(\alpha+\mathbf{m})}{\left(\frac{n}{r}\right)_{\mathbf{m}}(2\sin\theta)^{r\alpha}}\delta_{\mathbf{mn}}.$$

Here $q_{\mathbf{m}}^{(\alpha,\theta)}(\mathbf{s}) = e^{-i|\mathbf{m}|\theta}P_{\mathbf{m}}^{(\frac{\alpha}{2})}(\mathbf{s};\theta)$ and m is the Lebesgue measure on $\mathbb{R}^r$. The relations with the MCJ polynomials is the following. From (35), for any $\theta\in\mathbb{R}$ and $\mathbf{m}\in\mathscr{P}$, we have

$$\phi_{\mathbf{m}}^{(\alpha,\nu)}(e^{i\theta}e) = q_{\mathbf{m}}^{(\alpha,-\frac{\theta}{2})}\left(\nu - i\left(\frac{n}{2r}+\rho\right)\right). \tag{62}$$

3.2. Generating function

We present the generating functions of MCJ polynomials by using these unitary isomorphisms.

Theorem 3.3. *We assume $z = u\sum_{j=1}^r a_j c_j \in \mathcal{D}$ with $u\in U$, $1>a_1\geq\ldots\geq a_r\geq 0$ and $a_1<\frac{1}{3}$.*
(1) *For all $t\in V$,*

$$\sum_{\mathbf{m}\in\mathscr{P}} \Psi_{\mathbf{m}}^{(\alpha,\nu)}(t)\Phi_{\mathbf{m}}(z) = \Delta(e-z)^{-\alpha}\int_K \Delta((e+z)(e-z)^{-1}-ikt)^{-\frac{1}{2}\left(\alpha+\frac{n}{r}\right)-i\nu}\,dk. \tag{63}$$

(2) *For any $\sigma\in\Sigma$,*

$$\sum_{\mathbf{m}\in\mathscr{P}} \phi_{\mathbf{m}}^{(\alpha,\nu)}(\sigma)\Phi_{\mathbf{m}}(z) = \Delta(e-z)^{-\alpha}\Delta(e-\sigma)^{-\frac{1}{2}\left(\alpha+\frac{n}{r}\right)-i\nu}$$
$$\cdot\int_K \Delta(z(e-z)^{-1}+k(e-\sigma)^{-1})^{-\frac{1}{2}\left(\alpha+\frac{n}{r}\right)-i\nu}\,dk. \tag{64}$$

Proof. (1) By Lemmas 2.4, 2.8, 2.11 and 2.12,

$$\sum_{\mathbf{m}\in\mathscr{P}} |L_{\mathbf{m}}^{(\alpha-\frac{n}{r})}(u)\Phi_{\mathbf{m}}(z)| \leq \sum_{\mathbf{m}\in\mathscr{P}}\sum_{\mathbf{k}\subset\mathbf{m}}\left| d_{\mathbf{m}}\frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}\binom{\mathbf{m}}{\mathbf{k}}\frac{(-1)^{|\mathbf{k}|}}{(\alpha)_{\mathbf{k}}}\Phi_{\mathbf{k}}(u)\right|\Phi_{\mathbf{m}}(a_1)$$
$$\leq \sum_{\mathbf{k}\in\mathscr{P}} d_{\mathbf{k}}\frac{1}{\left(\frac{n}{r}\right)_{\mathbf{k}}}\frac{1}{(|\alpha|+d(r-1))_{\mathbf{k}}}\Phi_{\mathbf{k}}(u)$$
$$\sum_{\mathbf{m}\in\mathscr{P}} d_{\mathbf{m}}\frac{(|\alpha|+d(r-1))_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}\gamma_{\mathbf{k}}(\mathbf{m}-\rho)\Phi_{\mathbf{m}}(a_1)$$

$$= (1-a_1)^{-r|\alpha|-dr(r-1)} \sum_{\mathbf{k}\in\mathscr{P}} d_{\mathbf{k}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{k}}} \Phi_{\mathbf{k}}\left(\frac{a_1}{1-a_1}u\right)$$
$$= (1-a_1)^{-r|\alpha|-dr(r-1)} e^{\frac{a_1}{1-a_1}\operatorname{tr} u} < \infty.$$

Hence, we have

$$\mathcal{F}^{-1}_{\alpha,\nu}\left(\sum_{\mathbf{m}\in\mathscr{P}} |\psi^{(\alpha)}_{\mathbf{m}}(u)\Phi_{\mathbf{m}}(z)|\right)(t) \tag{65}$$
$$\le (1-a_1)^{-r|\alpha|-dr(r-1)} \mathcal{F}^{-1}_{\alpha,\nu}(e^{-\frac{1-3a_1}{1-a_1}\operatorname{tr} u})(t)$$
$$= (1-a_1)^{-r|\alpha|-dr(r-1)} \Delta\left(\frac{1-3a_1}{1-a_1}e - it\right)^{-\frac{1}{2}\left(\alpha+\frac{n}{r}\right)-i\nu} < \infty.$$

The exchange of integration and summation is justified and gives

$$\sum_{\mathbf{m}\in\mathscr{P}} \Psi^{(\alpha,\nu)}_{\mathbf{m}}(t)\Phi_{\mathbf{m}}(z) = \sum_{\mathbf{m}\in\mathscr{P}} \mathcal{F}^{-1}_{\alpha,\nu}(\psi^{(\alpha)}_{\mathbf{m}})(t)\Phi_{\mathbf{m}}(z)$$
$$= \mathcal{F}^{-1}_{\alpha,\nu}\left(\sum_{\mathbf{m}\in\mathscr{P}} \psi^{(\alpha)}_{\mathbf{m}}(u)\Phi_{\mathbf{m}}(z)\right)(t)$$
$$= \Delta(e-z)^{-\alpha}\mathcal{F}^{-1}_{\alpha,\nu}\left(\int_K e^{-(ku|(e+z)(e-z)^{-1})}\,dk\right)(t).$$

Moreover, by Lemma 2.6,

$$\mathcal{F}^{-1}_{\alpha,\nu}\left(\int_K e^{-(ku|(e+z)(e-z)^{-1})}\,dk\right)(t)$$
$$= \frac{1}{\Gamma_\Omega\left(\frac{1}{2}\left(\alpha+\frac{n}{r}\right)+i\nu\right)} \int_\Omega e^{i(t|u)}\Delta(u)^{\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu} \int_K e^{-(ku|(e+z)(e-z)^{-1})}\,dkdu$$
$$= \frac{1}{\Gamma_\Omega\left(\frac{1}{2}\left(\alpha+\frac{n}{r}\right)+i\nu\right)} \int_K\int_\Omega e^{-(u|k(e+z)(e-z)^{-1}-it)}\Delta(u)^{\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu}\,dudk$$
$$= \int_K \Delta(k(e+z)(e-z)^{-1}-it)^{-\frac{1}{2}\left(\alpha+\frac{n}{r}\right)-i\nu}\,dk$$
$$= \int_K \Delta((e+z)(e-z)^{-1}-ikt)^{-\frac{1}{2}\left(\alpha+\frac{n}{r}\right)-i\nu}\,dk.$$

(2) Applying the modified Cayley transform $\mathcal{C}^{-1}_{\alpha,\nu}$ to (63), we obtain

$$\sum_{\mathbf{m}\in\mathscr{P}} \phi^{(\alpha,\nu)}_{\mathbf{m}}(\sigma)\Phi_{\mathbf{m}}(z)$$
$$= \Delta(e-z)^{-\alpha}\mathcal{C}^{-1}_{\alpha,\nu}\left(\int_K \Delta((e+z)(e-z)^{-1}-ikt)^{-\frac{1}{2}\left(\alpha+\frac{n}{r}\right)-i\nu}\,dk\right)(\sigma)$$

$$= \Delta(e-z)^{-\alpha}\Delta\left(\frac{e-\sigma}{2}\right)^{-\frac{1}{2}\left(\alpha+\frac{n}{r}\right)-i\nu}$$
$$\cdot\int_K \Delta((e+z)(e-z)^{-1}-ikc(\sigma))^{-\frac{1}{2}\left(\alpha+\frac{n}{r}\right)-i\nu}\,dk.$$

Since

$$(e+z)(e-z)^{-1}-ikc(\sigma)=2z(e-z)^{-1}+2k(e-\sigma)^{-1},$$

we have the second conclusion. □

3.3. Differential equation for $\Psi_{\mathbf{m}}^{(\alpha,\nu)}$

Considering the pseudo-differential operator

$$\mathrm{tr}\,(\nabla_t^{-1})e^{(t|u)}=\mathrm{tr}\,(u^{-1})e^{(t|u)}\quad(\text{any } t\in V, u\in\Omega), \tag{66}$$

we obtain explicit (pseudo-) differential equations for $\Psi_{\mathbf{m}}^{(\alpha,\nu)}$.

THEOREM 3.4. *We define the operator $D_{\alpha,\nu}^{(2)}$ on V by*

$$D_{\alpha,\nu}^{(2)}=\mathrm{tr}\left(-i(e+t^2)\nabla_t+\left(2\nu-i\frac{n}{r}\right)t-\alpha e+i\left(\frac{1}{4}\left(\alpha-\frac{n}{r}\right)^2+\nu^2\right)\nabla_t^{-1}\right). \tag{67}$$

The modified Cayley transform of the MCJ polynomial, $\Psi_{\mathbf{m}}^{(\alpha,\nu)}(t)$, satisfies the following differential equation:

$$D_{\alpha,\nu}^{(2)}\Psi_{\mathbf{m}}^{(\alpha,\nu)}(t)=2|\mathbf{m}|\Psi_{\mathbf{m}}^{(\alpha,\nu)}(t). \tag{68}$$

PROOF. By the definition of the modified Fourier transform $\mathcal{F}_{\alpha,\nu}^{-1}$ and the inner product of $L_\alpha^2(\Omega)$, we write

$$(\mathcal{F}_{\alpha,\nu}^{-1}\psi)(t)=(e^{i(t|u)}\Delta(u)^{-\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu}|\overline{\psi})_{L_\alpha^2(\Omega)}. \tag{69}$$

From [**8**, Lemma 3.13], $D_\alpha^{(1)}=\overline{D_\alpha^{(1)}}$ is a self-adjoint operator with respect to the measure $\Delta(u)^{\alpha-\frac{n}{r}}\,du$. Based on Lemma 2.13, let us perform

$$\begin{aligned}
&\mathrm{tr}\,(u\nabla_u^2)(e^{i(t|u)}\Delta(u)^{-\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu})\\
&\quad=\mathrm{tr}\,(u(\nabla_u^2e^{i(t|u)}))\Delta(u)^{-\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu}+2\,\mathrm{tr}\,(u(\nabla_ue^{i(t|u)})(\nabla_u\Delta(u)^{-\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu}))\\
&\qquad+e^{i(t|u)}\,\mathrm{tr}\,(u\nabla_u^2\Delta(u)^{-\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu})\\
&\quad=e^{i(t|u)}\Delta(u)^{-\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu}\left\{\mathrm{tr}\,(-ut^2)-i\left(\left(\alpha-\frac{n}{r}\right)-2i\nu\right)\mathrm{tr}\,(t)\right.
\end{aligned}$$

$$+\left(\frac{1}{2}\left(\alpha-\frac{n}{r}\right)-i\nu\right)\left(\frac{1}{2}\left(\alpha+\frac{n}{r}\right)-i\nu\right)\operatorname{tr}(u^{-1})\Big\}$$

and

$$\begin{aligned}
&\operatorname{tr}(\alpha\nabla_u)(e^{i(t|u)}\Delta(u)^{-\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu})\\
&\quad=\alpha\operatorname{tr}(\nabla_u e^{i(t|u)})\Delta(u)^{-\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu}+\alpha e^{i(t|u)}\operatorname{tr}(\nabla_u\Delta(u)^{-\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu})\\
&\quad=\operatorname{tr}\left(i\alpha t-\alpha\left(\frac{1}{2}\left(\alpha-\frac{n}{r}\right)-i\nu\right)u^{-1}\right)e^{i(t|u)}\Delta(u)^{-\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu}.
\end{aligned}$$

Observe that for any $p\in\mathbb{Z}_{\geq 0}$ $\operatorname{tr}((\nabla_u^p e^{i(t|u)}))=\operatorname{tr}((it)^p)e^{i(t|u)}$ and

$$\begin{aligned}
&\operatorname{tr}(u\nabla_u^2\Delta(u)^{-\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu})\\
&=\left(-\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu\right)\Big\{\operatorname{tr}(u(\nabla_u u^{-1}))\Delta(u)^{-\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu}+\operatorname{tr}(\nabla_u\Delta(u)^{-\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu})\Big\}\\
&=\left(\frac{1}{2}\left(\alpha-\frac{n}{r}\right)-i\nu\right)\left(\frac{1}{2}\left(\alpha+\frac{n}{r}\right)-i\nu\right)\operatorname{tr}(u^{-1})\Delta(u)^{-\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu}
\end{aligned}$$

and

$$\begin{aligned}
&\operatorname{tr}((\nabla_u e^{i(t|u)})(u\nabla_u\Delta(u)^{-\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu}))\\
&\quad=\left(-\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu\right)\operatorname{tr}(it)e^{i(t|u)}\Delta(u)^{-\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu}.
\end{aligned}$$

Hence

$$\begin{aligned}
&D_\alpha^{(1)}(e^{i(t|u)}\Delta(u)^{-\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu})\\
&=\operatorname{tr}(-u\nabla_u^2-\alpha\nabla_u+u-\alpha e)(e^{i(t|u)}\Delta(u)^{-\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu})\\
&=\operatorname{tr}\left((e+t^2)u+\left(\left(2\nu-i\frac{n}{r}\right)t-\alpha e\right)+\left(\frac{1}{4}\left(\alpha-\frac{n}{r}\right)^2+\nu^2\right)u^{-1}\right)e^{i(t|u)}\Delta(u)^{-\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu}\\
&\quad=D_{\alpha,\nu}^{(2)}e^{i(t|u)}\Delta(u)^{-\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu}.
\end{aligned}$$

Therefore

$$\begin{aligned}
(\mathcal{F}_{\alpha,\nu}^{-1}D_\alpha^{(1)}\psi_{\mathbf{m}}^{(\alpha)})(t)&=(D_{\alpha,\nu}^{(2)}e^{i(t|u)}\Delta(u)^{-\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu}|\overline{\psi_{\mathbf{m}}^{(\alpha)}})_{L_\alpha^2(\Omega)}\\
&=D_{\alpha,\nu}^{(2)}(e^{i(t|u)}\Delta(u)^{-\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu}|\overline{\psi_{\mathbf{m}}^{(\alpha)}})_{L_\alpha^2(\Omega)}\\
&=D_{\alpha,\nu}^{(2)}((\mathcal{F}_{\alpha,\nu}^{-1}\psi)(t)).
\end{aligned}$$

The second equality is justified by $e^{i(t|u)}\Delta(u)^{\frac{1}{2}\left(\alpha-\frac{n}{r}\right)+i\nu}\psi_{\mathbf{m}}^{(\alpha)}\in L^1(\Omega)$.

Finally, from Theorem 3.1 and (52), we have

$$D^{(2)}_{\alpha,\nu}\Psi^{(\alpha,\nu)}_{\mathbf{m}}(t)=D^{(2)}_{\alpha,\nu}\mathcal{F}^{-1}_{\alpha,\nu}(\psi^{(\alpha)}_{\mathbf{m}})(t)=\mathcal{F}^{-1}_{\alpha,\nu}(D^{(1)}_{\alpha}\psi^{(\alpha)}_{\mathbf{m}})(t)=2|\mathbf{m}|\Psi^{(\alpha,\nu)}_{\mathbf{m}}(t).$$

□

We can also define $D^{(3)}_{\alpha,\nu}$ by the relation $D^{(3)}_{\alpha,\nu}\mathcal{C}^{-1}_{\alpha,\nu}=\mathcal{C}^{-1}_{\alpha,\nu}D^{(2)}_{\alpha,\nu}$. But there are difficulties in deriving the modified Cayley transform of ∇_t^{-1} and we could not obtain the explicit expression for $D^{(3)}_{\alpha,\nu}$ like that in the above theorem. On the other hand, when $\alpha=\frac{n}{r},\nu=0$, $\Psi^{(\frac{n}{r},0)}_{\mathbf{m}}(t)$ becomes

$$\begin{aligned}\Psi^{(\frac{n}{r},0)}_{\mathbf{m}}(t)&=\Delta(e-it)^{-\frac{n}{r}}d_{\mathbf{m}}\sum_{\mathbf{k}\subset\mathbf{m}}(-1)^{|\mathbf{k}|}\binom{\mathbf{m}}{\mathbf{k}}\Phi_{\mathbf{k}}(2(e-it)^{-1})\\&=\Delta(e-it)^{-\frac{n}{r}}d_{\mathbf{m}}\Phi_{\mathbf{m}}(c^{-1}(t)).\end{aligned}\tag{70}$$

The term of the pseudo-differential operator vanishes for $D^{(2)}_{\frac{n}{r},0}$, so

$$D^{(2)}_{\frac{n}{r},0}=-i\,\mathrm{tr}\,((e+t^2)\partial_t)-\frac{n}{r}\,\mathrm{tr}\,(e+it).\tag{71}$$

This gives the following explicit expression for $D^{(3)}_{\frac{n}{r},0}$ from the modified Cayley transform of $D^{(2)}_{\frac{n}{r},0}$:

$$D^{(3)}_{\frac{n}{r},0}=2\,\mathrm{tr}\,(\sigma\nabla_\sigma).\tag{72}$$

3.4. Determinant formulas

Now we assume $d=2$. Then $\frac{n}{r}=r$, and the spherical polynomials $\Phi_{\mathbf{m}}$ are proportional to the Schur polynomials $s_{\mathbf{m}}$ as in (26). We obtain determinant formulas for $\Psi^{(\alpha,\nu)}_{\mathbf{m}}$ and $\phi^{(\alpha,\nu)}_{\mathbf{m}}$. To prove them, we use these two results.

LEMMA 3.5 ([**9**, Theorem 1.2.1]). *Consider r power series of single variable $z\in\mathbb{C}$*

$$f_j(z)=\sum_{m\geq 0}A^{(j)}_m z^m\quad(j=1,\cdots,r).$$

Then

$$\frac{\det\,(f_j(z_l))}{V(z_1,\ldots,z_r)}=\sum_{\mathbf{m}\in\mathscr{P}}A_{\mathbf{m}}s_{\mathbf{m}}(z_1,\ldots,z_r),\tag{73}$$

where $V(z_1,\dots,z_r)$ *denote the Vandermonde determinant*

$$V(z_1,\dots,z_r) := \prod_{1\le p<q\le r}(z_p - z_q)$$

and

$$A_{\mathbf{m}} := \det\,(A^{(q)}_{m_p+r-p}).$$

Corollary 3.6. *Let* $w_1,\dots,w_r,z_1,\dots,z_r \in \mathbb{C}\setminus\{0\}$*. For* $w=\sum_{j=1}^r w_jc_j$ *and* $z=\sum_{j=1}^r z_jc_j$*,*

$$\Delta(w)^{-\alpha}\int_K \Delta(kw^{-1}-z)^{-\alpha}dk = \delta!\prod_{j=1}^r \frac{1}{(\alpha-r+1)_{j-1}}\frac{\det\left((1-w_pz_q)^{-(\alpha-r+1)}\right)}{V(w_1,\dots,w_r)V(z_1,\dots,z_r)}. \tag{74}$$

Here

$$\delta! := (r-1)!(r-2)!\cdots 2!1!.$$

Proof. We assume $|z_j|,|w_j|<1$ for $j=1,\dots,r$ and consider the r power series

$$f_j(w) = (1-wz_j)^{-(\alpha-r+1)} = \sum_{m\ge 0}\frac{(\alpha-r+1)_m}{m!}w^m z_j^m \quad (j=1,\cdots,r).$$

From Lemma 3.5, we have

$$\begin{aligned}
&\frac{\det\left((1-w_pz_q)^{-(\alpha-r+1)}\right)}{V(w_1,\dots,w_r)V(z_1,\dots,z_r)}\\
&\quad= \sum_{\mathbf{m}\in\mathscr{P}}\det\left(\frac{(\alpha-r+1)_{m_p+r-p}}{(m_p+r-p)!}w_q^{m_p+r-p}\right)\frac{s_{\mathbf{m}}(z_1,\dots,z_r)}{V(w_1,\dots,w_r)}\\
&\quad= \sum_{\mathbf{m}\in\mathscr{P}}\left\{\prod_{j=1}^r\frac{(\alpha-r+1)_{m_j+r-j}}{m_j+r-j}\right\}s_{\mathbf{m}}(w_1,\dots,w_r)s_{\mathbf{m}}(z_1,\dots,z_r)\\
&\quad= \frac{1}{\delta!}\prod_{j=1}^r(\alpha-r+1)_{j-1}\sum_{\mathbf{m}\in\mathscr{P}}\frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}s_{\mathbf{m}}(w_1,\dots,w_r)s_{\mathbf{m}}(z_1,\dots,z_r).
\end{aligned}$$

Furthermore, by (22), (26) and (30), we prove (74) for $|z_1|,\dots,|z_r|,|w_1|,\dots,|w_r|<1$. By analytic continuation, this corollary holds for $w_1,\dots,w_r,z_1,\dots,z_r\in\mathbb{C}\setminus\{0\}$. □

Theorem 3.7. (1) *Let* $t=\sum_{j=1}^r t_jc_j\in V$. *We have*

$$\Psi_{\mathbf{m}}^{(\alpha,\nu)}(t)=\frac{s_{\mathbf{m}}(1,\dots,1)}{(-2i)^{\frac{r(r-1)}{2}}}\delta!\prod_{j=1}^r\frac{1}{\left(\frac12(\alpha-r)+i\nu+1\right)_{j-1}}\frac{\det\left(\Psi_{m_p+r-p}^{(\alpha-r+1,\nu)}(t_q)\right)}{V(t_1,\dots,t_r)}. \quad (75)$$

(2) *Let* $\sigma=\sum_{j=1}^r\sigma_jc_j\in\Sigma$. *We have*

$$\phi_{\mathbf{m}}^{(\alpha,\nu)}(\sigma)=s_{\mathbf{m}}(1,\dots,1)\delta!\prod_{j=1}^r\frac{1}{\left(\frac12(\alpha-r)+i\nu+1\right)_{j-1}}\frac{\det\left(\phi_{m_p+r-p}^{(\alpha-r+1,\nu)}(\sigma_q)\right)}{V(\sigma_1,\dots,\sigma_r)}. \quad (76)$$

Proof. (1) We start with the generating formula for $\Psi_{\mathbf{m}}^{(\alpha,\nu)}(t)$. For $z=\sum_{j=1}^r z_jc_j$, $0<z_1,\dots,z_r<1$,

$$\begin{aligned}&\sum_{\mathbf{m}\in\mathscr{P}}\Psi_{\mathbf{m}}^{(\alpha,\nu)}(t)\Phi_{\mathbf{m}}(z)\\&=\Delta(e-z)^{-\alpha}\int_K\Delta((e+z)(e-z)^{-1}-ikt)^{-\frac12(\alpha+r)-i\nu}\,dk\\&=\prod_{j=1}^r(1-z_j)^{-\alpha}\left(\frac{1+z_j}{1-z_j}\right)^{-\frac12(\alpha+r)-i\nu}\delta!\prod_{l=1}^r\frac{1}{\left(\frac12(\alpha-r)+i\nu+1\right)_{l-1}}\\&\cdot\frac{\det\left(\left(1-\frac{1-z_p}{1+z_p}it_q\right)^{-\frac12(\alpha-r)-i\nu-1}\right)}{V\left(\frac{1-z_1}{1+z_1},\dots,\frac{1-z_r}{1+z_r}\right)V(it_1,\dots,it_r)}.\end{aligned}$$

The second equality follows from (74). Noticing that

$$\frac{1-z_p}{1+z_p}-\frac{1-z_q}{1+z_q}=-2\frac{z_p-z_q}{(1+z_p)(1+z_q)},$$

we obtain

$$\begin{aligned}\sum_{\mathbf{m}\in\mathscr{P}}\Psi_{\mathbf{m}}^{(\alpha,\nu)}(t)\Phi_{\mathbf{m}}(z)&=(-2i)^{-\frac{r(r-1)}{2}}\delta!\prod_{j=1}^r\frac{1}{\left(\frac12(\alpha-r)+i\nu+1\right)_{j-1}}\\&\cdot\frac{\det\left((1-z_p)^{-(\alpha-r+1)}\left(\frac{1+z_p}{1-z_p}-it_q\right)^{-\frac12(\alpha-r+1)-\frac12-i\nu}\right)}{V(z_1,\dots,z_r)V(t_1,\dots,t_r)}.\end{aligned}$$

By (63), we have

$$f_q(z)=(1-z)^{-(\alpha-r+1)}\left(\frac{1+z}{1-z}-it_q\right)^{-\frac12(\alpha-r+1)-\frac12-i\nu}=\sum_{m\geq0}\Psi_m^{(\alpha-r+1,\nu)}(t_q)z^m.$$

Now, we expand the above determinant expression in Schur function series by using Lemma 3.5:

$$\sum_{\mathbf{m}\in\mathscr{P}} \Psi_{\mathbf{m}}^{(\alpha,\nu)}(t)\Phi_{\mathbf{m}}(z) = (-2i)^{-\frac{r(r-1)}{2}}\delta! \prod_{j=1}^{r} \frac{1}{\left(\frac{1}{2}(\alpha-r)+i\nu+1\right)_{j-1}} \cdot \sum_{\mathbf{m}\in\mathscr{P}} \frac{\det\left(\Psi_{m_p+r-p}^{(\alpha-r+1,\nu)}(t_q)\right)}{V(t_1,\dots,t_r)} s_{\mathbf{m}}(z_1,\dots,z_r).$$

Finally, by comparing of $s_{\mathbf{m}}(z_1,\dots,z_r) = s_{\mathbf{m}}(1,\dots,1)\Phi_{\mathbf{m}}(z)$ in the above equation for $\mathbf{m}\in\mathscr{P}$, we obtain (75).
(2) Applying the modified Cayley transform $\mathcal{C}_{\alpha,\nu}^{-1}$ to (75), we obtain

$$\phi_{\mathbf{m}}^{(\alpha,\nu)}(\sigma) = (-2i)^{-\frac{r(r-1)}{2}} s_{\mathbf{m}}(1,\dots,1)\delta! \prod_{j=1}^{r} \frac{1}{\left(\frac{1}{2}(\alpha-r)+i\nu+1\right)_{j-1}} \cdot \prod_{j=1}^{r}\left(\frac{1-\sigma_j}{2}\right)^{-\frac{1}{2}(\alpha+r)-i\nu} \frac{\det\left(\Psi_{m_p+r-p}^{(\alpha-r+1,\nu)}\left(i\frac{1+\sigma_q}{1-\sigma_q}\right)\right)}{V\left(i\frac{1+\sigma_1}{1-\sigma_1},\dots,i\frac{1+\sigma_r}{1-\sigma_r}\right)}.$$

Since

$$i\frac{1+\sigma_p}{1-\sigma_p} - i\frac{1+\sigma_q}{1-\sigma_q} = 2i\frac{\sigma_p-\sigma_q}{(1-\sigma_p)(1-\sigma_q)}$$

and

$$\phi_{m_p+r-p}^{(\alpha-r+1,\nu)}(\sigma_q) = \left(\frac{1-\sigma_q}{2}\right)^{-\frac{1}{2}((\alpha-r+1)+1)-i\nu} \Psi_{m_p+r-p}^{(\alpha-r+1,\nu)}\left(i\frac{1+\sigma_q}{1-\sigma_q}\right),$$

we have

$$(-2i)^{-\frac{r(r-1)}{2}} \frac{\det\left(\Psi_{m_p+r-p}^{(\alpha-r+1,\nu)}\left(i\frac{1+\sigma_q}{1-\sigma_q}\right)\right)}{V\left(i\frac{1+\sigma_1}{1-\sigma_1},\dots,i\frac{1+\sigma_r}{1-\sigma_r}\right)} = \prod_{j=1}^{r}\left(\frac{1-\sigma_j}{2}\right)^{\frac{1}{2}(\alpha+r)+i\nu} \frac{\det\left(\phi_{m_p+r-p}^{(\alpha-r+1,\nu)}(\sigma_q)\right)}{V(\sigma_1,\dots,\sigma_r)}.$$

This gives the conclusion. □

3.5. One variable case

In this subsection, we assume that $r=1$.

Then (53) becomes

$$\phi_m^{(\alpha,\nu)}(\sigma) := \frac{(\alpha)_m}{m!}\sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{\left(\frac{1}{2}(\alpha+1)+i\nu\right)_k}{(\alpha)_k}(1-\sigma)^k \tag{77}$$

$$= \frac{(\alpha)_m}{m!}\,{}_2F_1\left(\begin{matrix}-m, \frac{1}{2}(\alpha+1)+i\nu \\ \alpha\end{matrix}; 1-\sigma\right) \tag{78}$$

$$= \frac{\left(\frac{\alpha-1}{2}-i\nu\right)}{m!}\,{}_2F_1\left(\begin{matrix}-m, \frac{1}{2}(\alpha+1)+i\nu \\ -m-\frac{\alpha-3}{2}+i\nu\end{matrix}; \sigma\right). \tag{79}$$

For $\alpha > 0$, $\nu \in \mathbb{R}$, (58) degenerates to

$$\frac{1}{2\pi i}\int_{\Sigma}\phi_m^{(\alpha,\nu)}(\sigma)\overline{\phi_n^{(\alpha,\nu)}(\sigma)}|(1-\sigma)^{\frac{\alpha-1}{2}+i\nu}|^2\,d\mu(\sigma) = \frac{\Gamma(\alpha+m)}{m!}\frac{1}{\left|\Gamma(\frac{\alpha+1}{2}+i\nu)\right|^2}\delta_{mn}. \tag{80}$$

That is a one-parameter deformation of the usual circular Jacobi polynomial that coincides with $\phi_m^{(\alpha,0)}(\sigma)$. In particular, $\phi_m^{(1,0)}(\sigma) = \sigma^m$ and

$$\frac{1}{2\pi i}\int_{\Sigma}\phi_m^{(1,0)}(\sigma)\overline{\phi_n^{(1,0)}(\sigma)}\,d\mu(\sigma) = \frac{1}{2\pi}\int_0^{2\pi}e^{im\theta}e^{-in\theta}\,d\theta = \delta_{mn}.$$

We remark that the rank 1 case of (62) is

$$\phi_m^{(\alpha,\nu)}(e^{i\theta}) = q_m^{(\alpha,-\frac{\theta}{2})}\left(\nu+\frac{1}{2i}\right) = e^{m\frac{i\theta}{2}}P_m^{(\frac{\alpha}{2})}\left(\nu+\frac{1}{2i}; -\frac{\theta}{2}\right), \tag{81}$$

If θ is regarded as a parameter and ν is regarded as a variable for the circular Jacobi polynomial, then the circular Jacobi polynomials can be considered as the Meixner-Pollaczek polynomials.

Moreover, the generating function of $\phi_m^{(\alpha,\nu)}(\sigma)$ is given by

$$\sum_{m\geq 0}\phi_m^{(\alpha,\nu)}(\sigma)z^m = (1-z)^{-\frac{1}{2}(\alpha-1)+i\nu}(1-\sigma z)^{-\frac{1}{2}(\alpha+1)-i\nu}. \tag{82}$$

Although we do not know an explicit second order differential relation for $\phi_{\mathbf{m}}^{(\alpha,\nu)}(\sigma)$ in the multivariate case, we obtain the following explicit result in the one variable case from the differential equation of ${}_2F_1$.

Proposition 3.8. *If*

$$\begin{aligned} D_{\alpha,\nu} :=& \sigma(1-\sigma)\partial_\sigma^2 + \left\{\left(-m+\frac{3}{2}+i\nu\right)(1-\sigma) - \frac{\alpha}{2}(1+\sigma)\right\}\partial_\sigma \\ &+ m\left(\frac{1}{2}(\alpha+1)+i\nu\right), \end{aligned} \tag{83}$$

then

$$D_{\alpha,\nu}\phi_m^{(\alpha,\nu)}(\sigma) = 0. \tag{84}$$

4. Open problems

Interesting problems related to the MCJ polynomials remain. Firstly, we may look for a differential equation for $\phi_{\mathbf{m}}^{(\alpha,\nu)}$ similar to Proposition 3.8. Actually, a modified Cayley transform of tr ∇_u^{-1} is not known yet. Secondly, it is important to study a generalization of our MCJ polynomials for an arbitrary real value of multiplicity $d > 0$. We can consider the MCJ polynomials and their orthogonality without using analysis on symmetric cones as follows. Let $n := r + \frac{d}{2}r(r-1)$,

$$\begin{aligned} d_{\mathbf{m}} &:= \prod_{j=1}^{r} \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2}j\right)\Gamma\left(\frac{d}{2}(j-1)+1\right)} \\ &\quad \cdot \prod_{1\le p<q\le r}\left(m_p - m_q + \frac{d}{2}(q-p)\right)\frac{\Gamma\left(m_p - m_q + \frac{d}{2}(q-p+1)\right)}{\Gamma\left(m_p - m_q + \frac{d}{2}(q-p-1)+1\right)}, \quad (85) \\ \Gamma_\Omega(\mathbf{s}) &:= (2\pi)^{\frac{n-r}{2}}\prod_{j=1}^{r}\Gamma\left(s_j - \frac{d}{2}(j-1)\right), \\ (\mathbf{s})_{\mathbf{k}} &:= \prod_{j=1}^{r}\left(s_j - \frac{d}{2}(j-1)\right)_{k_j}. \end{aligned}$$

Furthermore, $P_{\mathbf{k}}^{(\frac{2}{d})}(\lambda_1,\ldots,\lambda_r)$ is an r-variable Jack polynomial and

$$\Phi_{\mathbf{k}}^{(d)}(\lambda_1,\ldots,\lambda_r) := \frac{P_{\mathbf{k}}^{(\frac{2}{d})}(\lambda_1,\ldots,\lambda_r)}{P_{\mathbf{k}}^{(\frac{2}{d})}(1,\ldots,1)}. \tag{86}$$

We introduce the generalized (Jack) binomial coefficients based on [**12**] by

$$\Phi_{\mathbf{m}}^{(d)}(1+\lambda_1,\ldots,1+\lambda_r) = \sum_{\mathbf{k}\subset\mathbf{m}}\binom{\mathbf{m}}{\mathbf{k}}_{\frac{d}{2}}\Phi_{\mathbf{k}}^{(d)}(\lambda_1,\ldots,\lambda_r).$$

Definition 4.1. *We define the generalized MCJ (GMCJ) polynomial as follows:*

$$\begin{aligned} &\phi_{\mathbf{m}}^{(d\,;\,\alpha,\nu)}(e^{i\theta})(=\phi_{\mathbf{m}}^{(d\,;\,\alpha,\nu)}(e^{i\theta_1},\ldots,e^{i\theta_r})) \\ &\quad := d_{\mathbf{m}}\frac{(\alpha)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}\sum_{\mathbf{k}\subset\mathbf{m}}(-1)^{|\mathbf{k}|}\binom{\mathbf{m}}{\mathbf{k}}_{\frac{d}{2}}\frac{\left(\frac{1}{2}\left(\alpha+\frac{n}{r}\right)+i\nu\right)_{\mathbf{k}}}{(\alpha)_{\mathbf{k}}}\Phi_{\mathbf{k}}^{(d)}(1-e^{i\theta_1},\ldots,1-e^{i\theta_r}). \end{aligned} \tag{87}$$

We present the following conjecture.

Conjecture 4.2. *If* $\alpha > \frac{n}{r} - 1$, $\nu \in \mathbb{R}$, $d > 0$, *then*

$$\frac{\widetilde{c_0}}{(2\pi)^n} \int_{\mathbb{T}^r} \phi_{\mathbf{m}}^{(d\,;\,\alpha,\nu)}(e^{i\theta}) \overline{\phi_{\mathbf{n}}^{(d\,;\,\alpha,\nu)}(e^{i\theta})}$$

$$\prod_{j=1}^{r} |(1 - e^{i\theta_j})^{\frac{1}{2}\left(\alpha - \frac{n}{r}\right) + i\nu}|^2 \prod_{1 \le p < q \le r} |e^{i\theta_p} - e^{i\theta_q}|^d \, d\theta_1 \cdots d\theta_r$$

$$= d_{\mathbf{m}} \frac{\Gamma_\Omega(\alpha + \mathbf{m})}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \frac{1}{\left|\Gamma_\Omega\left(\frac{1}{2}\left(\alpha + \frac{n}{r}\right) + i\nu\right)\right|^2} \delta_{\mathbf{mn}}. \tag{88}$$

For the following special cases, we prove the conjecture.

Proposition 4.3. (1) *If* $d = 1, 2, 4$ *or* $r = 2$, $d \in \mathbb{Z}_{>0}$ *or* $r = 3$, $d = 8$, *then this conjecture is true.*
(2) *When* $\alpha = \frac{n}{r}$ *and* $\nu = 0$, *the conjecture is true as well.*

Proof. (1) This follows immediately from Theorem 3.1 and the classification of irreducible symmetric cones.
(2) When $\alpha = \frac{n}{r}$, $\nu = 0$,

$$\phi_{\mathbf{m}}^{\left(d\,;\,\frac{n}{r},0\right)}\left(e^{i\theta}\right) = d_{\mathbf{m}} \sum_{\mathbf{k} \subset \mathbf{m}} (-1)^{|\mathbf{k}|} \binom{\mathbf{m}}{\mathbf{k}}_{\frac{d}{2}} \frac{P_{\mathbf{k}}^{\left(\frac{2}{d}\right)}(1 - e^{i\theta_1}, \ldots, 1 - e^{i\theta_r})}{P_{\mathbf{k}}^{\left(\frac{2}{d}\right)}(1, \ldots, 1)}$$

$$= d_{\mathbf{m}} \frac{P_{\mathbf{m}}^{\left(\frac{2}{d}\right)}(e^{i\theta_1}, \ldots, e^{i\theta_r})}{P_{\mathbf{m}}^{\left(\frac{2}{d}\right)}(1, \ldots, 1)}. \tag{89}$$

For the Jack polynomials, the following formulas are known (see [**12**, (6.4)] and [**11**, (10.38)], respectively).

$$P_{\mathbf{m}}^{\left(\frac{2}{d}\right)}(1, \ldots, 1) = \prod_{j=1}^{r} \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2}j\right)} \prod_{1 \le p < q \le r} \frac{\Gamma\left(m_p - m_q + \frac{d}{2}(q - p + 1)\right)}{\Gamma\left(m_p - m_q + \frac{d}{2}(q - p)\right)}, \tag{90}$$

$$\|P_{\mathbf{m}}^{\left(\frac{2}{d}\right)}\|_{r,\frac{2}{d}}^2 := \frac{1}{(2\pi)^r} \frac{1}{r!} \int_{\mathbb{T}^r} |P_{\mathbf{m}}^{\left(\frac{2}{d}\right)}(e^{i\theta_1}, \ldots, e^{i\theta_r})|^2 \prod_{1 \le p < q \le r} |e^{i\theta_p} - e^{i\theta_q}|^d \, d\theta_1 \cdots d\theta_r$$

$$= \prod_{1 \le p < q \le r} \frac{\Gamma\left(m_p - m_q + \frac{d}{2}(q - p + 1)\right) \Gamma\left(m_p - m_q + \frac{d}{2}(q - p - 1) + 1\right)}{\Gamma\left(m_p - m_q + \frac{d}{2}(q - p)\right) \Gamma\left(m_p - m_q + \frac{d}{2}(q - p) + 1\right)}. \tag{91}$$

From (85), (90) and (91), we have

$$d_{\mathbf{m}}\|P_{\mathbf{m}}^{\left(\frac{2}{d}\right)}\|_{r,\frac{2}{d}}^2 = \left\{\prod_{j=1}^{r} \frac{1}{\Gamma\left(\frac{d}{2}(j-1)+1\right)} \frac{\Gamma\left(\frac{d}{2}j\right)}{\Gamma\left(\frac{d}{2}\right)}\right\} P_{\mathbf{m}}^{\left(\frac{2}{d}\right)}(1,\ldots,1)^2. \tag{92}$$

Therefore, by (89) and the orthogonality of the Jack polynomials, we obtain

$$\begin{aligned}
&\frac{\widetilde{c_0}}{(2\pi)^n}\int_{\mathbb{T}^r} \phi_{\mathbf{m}}^{\left(d\,;\,\frac{n}{r},0\right)}\left(e^{i\theta}\right) \overline{\phi_{\mathbf{n}}^{\left(d\,;\,\frac{n}{r},0\right)}\left(e^{i\theta}\right)} \prod_{1\le p<q\le r} |e^{i\theta_p}-e^{i\theta_q}|^d \, d\theta_1\cdots d\theta_r \\
&= \frac{1}{(2\pi)^n}\frac{(2\pi)^{\frac{n-r}{2}}}{r!}\left\{\prod_{j=1}^{r}\frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2}j\right)}\right\} \frac{d_{\mathbf{m}}^2}{P_{\mathbf{m}}^{\left(\frac{2}{d}\right)}(1,\ldots,1)^2}(2\pi)^r r!\|P_{\mathbf{m}}^{\left(\frac{2}{d}\right)}\|_{r,\frac{2}{d}}^2\delta_{\mathbf{mn}} \\
&= \frac{d_{\mathbf{m}}}{(2\pi)^{\frac{n-r}{2}}}\prod_{j=1}^{r-1}\frac{1}{\Gamma\left(1+\frac{d}{2}(r-j)\right)}\delta_{\mathbf{mn}} = d_{\mathbf{m}}\frac{1}{\Gamma_\Omega\left(\frac{n}{r}\right)}\delta_{\mathbf{mn}}.
\end{aligned}$$

□

To summarize, we draw the following diagram of GMCJ polynomial.

		α,ν**-deform**	
	Spherical poly : $\Phi_{\mathbf{m}}$	$\Longrightarrow$	(1) MCJ poly : $\phi_{\mathbf{m}}^{(\alpha,\nu)}$
d**-deform**	$\Downarrow$		$\Downarrow$
	(2) Jack poly : $P_{\mathbf{m}}^{\left(\frac{2}{d}\right)}$	$\Longrightarrow$	GMCJ poly : $\phi_{\mathbf{m}}^{(d\,;\,\alpha,\nu)}$
q,t**-deform**	$\Downarrow$		$\Downarrow$
	Macdonald poly : $P_{\mathbf{m}}^{(q,t)}$	$\Longrightarrow$	unknown poly : ?

Unfortunately, our method cannot be applied to the most general cases. It may be necessary to use quantum integrable systems. That means we need to construct commuting families of differential or pseudo-differential operators whose simultaneous eigenfunctions become GMCJ polynomials. In fact, for the multivariate Laguerre polynomials, there exists a commuting family of some differential operators $D_1,\ldots,D_r$ such that

$$D_k\psi_{\mathbf{m}}^{(\alpha)}(u) = \lambda_{k,\mathbf{m}}\psi_{\mathbf{m}}^{(\alpha)}(u),$$

for all $k=1,\ldots,r$. When $d=1,2,4$ or $r=2$, $d\in\mathbb{Z}_{>0}$ or $r=3$, $d=8$, we put

$\widetilde{D_k} := (\mathcal{C}_{\alpha,\nu}^{-1} \circ \mathcal{F}_{\alpha,\nu}^{-1}) \circ D_k \circ (\mathcal{F}_{\alpha,\nu} \circ \mathcal{C}_{\alpha,\nu})$. Then these (pseudo?) differential operators $\widetilde{D_k}$ commute with each other and for all $k = 1, \dots, r$,

$$\widetilde{D_k}\phi_{\mathbf{m}}^{(d\,;\,\alpha,\nu)}(\sigma) = \lambda_{k,\mathbf{m}}\phi_{\mathbf{m}}^{(d\,;\,\alpha,\nu)}(\sigma).$$

That is to say, MCJ polynomials give a new quantum integrable system. Hence we desire to construct commuting families of pseudo-differential operators for GMCJ polynomials. Since commuting families of differential operators for the Jack polynomials $\phi_{\mathbf{m}}^{(d\,;\,\frac{n}{r},0)}$ are constructed by using the degenerate double affine Hecke algebra (DDAHA) of type A, we expect existence of a particular algebraic structure related to DDAHA for our polynomials. Once we obtain such an interpretation, we may succeed in proving the above conjecture.

Lastly, we would like to raise the issue of applications of MCJ polynomials. Since the weight function of the orthogonality relation for MCJ polynomials coincides with a circular Jacobi ensemble, we expect an application to the random matrix model whose density function is a circular Jacobi ensemble.

We intend to investigate these topics in the future.

References

[1] R. Askey (1982), Discussion of Szegö's paper *Beiträge zur Theorie der Toeplitzschen Formen*, In: *G. Szegö, Collected Works, Vol. I*, Birkhäuser, Boston: 303–305.

[2] T. H. Baker and P. J. Forrester (1997), The Calogero-Sutherland model and generalized classical polynomials, *Commun. Math. Phys.*, 188: 175–216.

[3] P. Bourgade, A. Nikeghbali and A. Rouault (2009), Circular Jacobi ensembles and deformed Verblunsky coefficients, *IMRN.*, 23: 4357–4394.

[4] H. Dib (1990), Fonctions de Bessel sur une algèbre de Jordan, *J. Math. pures et appl.*, 69: 403–448.

[5] H. Ding (2007), Differential equations and recursion relations for Laguerre functions on symmetric cones, *Trans. Amer. Math. Soc.*, 359: 3239–3250.

[6] J. Faraut (1992), Prolongement analytique des series de Taylor spheriques, *Contemp. Math.*, 138: 139–149.

[7] J. Faraut and A. Koranyi (1994), *Analysis on Symmetric Cones*, Clarendon Press, Oxford.

[8] J. Faraut and M. Wakayama, *Hermitian symmetric spaces of tube type and multivariate Meixner-Pollaczek polynomials*, arXiv:0812.1292.

[9] L. Hua (1963), *Harmonic analysis of functions of several complex variables in the classical domains*, Amer. Math. Soc.

[10] M. Ismail (2009), *Classical and Quantum Orthogonal Polynomials in One Variable*, Encyclopedia of Mathematics and its Applications, 98: Cambridge University Press.

[11] I. G. Macdonald (1995), *Symmetric Functions and Hall Polynomials*, Oxford University Press.

[12] A. Okounkov and G. Olshanski (1997), Shifted Jack polynomials, binomial formula, and applications, *Math. Res. Letters*, 4: 69–78.

[13] P. Sawyer (1997), Spherical functions on symmetric cones, *Trans. Amer. Math. Soc.*, 349-9: 3569–3584.

[14] L. Shen (2001), Orthogonal polynomials on the unit circle associated with the Laguerre polynomials, *Proc. Amer. Math. Soc.*, 129: 873–879.

[15] G. Shibukawa (2016), Multivariate Meixner, Charlier and Krawtchouk polynomials according to analysis on symmetric cones, *J. Lie Theory*, 26: 439–477.

[16] H. Upmeier (1983), Toeplitz operators on bounded symmetric domains, *Trans. Amer. Math. Soc.*, 280-1: 221–237.

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