Josai Mathematical Monographs
vol. 10 (2017), pp. 81 – 95

Toward a combinatorial formula for an irregular conformal block of rank one

Hajime NAGOYA and Yasuhide NUMATA

Abstract. In this note, we give a combinatorial formula for a particular three-point irregular conformal block of rank one using the Littlewood-Richardson numbers and propose a conjectural formula for the general three-point irregular conformal block of rank one.

1. Introduction

The aim of this note is to study explicit presentations for series expansions of irregular conformal blocks at irregular singular points for the Virasoro algebra. Our study is motivated by AGT correspondence [**1**] which gives an explicit presentation of the four-point conformal block in terms of Nekrasov partition function. This correspondence has some degenerate versions, the investigation of which was started by Gaiotto [**4**].

As is well-known, the ordinary conformal block is a building block of conformal field theory with Virasoro symmetry. The degeneration procedure mentioned above yields explicit presentations for series expansions of 'degenerate' conformal blocks at regular singular points. Such conformal blocks could have been investigated since the beginning of conformal field theories, but actually very few studies, such as [**10**], [**16**], have been done before AGT. 'Degenerate' conformal blocks are now called *irregular* conformal blocks since the Ward equations (some people call it the BPZ equations) have irregular singularities as ordinary differential equations.

This introductory section 1 will give some definitions and basic facts necessary for our study. We begin with the definition of vertex operators for irregular conformal blocks. In the second subsection irregular conformal blocks will be defined. The third subsection gives a brief explanation of the consequence of AGT correspondence in our situation.

1.1. Vertex operator

First of all, let us recall the definition of vertex operators of Virasoro algebra on irregular Verma modules and irregular conformal blocks.

2010 *Mathematics Subject Classification.* Primary 81R10, Secondary 33E99
Key words and phrases. Irregular conformal blocks; Painlevé equations; Virasoro algebra.

Denote the Virasoro algebra by $\mathrm{Vir} = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}L_n \oplus \mathbb{C}c$, which is the Lie algebra with the commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m+n,0}\frac{n^3-n}{12}c, \quad [L_m, c] = 0.$$

For $r \in \mathbb{Z}_{\geq 0}$, we define an irregular Verma module $M_\Lambda^{[r]}$ as a representation of Vir generated by an element $|\Lambda\rangle$ ($\Lambda = (\Lambda_r, \dots, \Lambda_{2r}) \in \mathbb{C}^{r+1}$) such that

$$L_n|\Lambda\rangle = \Lambda_n|\Lambda\rangle \quad (n = r, r+1, \dots, 2r), \quad L_n|\Lambda\rangle = 0 \quad (n \geq 2r+1).$$

Hereafter, we call $|\Lambda\rangle$ the *irregular vector*. The module $M_\Lambda^{[r]}$ is spanned by linearly independent vectors of the form

$$L_{i_1} \cdots L_{i_k}|\Lambda\rangle \quad (i_1 \leq \cdots \leq i_k < r).$$

When $r = 0$, $M_\Lambda^{[0]}$ is the usual Verma module. The non-negative integer r corresponds to the rank of singularity, as explained below.

Definition 1.1. *A vertex operator $\Phi^{\Delta}_{\Lambda',\Lambda}(z) : M_\Lambda^{[r]} \to M_{\Lambda'}^{[r]}$ is an linear operator satisfying*

$$[L_n, \Phi^{\Delta}_{\Lambda',\Lambda}(z)] = z^n \left(z\frac{d}{dz} + (n+1)\Delta \right) \Phi^{\Delta}_{\Lambda',\Lambda}(z),$$
$$\Phi^{\Delta}_{\Lambda',\Lambda}(z)|\Lambda\rangle = z^\alpha \exp\left(\sum_{n=1}^{r} \frac{\beta_n}{z^n} \right) \sum_{n=0}^{\infty} w_n z^n,$$

where $\alpha, \beta_n \in \mathbb{C}$, $w_n \in M_{\Lambda'}^{[r]}$ and $w_0 = |\Lambda'\rangle$.

In the case of $r = 0$, the vertex operator $\Phi^{\Delta_2}_{\Delta_3,\Delta_1}(z) : M_{\Delta_1}^{[0]} \to M_{\Delta_3}^{[0]}$ was introduced in [**2**] and exists if $M_{\Delta_3}^{[0]}$ is irreducible.

In the case of $r > 0$, we have the following theorem.

Theorem 1.2 ([**18**]). *For a positive integer r, if $\Lambda_{2r} \neq 0$, then the vertex operator $\Phi^{\Delta}_{\Lambda',\Lambda}(z)$: $M_\Lambda^{[r]} \to M_{\Lambda'}^{[r]}$ exists and is uniquely determined by the given parameters Λ, Δ, β_r with $\alpha = -(r+1)\Delta + \tilde{\alpha}(\beta_r, \Lambda)$, $\beta_n = \beta_n(\beta_r, \Lambda)$ ($n = 1, \dots, r-1$) and*

$$\Lambda'_n = \Lambda_n - \delta_{n,r} r\beta_r \quad (n = r, \dots, 2r).$$

Moreover, the coefficients w_n are polynomials of Λ, β_r, Δ and Λ_{2r}^{-1}.

We remark that $M_\Lambda^{[r]}$ is irreducible if and only if $\Lambda_{2r-1} \neq 0$ or $\Lambda_{2r} \neq 0$ [**11**],[**3**]. We expect the irregular Verma module $M_\Lambda^{[r]}$ with $\Lambda_{2r} = 0$ and $\Lambda_{2r-1} \neq 0$ can be used to describe a system in a ramified case. Because the pairing of the Whittaker (or Gaiotto) vector is the building block of a short-distance expansion of Painlevé III_3 tau function [**7**], which has only ramified singular points. Here, the Whittaker vector is the rank one irregular vector $|\Lambda_1, 0\rangle$ embedded in a Virasoro Verma module M_Δ.

Denote a dual irregular Verma module by $M_\Lambda^{*,[r]}$ with irregular vector $\langle\Lambda|$ such that

$$\langle\Lambda|L_n = \Lambda_n\langle\Lambda| \quad (n = -r, -r-1, \ldots, -2r),$$

and $M_\Delta^{*,[r]}$ is spanned by linearly independent vectors of the form

$$\langle\Lambda|L_{i_1}\cdots L_{i_k} \quad (-r < i_1 \leq \cdots \leq i_k).$$

A dual vertex operator $\Phi^{*,\Delta}_{\Lambda,\Lambda'}(z)\colon\ M_\Lambda^{*,[r]} \to M_{\Lambda'}^{*,[r]}$ is defined in the same manner and the similar statement to Theorem 1.2 holds for the dual vertex operator.

1.2. Conformal block

Recall the Shapovalov bilinear form on the usual Verma module. We denote it by $\langle\cdot\rangle\colon\ M_\Delta^{*,[0]} \times M_\Delta^{[0]} \to \mathbb{C}$. It satisfies

$$\begin{aligned}&\langle\Delta|\cdot|\Delta\rangle = 1,\\ &\langle u|L_n\cdot|v\rangle = \langle u|\cdot L_n|v\rangle \equiv \langle u|L_n|v\rangle,\end{aligned}$$

where $\langle u| \in M_\Delta^{*,[0]}$, $|v\rangle \in M_\Delta^{[0]}$. We also use a bilinear pairing $\langle\cdot\rangle\colon\ M_\Delta^{*,[0]} \times M_\Lambda^{[1]} \to \mathbb{C}$ as

$$\langle\Delta|\cdot|\Lambda\rangle = 1, \quad \langle u|L_n\cdot|v\rangle = \langle u|\cdot L_n|v\rangle \equiv \langle u|L_n|v\rangle,$$

where $\langle u| \in M_\Delta^{*,[0]}$, $|v\rangle \in M_\Lambda^{[1]}$. It is well-defined because L_n acts on $|\Lambda\rangle$ diagonally for $n > 0$ and L_n acts on $\langle\Delta|$ diagonally for $n \leq 0$.

A four-point regular conformal block is defined as an expectation value of a composition of vertex operators. For example,

$$\langle\Delta_\infty|\cdot\left(\Phi^{\Delta_1}_{\Delta_\infty,\Delta}(1)\circ\Phi^{\Delta_z}_{\Delta,\Delta_0}(z)|\Delta_0\rangle\right) \tag{1}$$

is a regular conformal block for the points $0, 1, z, \infty$ and a series expansion at $z = 0$. By definition, it coincides with

$$\langle\Delta_\infty|\Phi^{*,\Delta_1}_{\Delta_\infty,\Delta}(1)\cdot\Phi^{\Delta_z}_{\Delta,\Delta_0}(z)|\Delta_0\rangle.$$

Here, we suppose that Verma modules M_{Δ_∞}, M_Δ and M_{Δ_0} are irreducible. The following

$$\langle\Delta_\infty|\Phi^{*,\Delta_z}_{\Delta_\infty,\widetilde{\Delta}}(z)\cdot\Phi^{\Delta_1}_{\widetilde{\Delta},\Delta_0}(1)|\Delta_0\rangle \tag{2}$$

is also a regular conformal block for the points $0,1,z,\infty$ and a series expansion at $z=\infty$. The connection problem of conformal blocks has been still an open problem. See, for example, [**20**], [**9**]. In addition, we have not understood convergence of regular conformal blocks neither.

A three-point irregular conformal block is also defined as an expectation value of a composition of vertex operators. The following

$$\langle\Delta_\infty|\cdot\Phi^{\Delta_z}_{\Lambda',\Lambda}(z)|\Lambda\rangle \tag{3}$$

is a three-point irregular conformal block for two regular singular points z,∞ and one irregular singular point 0 and a series expansion at $z=0$. A series expansion at $z=\infty$ of the three-point irregular conformal block is given by

$$\langle\Delta_\infty|\Phi^{*,\Delta_z}_{\Delta_\infty,\widetilde{\Delta}}(z)\cdot|\Lambda\rangle. \tag{4}$$

If the parameters Δ_z, Λ' and $\widetilde{\Delta}$ take certain special values, then the irregular conformal blocks (3) and (4) are solutions to a second-order ordinary differential equation equivalent to the Kummer confluent hypergeometric equation. See Subsection III B 1 in [**18**]. In this case, the irregular conformal block (3) is a divergent series and (4) is absolutely convergent in $\mathbb{C}$.

1.3. Combinatorial formula

By AGT correspondence [**1**], the four-point conformal block (2) with $c=1$ is expressed as

$$\langle\theta_\infty^2|\Phi^{*,\theta_z^2}_{\theta_\infty^2,\tilde\sigma}(z)\cdot\Phi^{\theta_1^2}_{\tilde\sigma^2,\theta_0^2}(1)|\theta_0^2\rangle = z^{\theta_\infty^2-\theta_z^2-\tilde\sigma^2}(1-z)^{2\theta_1\theta_z}\sum_{\lambda,\mu\in\mathbb{Y}}\mathcal{F}_{\lambda,\mu}\begin{pmatrix}\theta_z,\theta_1\\ \theta_\infty,\tilde\sigma,\theta_0\end{pmatrix}z^{-|\lambda|-|\mu|}, \tag{5}$$

where

$$\mathcal{F}_{\lambda,\mu}\begin{pmatrix}\theta_z,\theta_1\\ \theta_\infty,\tilde\sigma,\theta_0\end{pmatrix} = \prod_{(i,j)\in\lambda}\frac{((\theta_1+\tilde\sigma+i-j)^2-\theta_0^2)((\theta_z+\tilde\sigma+i-j)^2-\theta_\infty^2)}{h_\lambda^2(i,j)(\lambda'_j+\mu_i-i-j+1+2\tilde\sigma)^2}$$
$$\times\prod_{(i,j)\in\mu}\frac{((\theta_1-\tilde\sigma+i-j)^2-\theta_0^2)((\theta_z-\tilde\sigma+i-j)^2-\theta_\infty^2)}{h_\mu^2(i,j)(\mu'_j+\lambda_i-i-j+1-2\tilde\sigma)^2}.$$

Here $\mathbb{Y}$ denotes the set of all partitions, λ' denotes the transposition of λ and $h_\lambda(i,j)$ is the hook length defined by $h_\lambda(i,j)=\lambda_i+\lambda'_j-i-j+1$. It is natural to expect that our irregular conformal blocks (3) and (4) have explicit combinatorial formulas, since *degenerate* Nekrasov partition functions are known [**4**].

If we use the explicit representation (5) of the four-point conformal block, computations of degeneration limits are easy. We can take a limit of (5) by

$$\theta_1 + \theta_0 = \epsilon^{-1}, \quad \theta_1 - \theta_0 = \theta_*, \quad z \to \epsilon z, \quad \epsilon \to 0, \tag{6}$$

which corresponds to confluence process moving the regular singular point 1 to 0, and obtain a combinatorial formula for the three-point irregular conformal block (4) with $c = 1$ [**14**], [**7**]. This is also true for any c.

On the other hand, by the limit (6), the four-point regular conformal block (1) with $c = 1$ diverges although it has a similar expression to (5). It turns out that we have to also move $\Delta = \sigma^2$ to infinity and multiply some function such as $(1 - z)^A$ in order to obtain a limit of (1) [**12**]. This multiplication breaks the combinatorial structure of the four-point regular conformal block. As a result, we cannot obtain a combinatorial formula for the three-point irregular conformal block (3) by the limit.

An explanation why the four-point regular conformal block with $c = 1$ admits the combinatorial formula (5) is given by expanding the integral representation of it in terms of Schur functions [**15**]. In the following, we discuss the expansion of the integral representation of our irregular conformal block (3) to construct an explicit combinatorial formula.

2. Integral representation

The integral formula for the three-point irregular conformal block (3) with $c = 1$ is

$$z^{2\alpha_0\alpha_z} e^{-\alpha_z/z} \int_0^\infty dy_1 \cdots \int_0^\infty dy_n \int_0^t dx_1 \cdots \int_0^t dx_m \prod_{i=1}^m \prod_{j=1}^n (x_i - y_j)^2$$
$$\times \prod_{i<j} (y_i - y_j)^2 \prod_{i<j} (x_i - x_j)^2 \prod_{i=1}^m x_i^{2\alpha_0} e^{-1/x_i} (z - x_i)^{2\alpha_z} \prod_{j=1}^n y_j^{2\alpha_0} e^{-1/y_j} (y_i - z)^{2\alpha_z}. \tag{7}$$

Precisely, the expansion of the integral above at $z = 0$ is equal to

$$\langle (\alpha_\infty + n)^2 | \cdot \Phi^{\alpha_z^2}_{(\alpha_0+\alpha_z+m,1/4),(\alpha_0,1/4)}(z) | ((\alpha_0, 1/4)\rangle \tag{8}$$

up to scalar, where $\alpha_0 + \alpha_z + \alpha_\infty + m + n = 0$. The integral formula above is obtained by the free field realization of vertex operators. Because the construction is a standard one, we omit the explanation.

Let us recall the Selberg integral formulas with Schur polynomials inserted in the integrand:

$$\int_0^1 dx_1 \cdots \int_0^1 dx_n s_\lambda(x) \prod_{i<j}(x_i - x_j)^2 \prod_{i=1}^n x_i^u (1-x_i)^v$$
$$= n! f_n(\lambda) \prod_{i=1}^n \frac{\Gamma(u+(n-i)+\lambda_i)\Gamma(v+(n-i))}{\Gamma(u+v+(2n-i-1)+\lambda_i)},$$

where $s_\lambda(x)$ is the Schur polynomial defined for a Young diagram λ, $\Gamma(u)$ is the Gamma function and

$$f_n(\lambda) = \prod_{i<j}(\lambda_i - \lambda_j + j - i).$$

Thus, we expect that we obtain a combinatorial formula for the special three-point irregular conformal block (8), if we expand the integral representation (7) at $z=0$ in terms of Schur polynomials.

By Theorem 1.2, the coefficients of the three-point irregular conformal block (3) are polynomials of Δ_0, Λ_1, β_1 and Δ_∞. For example, in the case of $c=1$ we have

$$\begin{aligned}
&\langle \theta_0^2| \cdot \left(\Phi^{\theta_z^2}_{(\theta,1/4),(\theta-\beta,1/4)}(z)|(\theta,1/4)\rangle\right) \\
&= z^{2\theta_z^2+2\beta(\theta-\beta)} e^{\beta/z} \left(1 + 2\left(2\beta^3 - 3\beta^2\theta + \beta\theta^2 - \beta\theta_0^2 - \beta\theta_z^2 + \theta\theta_z^2\right) z\right. \\
&+ 2\left(4\beta^6 - 12\beta^5\theta + 13\beta^4\theta^2 - 4\beta^4\theta_0^2 - 4\beta^4\theta_z^2 + 5\beta^4 - 6\beta^3\theta^3 + 6\beta^3\theta\theta_0^2\right. \\
&+ 10\beta^3\theta\theta_z^2 - 10\beta^3\theta + \beta^2\theta^4 - 2\beta^2\theta^2\theta_0^2 - 8\beta^2\theta^2\theta_z^2 + 6\beta^2\theta^2 + \beta^2\theta_0^4 \\
&+ 2\beta^2\theta_0^2\theta_z^2 - 3\beta^2\theta_0^2 + \beta^2\theta_z^4 - 3\beta^2\theta_z^2 + 2\beta\theta^3\theta_z^2 - \beta\theta^3 \\
&\left.\left. -2\beta\theta\theta_0^2\theta_z^2 + \beta\theta\theta_0^2 - 2\beta\theta\theta_z^4 + 5\beta\theta\theta_z^2 + \theta^2\theta_z^4 - 2\theta^2\theta_z^2 + \theta_0^2\theta_z^2\right) z^2 + \cdots\right).
\end{aligned}$$

We remark that Λ_2 can be taken as any non-zero complex number by scaling z. Therefore, the combinatorial formula for the special three-point irregular conformal block (8) will give a combinatorial formula for the general three-point irregular conformal block (3).

By the following change of the integral variables

$$x_i \to \frac{z}{1-zx_i}, \quad y_i \to \frac{1}{y_i},$$

the integral becomes

$$\begin{aligned}
&z^{2(\alpha_z+m)(\alpha_z+m+\alpha_0)-2\alpha_z^2} e^{-(\alpha_z+m)/z} \\
&\times \int_0^\infty dy_1 \cdots \int_0^\infty dy_n \int_0^\infty dx_1 \cdots \int_0^\infty dx_m \prod_{j=1}^n y_j^{-2\alpha_0-4} e^{-y_j} \prod_{i=1}^m x_i^{2\alpha_z} e^{x_i}
\end{aligned}$$

$$\times \prod_{i=1}^{m}(1-zx_i)^{-2(\alpha_0+\alpha_z+m+n)} \prod_{j=1}^{n}(1-zy_j)^{2\alpha_z} \prod_{i=1}^{m}\prod_{j=1}^{n}(1-z(x_i+y_j))^2.$$

Thus, the problem reduces to how we expand

$$\prod_{i=1}^{m}(1-zx_i)^{-u} \prod_{j=1}^{n}(1-zy_j)^{-v} \prod_{i=1}^{m}\prod_{j=1}^{n}(1-z(x_i+y_j))^2 \tag{9}$$

in terms of Schur polynomials for $u, v \in \mathbb{Z}_{\geq 0}$.

3. Combinatorial formula

3.1. By the Littlewood-Richardson numbers

Denote by $d_\lambda(n)$, the number of tableaux on the shape λ whose entries are taken from $\{1, \dots, n\}$. Then, we have

$$\prod_{i=1}^{m}(1-x_i)^{-n} = \sum_{\lambda, l(\lambda)\leq m} d_\lambda(n) s_\lambda(x) \quad (n \in \mathbb{Z}_{\geq 1}). \tag{10}$$

Indeed, the left one is the sum of all products of $x_i^{a_{ij}}$ over all m by n matrices $A = (a_{ij})$ with non-negative integer entries. By Robinson-Schensted-Knuth correspondence, A corresponds to pair (P, Q) of tableaux of the same shape, with P having its entries in $\{1, \dots, n\}$ and Q in $\{1, \dots, m\}$. The original product is x^Q. Recall the formula

$$d_\lambda(u) = \prod_{(i,j)\in\lambda} \frac{u+j-i}{h_\lambda(i,j)}.$$

Set

$$d^n_{a,b} = \det\left(\binom{a_i+n-i}{b_j+n-j}\right)_{1\leq i,j\leq n},$$

where $n \in \mathbb{N}$, $a = (a_1, \dots, a_n)$, $b = (b_1, \dots, b_n) \in \mathbb{Z}^n_{\geq 0}$ and $\binom{a_i}{b_j}$ is the binomial coefficient. On the expansion of the final part of (9), the following formula

$$\prod_{i=1}^{m}\prod_{j=1}^{n}(1+x_i+y_j) = \sum_{\mu\subset\lambda\subset(n^m)} d^n_{\lambda,\mu} s_\mu(x) s_{\tilde{\lambda}'}(y) \tag{11}$$

is known, where $\tilde{\lambda}' = (m-\lambda'_n, \dots, m-\lambda'_1)$. See p.65 in [**13**].

From the formulas (10) and (11), we can expand the product (9) in terms of Schur polynomials using the Littlewood-Richardson numbers $c^\nu_{\lambda,\mu}$.

Proposition 3.1. *We have*

$$
\begin{aligned}
&\prod_{i=1}^{m}(1-zx_i)^{-u}\prod_{j=1}^{n}(1-zy_j)^{-v}\prod_{i=1}^{m}\prod_{j=1}^{n}(1-z(x_i+y_j))^2 \\
&=\sum(-1)^{|\tilde{\alpha}'|+|\beta|+|\tilde{\gamma}'|+|\delta|}d_\lambda(u)d_\mu(v)d^n_{\alpha,\beta}d^n_{\gamma,\delta}c^\kappa_{\beta,\delta}c^\tau_{\tilde{\alpha}',\tilde{\gamma}'}c^\nu_{\lambda,\kappa}c^\eta_{\mu,\tau}s_\nu(x)s_\eta(y), \qquad (12)
\end{aligned}
$$

where the sum is taken over all Young diagrams α, β, γ, δ, λ, μ, ν, η, κ and τ such that $\beta \subset \alpha \subset (n^m)$, $\delta \subset \gamma \subset (n^m)$.

Hence, we have a combinatorial formula for the particular irregular conformal block (8):

$$\langle(\alpha_\infty+n)^2|\cdot\Phi^{\alpha_z^2}_{(\alpha_0+\alpha_z+m,1/4),(\alpha_0,1/4)}(z)|((\alpha_0,1/4)\rangle$$

for $m,n\in\mathbb{Z}_{\geq 0}$, $\alpha_0,\alpha_z\in\mathbb{C}$. We want to replace m,n by complex parameters. In order to do that, we show that the determinant $d^n_{\alpha,\beta}$ appeared in (12) is a polynomial in m,n. Note that in the formula (11), the coefficient of $s_\mu(x)s_\lambda(y)$ is

$$d^n_{\tilde{\lambda}',\mu}=\det\left(\binom{m-\lambda'_{n+1-i}+n-i}{\mu_j+n-j}\right)_{1\leq i,j\leq n}.$$

We see that the determinant $d^n_{\tilde{\lambda}',\mu}$ is a polynomial in m whose degree may depend on n. Since we also have the formula

$$\prod_{i=1}^{m}\prod_{j=1}^{n}(1+x_i+y_j)=\sum_{\lambda\subset\tilde{\mu}'\subset(m^n)}d^m_{\tilde{\mu}',\lambda}s_\lambda(y)s_\mu(x), \qquad (13)$$

the determinant $d^n_{\tilde{\lambda}',\mu}$ should be equal to $d^m_{\tilde{\mu}',\lambda}$ which is a polynomial in n. We expect the degrees of the determinant $d^n_{\tilde{\lambda}',\mu}$ as a polynomial in m, n do not depend on n, m, respectively.

Set

$$d^{m,n}_{a,b}=\det\left(\binom{m-a_{n+1-i}+n-i}{b_j+n-j}\right)_{1\leq i,j\leq n}.$$

Proposition 3.2. *The determinant $d^{m,n}_{a,b}$ is a polynomial in m with degree $b_1+\cdots+b_n$.*

Proof. Instead of dealing with $d^{m,n}_{a,b}$ itself, we consider the determinant $\tilde{d}^{m,n}_{a,b}$ whose (i,j) entries are obtained by extracting the part $\sum_{k=b_j-j+2}^{b_j+n-j}c_{ijk}m^k$ from (i,j) entries $\sum_{k=0}^{b_j+n-j}c_{ijk}m^k$ of $d^{m,n}_{a,b}$. We show $\tilde{d}^{m,n}_{a,b}=0$. Then the proposition follows by induction on n.

All the coefficients c_{ijk} are written as

$$c_{ijk} = \sum_{\ell=k}^{b_j+n-j} f_i^{b_j+n-j-\ell} g_{jkl},$$

where each $f_i = -a_{n+1-i} + n - i$ depends only on i and each g_{jkl} is independent of i. Then by subtracting the first row from the other rows, (i,j) entries for $i \geq 2$ of $\tilde{d}_{a,b}^{m,n}$ are computed as

$$\begin{aligned}
&\sum_{k=b_j-j+2}^{b_j+n-j} c_{ijk} m^k - \sum_{k=b_j-j+2}^{b_j+n-j} c_{1jk} m^k \\
&= \sum_{k=b_j-j+2}^{b_j+n-j} \left(\sum_{\ell=k}^{b_j+n-j} \left(f_i^{b_j+n-j-\ell} - f_1^{b_j+n-j-\ell} \right) g_{jkl} \right) m^k \\
&= \sum_{k=b_j-j+2}^{b_j+n-j} \left(\sum_{\ell=k}^{b_j+n-j-1} \left(f_i^{b_j+n-j-\ell} - f_1^{b_j+n-j-\ell} \right) g_{jkl} \right) m^k \\
&= (f_i - f_1) \sum_{k=b_j-j+2}^{b_j+n-j} \left(\sum_{\ell=k}^{b_j+n-j-1} \sum_{s=0}^{b_j+n-j-l-1} f_i^{b_j+n-j-l-1-s} f_1^s g_{jkl} \right) m^k.
\end{aligned}$$

Hence, we can write (i,j) entries for $i \geq 2$ of $\tilde{d}_{a,b}^{m,n} / \prod_{k=2}^{n} (f_k - f_1)$ as

$$\sum_{k=b_j-j+2}^{b_j+n-j-1} \left(\sum_{\ell=k}^{b_j+n-j-1} f_i^{b_j+n-j-1-l} g'_{jkl} \right) m^k,$$

where g'_{jkl} is independent of i. We repeat the process above, and finally when we subtract the $(n-2)$-th row from the $(n-1)$-th row and the n-th row, (i,j) entries for $i = n-1, n$ of $\tilde{d}_{a,b}^{m,n} / \prod_{q=1}^{n-2} \prod_{p=q+1}^{n} (f_p - f_q)$ are

$$\sum_{k=b_j-j+2}^{b_j+n-j-(n-2)} \left(\sum_{\ell=k}^{b_j+n-j-(n-2)} f_i^{b_j+n-j-(n-2)-l} g_{jkl}^{(n-2)} \right) m^k,$$

which is independent of i. Therefore, the determinant $\tilde{d}_{a,b}^{m,n}$ is equal to zero. □

Corollary 3.3. *The determinant $d_{\tilde{\lambda}',\mu}^{n}$ is a polynomial in m, n with degree $|\mu|$, $|\lambda|$, respectively. In other words, The coefficient of $s_\mu(x)s_\lambda(y)$ in the expansion of $\prod_{i=1}^{m} \prod_{j=1}^{n} (1 + x_i + y_j)$ in terms of Schur polynomials is a polynomial in m, n with degree $|\mu|$, $|\lambda|$, respectively.*

From Corollary 3.3, we can regard m and n as complex numbers in principle. Unfortunately, the determinant expression of $d^{m,n}_{a,b}$ is not useful if n is not a positive integer. We should know an explicit polynomial expression of the determinant $d^{m,n}_{a,b}$ if we want to compute $d^{m,n}_{a,b}$ in such a case.

3.2. Conjecture

In this subsection, we give another approach for an expansion of (9). Put

$$M_{\lambda,\mu} = \prod_{(i,j)\in\lambda} (2(\beta-\theta)+i-j) \prod_{(i,j)\in\mu} (-2\beta+i-j),$$
$$N_{\lambda,\mu} = (-1)^{|\mu|} \prod_{(i,j)\in\lambda} \frac{(\beta+i-j)^2-\theta_t^2}{h_\lambda(i,j)^2} \prod_{(i,j)\in\mu} \frac{(\theta-\beta+i-j)^2-\theta_0^2}{h_\mu(i,j)^2}.$$

Conjecture 3.4 ([**19**]). *The three-point irregular conformal block with two regular singular points z, ∞ and one irregular singular point 0 of rank one admits the following combinatorial formula*

$$\begin{aligned}&\langle\theta_0^2| \cdot \left(\Phi^{\theta_z^2}_{(\theta,1/4),(\theta-\beta,1/4)}(z)|(\theta,1/4)\rangle\right)\\ &= z^{-2\theta_z^2-2\beta(\theta-\beta)} e^{\frac{\beta}{z}} \sum_{\lambda,\mu\in\mathbb{Y}} z^{|\lambda|+|\mu|} \sum_{\substack{\nu\subset\lambda,\eta\subset\mu,\\|\nu|=|\eta|}} (-1)^{|\nu|} a^{\nu,\eta}_{\lambda,\mu} M_{\lambda/\nu,\mu/\eta} N_{\lambda,\mu},\end{aligned} \tag{14}$$

where $a^{\nu,\eta}_{\lambda,\mu}\in\mathbb{Z}_{\geq 0}$, as an expansion at the irregular singular point 0.

Conjecture 3.4 can be obtained from the next conjecture.

Conjecture 3.5. *For $u,v,w\in\mathbb{Z}_{\geq 0}$, we have*

$$\begin{aligned}&\prod_{i=1}^{m}(1-zx_i)^{-u}\prod_{j=1}^{n}(1-zy_j)^{-v}\prod_{i=1}^{m}\prod_{j=1}^{n}(1-z(x_i+y_j))^{-w}\\ &= \sum_{\lambda,\mu\in\mathbb{Y}} c_{\lambda,\mu}(u,v,w) s_\lambda(x)s_\mu(y) z^{|\lambda|+|\mu|},\end{aligned} \tag{15}$$

where the coefficients $c_{\lambda,\mu}(u,v,w)$ admits the unique expansion

$$c_{\lambda,\mu}(u,v,w) = \sum_{\substack{\nu\subset\lambda,\eta\subset\mu,\\|\nu|=|\eta|}} \frac{c^{\nu,\eta}_{\lambda,\mu}}{h_\lambda h_\mu} d_{\nu,\eta}(w) P_{\lambda/\nu,\mu/\eta}(u,v,w,m,n), \tag{16}$$

where $c^{\nu,\eta}_{\lambda,\mu}\in\mathbb{Q}$, $d_{\nu,\eta}(w)\in\mathbb{Q}[w]$ is a monic polynomial of degree $|\nu|$ and

$$P_{\lambda/\nu,\mu/\eta}(u,v,w,m,n) = \prod_{(i,j)\in\lambda/\nu}(u+nw+j-i)\prod_{(i,j)\in\mu/\eta}(v+mw+j-i),$$

$$h_\lambda = \prod_{(i,j)\in\lambda} (\lambda_i - j + \lambda'_j - i + 1).$$

In other words, for any λ*,* μ*,* ν *and* η *such that* $\nu \subset \lambda$*,* $\eta \subset \mu$ *and* $|\nu| = |\eta|$*, a rational number* $c^{\nu,\eta}_{\lambda,\mu}$ *and a monic polynomial* $d_{\nu,\eta}(w)$ *are determined uniquely by* (15) *and* (16)*.*

Conjectures 3.4 and 3.5 imply that the parts consisting of parameters θ_0, θ, θ_z, β are explicit and the remained problem, except for proving conjectures, is how to describe the coefficients $a^{\nu,\eta}_{\lambda,\mu}$ or $c^{\nu,\eta}_{\lambda,\mu}$ and the polynomials $d_{\nu,\eta}(w)$ combinatorially.

When $m = n = 1$, the computation of $c^{\nu,\eta}_{\lambda,\mu}$ and the polynomials $d_{\nu,\eta}(w)$ is easy because Schur polynomials appeared in the expansion are only x^a, y^b $(a, b \in \mathbb{Z}_{\geq 0})$.

Proposition 3.6. *For any non-negative integer* p*,* q*,* r *such that* $r \leq p, q$*, there exists uniquely a rational number* $c^r_{p,q}$ *and a monic polynomial* $d_r(w)$ *such that*

$$\begin{aligned}
&(1 - zx)^{-u}(1 - zy)^{-v}(1 - z(x + y))^{-w} \\
&= \sum_{p,q=0}^{\infty} \frac{1}{p!q!} x^p y^q z^{p+q} \sum_{0\leq r\leq \min(p,q)} c^r_{p,q} d_r(w) P_{(p)/(r),(q)/(r)}(u, v, w, 1, 1).
\end{aligned} \tag{17}$$

Moreover, the rational numbers $c^r_{p,q}$ *and the monic polynomials* $d_r(w)$ *are given as*

$$c^r_{p,q} = r! \binom{p}{r}\binom{q}{r}, \quad d_r(w) = w(w + 1)\cdots(w + r - 1). \tag{18}$$

Proof. Firstly, we show that the expansion (17) holds with (18). Let $f(p, q, u, v, w)$ be the coefficient of $x^p y^q z^{p+q}$ in the expansion, that is,

$$(1 - zx)^{-u}(1 - zy)^{-v}(1 - z(x + y))^{-w} = \sum_{p,q=0}^{\infty} f(p, q, u, v, w) x^p y^q z^{p+q}.$$

Denote by $R(w)$ the left hand side of (17). Then, we have a relation

$$R(w - 1) = (1 - z(x + y))R(w),$$

which is equivalent to all of recursive relations

$$f(p, q, u, v, w) = f(p, q, u, v, w - 1) + f(p - 1, q, u, v, w) + f(p, q - 1, u, v, w) \tag{19}$$

for $p, q \in \mathbb{Z}_{\geq 0}$. Since $f(0, 0, u, v, w) = 1$ and from (10), we have

$$f(p, q, u, v, 0) = P_{(p),(q)}(u, v, 0, 1, 1),$$

it is sufficient to verify that

$$\sum_{0\le r\le p,q} c^r_{p,q} d_r(w) \frac{P_{(p)/(r),(q)/(r)}(u,v,w,1,1)}{p!q!}$$
$$= \sum_{0\le r\le p,q} \binom{w+r-1}{r}\binom{u+w+p-1}{p-r}\binom{v+w+q-1}{q-r} \quad (p,q\in\mathbb{Z}_{\ge 0}) \qquad (20)$$

satisfy all of the recursive relations (19). Using the following relation

$$\sum_{r=0}^{i} (g(p,q,r,w-1) - g(p,q,r,w) + g(p-1,q,r,w) + g(p,q-1,r,w))$$
$$= \binom{w+i-1}{i}\binom{u+w+p-2}{p-i-1}\binom{v+w+q-2}{q-i-1},$$

where

$$g(p,q,r,w) = \binom{w+r-1}{r}\binom{u+w+p-1}{p-r}\binom{v+w+q-1}{q-r},$$

the computation for the left hand side of (20) satisfying (19) is straightforward.

Secondly, we show that the recursive relations (19) determine the rational numbers $c^r_{p,q}$ and the monic polynomials $d_r(w)$ uniquely. We immediately know that

$$c^0_{p,q} = 1, \quad d_0(w) = 1$$

for $p,q\in\mathbb{Z}_{\ge 0}$.

Suppose $p > q$ and $c^r_{p-1,q}$, $c^r_{p,q-1}$, $d_r(w)$ for $r = 1,\dots,q$ satisfy (18). Then, the recursive relation (19) reads as

$$\sum_{r=0}^{q} c^r_{p,q} d_r(w) P_{(p)/(r),(q)/(r)}(u,v,w,1,1)$$
$$= \sum_{r=0}^{q} c^r_{p,q} d_r(w-1) P_{(p)/(r),(q)/(r)}(u,v,w-1,1,1) + F(p,q,u,v,w), \qquad (21)$$

where $F(p,q,u,v,w)$ is a known polynomial in u,v,w. By the relations

$$\begin{aligned} d_r(w-1) =& d_r(w) - r d_{r-1}(w),\\ P_{(p)/(r),(q)/(r)}(u,v,w-1,1,1) =& P_{(p)/(r),(q)/(r)}(u,v,w,1,1)\\ &+ (r-p) P_{(p-1)/(r),(q-1)/(r-1)}(u,v,w,1,1), \end{aligned}$$

the recursive relation (21) becomes

$$\sum_{r=0}^{q} c^r_{p,q}\left(rd_{r-1}(w)P_{(p)/(r),(q)/(r)}(w)+(p-r)(d_r(w)-rd_{r-1}(w))P_{(p-1)/(r),(q-1)/(r-1)}(w)\right)$$
$$=F(p,q,u,v,w).$$

From the equation above, by comparing the coefficient of $w^{p+q-r-1}$, $c^r_{p,q}$ is uniquely determined as a rational number inductively.

Suppose $p=q$ and $c^r_{p-1,p}$, $c^r_{p,p-1}$, $d_r(w)$ for $r=1,\dots,p-1$ satisfy (18). Then, the recursive relation (19) reads as

$$\begin{aligned}&\sum_{r=0}^{p} c^r_{p,p}d_r(w)P_{(p)/(r),(p)/(r)}(u,v,w,1,1)\\&=\sum_{r=0}^{p} c^r_{p,p}d_r(w-1)P_{(p)/(r),(p)/(r)}(u,v,w-1,1,1)+F(p,p,u,v,w).\end{aligned}\tag{22}$$

By the same way to the case of $p>q$, $c^r_{p,q}$ for $r=1,\dots,p-1$ are uniquely determined by (22). Put $c^r_{p,p}d_p(w)=\sum_{i=0}^{p}a_iw^i$. Then, a_p, a_{p-1},…, a_0 are uniquely determined by (22) again, respectively.

□

When m or n is greater than 1, the computations of $c^{\nu,\eta}_{\lambda,\mu}$ and the polynomials $d_{\nu,\eta}(w)$ become much difficult. We give some observations for $c^{\nu,\eta}_{\lambda,\mu}$ and the polynomials $d_{\nu,\eta}(w)$ below:

$$\begin{aligned}c^{(1),(1)}_{\lambda,\mu}&=|\lambda||\mu|,\\
c^{(p),(p)}_{(m,n),(r)}&=\left(p!\binom{m}{p}+\sum_{i=0}^{p-1}\sum_{j=1}^{p-i}(-1)^{p+i+j}j(p-j)!\prod_{k=1}^{j-1}(p-k-i)\binom{m}{i}\binom{n}{j}\right)\binom{r}{p},\\
c^{(p,1),(p+1)}_{(m,n),(r)}&=\Bigg(pp!\binom{m}{p}\binom{n}{1}\\
&\quad+\sum_{i=0}^{p-1}\sum_{j=1}^{p+1-i}(-1)^{p+i+j}pj(p-j)!\prod_{k=1}^{j-1}(p-k-(1+\delta_{k,j-1}(j-1))i)\binom{m}{i}\binom{n}{j}\Bigg)\\
&\quad\times\binom{r}{p+1},\\
d_{\nu,(p)}(w)&=\prod_{(i,j)\in\nu}(w+j-i).\end{aligned}$$

Acknowledgments. This work was supported by JSPS KAKENHI Grant Numbers JP15K17560, JP25800009.

References

[1] L. F. Alday, D. Gaiotto and Y. Tachikawa, Liouville Correlation Functions from Four-dimensional Gauge Theories, *Lett. Math. Phys.* **91** (2010) 167–197

[2] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, Infinite conformal symmetry in two-dimensional quantum field theory, *Nucl. Phys.* **B241**, (1984), 333-380

[3] E. Felińska, Z. Jaskólski and M. Kosztolowicz, Whittaker pairs for the Virasoro algebra and the Gaiotto-BMT states, *J. Math. Phys.,* **53**, 033504 (2012)

[4] D. Gaiotto, Asymptotically free $\mathcal{N} = 2$ theories and irregular conformal blocks, *J. Phys.: Conference Series* **462** (2013)

[5] D. Gaiotto and J. Teschner, Irregular singularities in Liouville theory and Argyres-Douglas type gauge theories, I, *J. High Energy Phys.*, **2012**, no.12, 050, 76pp

[6] O. Gamayun, N. Iorgov and O. Lisovyy, Conformal field theory of Painlevé VI, *J. High Energy Phys.*, **2012**, no.10, 038, 24pp

[7] O. Gamayun, N. Iorgov and O. Lisovyy, How instanton combinatorics solves Painlevé VI, V and III's, *J. Phys. A*, **46**, no.33, 335203, 29pp, (2013)

[8] L. K. Hua, Harmonic analysis of functions of several complex variables in the classical domains, Translations of mathematical monographs, Vol. 6, Amer. Math. Soc., Providence, RI, 1963

[9] N. Iorgov, O. Lisovyy and Yu. Tykhyy, Painlevé VI connection problem and monodromy of $c = 1$ conformal blocks, *J. High Energy Phys.*, (2013) 2013: 29.

[10] M. Jimbo, H. Nagoya and J. Sun, Remarks on the confuent KZ equation for $\mathfrak{sl}_2$ and quantum Painlevé equations, *J. Phys. A: Math. Theor.* **41** (2008) 14pp

[11] R. Lu, X. Guo and K. Zhao, Irreducible modules over the Virasoro algebra, *Documenta Math.,* **16** (2011) 709–721

[12] O. Lisovyy and H. Nagoya, in preparation

[13] I. G. Macdonald, Symmetric functions and Hall polynomials, Oxford, 1995

[14] A. Marshakov, A. Mironov, and A. Morozov, On non-conformal limit of the AGT relations, *Phys. Lett. B* **682**, 125 (2009)

[15] A. Mironov, A. Morozov and Sh. Shakirov, A direct proof of AGT conjecture $\beta = 1$, *J. High Energy Phys.*, (2011), no.2, 067, 41pp

[16] H. Nagoya and J. Sun, Confluent primary fields in the conformal field theory, *J. Phys. A: Math. Theor.* **43** 465203 (2010)

[17] H. Nagoya and Y. Yamada, Symmetries of Quantum Lax Equations for the Painlevé Equations, *Ann.–Henri Poincaré* **15**, (2014), 313–344

[18] H. Nagoya, Irregular conformal blocks, with an application to the fifth and fourth Painlevé equations, *J. Math. Phys.* **56**, 123505 (2015)

[19] H. Nagoya, Conformal blocks and Painlevé functions, submitted to RIMS Kôkyûroku Bessatsu

[20] B. Ponsot and J. Teschner, Clebsch-Gordan and Racah-Wigner coeffcients for a continuous series of representations of $U_q(\mathfrak{sl}(2,\mathbb{R}))$, *Comm. Math. Phys.* 224, (2001), 613–655

Hajime Nagoya

School of Mathematics and Physics, Kanazawa University
Ishikawa, Japan
nagoya@se.kanazawa-u.ac.jp

Yasuhide Numata

Department of Mathematical Sciences, Shinshu University
Nagano, Japan
nu@math.shinshu-u.ac.jp