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Factorization spaces and moduli spaces over curves

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Abstract. The notion of factorization space is a non-linear counterpart of the factorization algebra, which was introduced by Beilinson and Drinfeld in the theory of chiral algebras developing a geometric framework of vertex algebras. We give a review of factorization spaces with an emphasis on the connection to moduli problems on algebraic curves, and explain a general construction of factorization spaces from moduli spaces.

0. Introduction

0.1.

The two-dimensional quantum conformal field theory (*CFT* for short) is intimately related to moduli problems on algebraic curves. The first understanding of this claim is that the Virasoro Lie algebra controlling the symmetry of CFT is nothing but the Lie algebra of the vector fields on the affine line. By the Kodaira-Spencer deformation theory, the sheaf of Lie algebras of vector fields on a given variety X describes the infinitesimal deformation of X. Thus we see that CFT should be related to the moduli space of algebraic curves. A more precise treatment of this viewpoint was presented in the paper [**5**] of Beilinson and Schechtman. The WZW model, or the CFT attached to affine Lie algebras, has a similar feature, as shown in the work of Tsuchiya, Ueno and Yamada [**16**] formulating the WZW model on the moduli space of pointed complex curves.

The theory of *vertex algebras* began its life as one of the mathematical formulations of CFT, and at present it is the best established one in the algebraic viewpoint. For instance, representation theoretic methods can be applied relatively easily in the vertex algebra context.

However there are still many subjects in CFT to study. One of them is the relation to moduli problems mentioned above. The purpose of this article is to give a review on a non-linear or geometric reformulation of vertex algebra which seems to be powerful to reveal the relation of CFT and moduli problems. The formulation was introduced by Beilinson and Drinfeld in [**3**] and is nowadays called *factorization space*.

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Since the notion of factorization space is of abstract and complicated nature, we will not start with its definition but with some explanation on the way how Beilinson and Drinfeld reached it. In the first step some recollection will be given on vertex algebras, and then two equivalent notions of vertex algebras will be explained. They are *chiral algebras* and *factorization algebras*. Then we introduce our main object, *factorization spaces*. After explaining standard examples, we will show a general construction of factorization spaces from deformation problems.

0.2. Organization

Let us explain the organization of this article. §1 gives some recollection of vertex algebras. After the recollection of the definition and some standard examples in §1.1 and §1.2, we explain the construction of vertex algebra bundles on algebraic curves in §1.4.

§2 is an introduction to chiral and factorization algebras. These are almost equivalent notions of vertex algebra bundles. The definitions require some knowledge on $\mathcal{D}$-modules, on which we give a brief summary in §2.1.

In §3 we introduce the main object, namely the factorization space. For explaining the motivation of the definition, we start with the reformulation of factorization algebras as sheaves on the Ran space, explained in §3.1 and §3.2. The definition is given in §3.4 after the recollection of ind-schemes in §3.3. A factorization space gives a factorization algebra after the linearization explained in §3.6.

§4 and §5 give standard examples of factorization spaces, the Beilinson-Drinfeld Grassmannian and the factorization space associated to moduli spaces of pointed curves.

In the final §6 we explain a general method to construct factorization spaces from moduli spaces. The examples in §4 and §5 are special cases of our construction.

Let us remark that the contents in §§1–5 are based on known facts. The claim in §6 is new.

0.3. Notation

We will work over the field $\mathbb{C}$ of complex numbers unless otherwise stated. The symbol $\otimes$ denotes the tensor product of vector spaces over $\mathbb{C}$.

For a scheme or an algebraic stack Z, $\mathcal{O}_Z$, Θ_Z, Ω_Z and $\mathcal{D}_Z$ denote the structure sheaf, the tangent sheaf, the sheaf of 1-forms and the sheaf of differential operators on Z respectively (if they are defined). By "an $\mathcal{O}$-module on Z" we mean a quasi-coherent sheaf on Z. By "a $\mathcal{D}$-module on Z" we mean a sheaf of $\mathcal{D}_Z$-modules quasi-coherent as $\mathcal{O}_Z$-modules.

For a morphism $f : Z_1 \to Z_2$, the symbols $f^{\cdot}$ and $f_{\cdot}$ denote the inverse and direct image functors of $\mathcal{O}$-modules respectively.

1. Recollection on vertex algebras

We begin with the recollection of vertex algebras. The basic reference is [**8**].

1.1. Vertex algebras

The notion of vertex algebra is introduced to encode the chiral part of two dimensional quantum conformal field theory, which arises from string theory and condensed matter theory in physics. Let us begin with the definition of quantum fields in the context of vertex algebras.

DEFINITION. For a vector space V, a *field* on V is a formal power series

$$A(z) = \sum_{i\in\mathbb{Z}} A_i z^{-i} \in (\operatorname{End} V)[[z^{\pm 1}]]$$

valued in the algebra $\operatorname{End} V$ of linear endomorphisms on V such that for any $v \in V$ we have

$$A(z) \cdot v = \sum_{i\in\mathbb{Z}} (A_i \cdot v) z^{-i} \in V((z)).$$

In other words, for any $v \in V$ we have $A_i \cdot v = 0$ for large enough i.

Next we want to restrict the way two quantum fields interact. The interaction is encoded the following description of singularity.

DEFINITION. Two fields $A(z)$ and $B(w)$ acting on a vector space V are called *local* with respect to each other if there exists $N \in \mathbb{Z}_{\geq 0}$ such that

$$(z-w)^N [A(z), B(w)] = 0$$

as a formal power series in $(\operatorname{End} V)[[z^{\pm 1}, w^{\pm 1}]]$.

It is known (see [**8**, §1.2]) that this definition is equivalent to the following: for any $v \in V$ and $\varphi \in V^*$ the elements $\langle \varphi, A(z)B(w)v\rangle$ and $\langle \varphi, B(w)A(z)v\rangle$ are the expansions of the same element

$$f_{v,\varphi} \in \mathbb{C}[[z,w]][z^{-1}, w^{-1}, (z-w)^{-1}]$$

in $\mathbb{C}(z))((w))$ and $\mathbb{C}((w))((z))$ respectively, and the pole order of $f_{v,\varphi}$ in $(z-w)$ is uniformly bounded for all v and φ.

Now the definition of vertex algebra is

DEFINITION. A *vertex algebra* is a collection $(V, |0\rangle, T, Y)$ consisting of

- a vector space V, called the *space of states*
- a vector $|0\rangle \in V$, called the *vacuum vector*
- a linear operator $T \in \operatorname{End} V$, called the *translation operator*

- a linear operation $Y(-,z) : V \to (\operatorname{End} V)[[z^{\pm 1}]]$ with the image $Y(A,z)$ always a field on V for any $A \in V$, called the *state-field correspondence*

satisfying the following axioms.

1. $Y(|0\rangle, z) = \mathrm{id}_V$ and $Y(A,z)|0\rangle = A + \cdots \in V[[z]]$ for any $A \in V$.
2. $[T, Y(A,z)] = \partial_z Y(A,z)$ for any $A \in V$ and $T|0\rangle = 0$.
3. All fields $Y(A,z)$ are local with respect to each other.

This definition has an obvious super version. It is given by replacing the vector space V with a super vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$, and replacing the locality condition with

$$(z-w)^N Y(A,z)Y(B,w) = (-1)^{p(A)p(B)}(z-w)^N Y(B,w)Y(A,z),$$

where $p(A) \in \{\bar{0}, \bar{1}\}$ denotes the parity of a homogeneous element $A \in V$. One should also require that $|0\rangle \in V_{\bar{0}}$, that T should have even parity, that for $A \in V_{\bar{i}}$ the field $Y(A,z)$ is a series of endomorphisms of V with parity $\bar{i}$. The obtained object is called a *vertex superalgebra*.

Before showing some examples of vertex algebras, we introduce

Definition. The *formal delta function* $\delta(z,w)$ is the series

$$\delta(z,w) := \sum_{n \in \mathbb{Z}} z^n w^{-n-1}.$$

As is well-known, this series satisfies $A(z)\delta(z,w) = A(w)\delta(z,w)$ for any formal power series $A(z) \in \mathbb{C}[[z^{\pm 1}]]$, and $(z-w)^{n+1}\partial_w^n \delta(z,w) = 0$ for $n \in \mathbb{Z}_{\geq 0}$.

1.2. Examples of vertex algebras

Here are some standard examples of vertex (super) algebras.

Example 1.1. 1. For $k \in \mathbb{C} \setminus \{0\}$, let $\mathcal{H}_k$ be the one-dimensional Heisenberg Lie algebra, given by the central extension

$$0 \longrightarrow \mathbb{C}1 \longrightarrow \mathcal{H}_k \longrightarrow \mathbb{C}((t)) \longrightarrow 0$$

of Lie algebras with the cocycle

$$c(f,g) := -k \operatorname*{Res}_{t=0} f dg.$$

Here the space $\mathbb{C}((t))$ of formal Laurent series is considered as a commutative Lie algebra. The standard set of topological generators is given by $b_n :=$

t^n $(n \in \mathbb{Z})$ and 1. The defining relation is

$$[b_n, b_m] = kn\delta_{n,-m}1, \quad [1, b_n] = 0.$$

Denote by $\pi_{k,\nu}$ the Fock representation of $\mathcal{H}_k$ with $\nu \in \mathbb{C}$. It is defined to be the induced representation

$$\pi_{k,\nu} := \widetilde{U}(\mathcal{H}_k) \otimes_{\widetilde{U}(\mathcal{H}_+)} \mathbb{C}_\nu.$$

Here $\widetilde{U}(\mathcal{H}_k)$ is the completion of the universal enveloping algebra $U(\mathcal{H}_k)$ with respect to the subspaces $t^N\mathbb{C}[[t]]$ with $N \in \mathbb{Z}$. $\mathcal{H}_+$ is the Lie subalgebra of $\mathcal{H}_k$ generated by $b_0, b_1, \ldots$. $\mathbb{C}_\nu$ is the one-dimensional representation of $\mathcal{H}_+$ with b_0 acting by ν and b_n acting trivially for $n > 0$.

Now set $\nu = 0$. As a vector space, we have an isomorphism

$$\pi_{k,0} \xrightarrow{\sim} \mathbb{C}[b_{-1}, b_{-2}, \ldots].$$

This vector space $\pi_{k,0}$ has a structure of vertex algebra with

- $|0\rangle := 1$.
- T is determined by $T \cdot 1 = 0$ and $T, b_i = -ib_{i-1}$.
- Y is determined by

  $$Y(b_{-1}, z) = b(z) := \sum_{n\in\mathbb{Z}} b_n z^{-n-1},$$
  $$Y(b_{i_1} \cdots b_{i_k}, z) := \frac{1}{(-j_1-1)! \cdots (-j_k-1)!} : \partial_z^{-j_1-1} b(z) \cdots \partial_z^{-j_k-1} b(z) : .$$

  Here the symbol : : denotes the normally ordered product defined by

  $$: b_i b_j := \begin{cases} b_j b_i & i > 0 \\ b_i b_j & \text{otherwise} \end{cases}$$

  for a two-length monomial, and for a longer monomial we define

  $$: b_i X :=: b_i(: X :) : .$$

  Finally we assume :: is linear over the space of formal series.

One can check the axioms of vertex algebra easily except the locality axiom. The locality follows from the formula

$$b(z)b(w) =: b(z)b(w) : +k \sum_{n>0} nz^{-n-1}w^{n-1},$$

and the fact that the last summation is the expansion of $(z-w)^{-2}$ in $\mathbb{C}((z))((w))$. The resulting vertex algebra $\pi_{k,0}$ will be called the *Heisenberg vertex algebra* (of rank one).

2. The next example is associated to the vacuum representation of a current Lie algebra (the derived Lie algebra of an affine Kac-Moody Lie algebra). Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra, and $L\mathfrak{g} = \mathfrak{g}((t)) = \mathfrak{g} \otimes \mathbb{C}((t))$ be the formal loop algebra with the commutator $[A \otimes f(t), B \otimes g(t)] = [A, B] \otimes f(t)g(t)$. Denote by $\widehat{\mathfrak{g}}$ the current Lie algebra, which is the central extension

$$0 \longrightarrow \mathbb{C}K \longrightarrow \widehat{\mathfrak{g}} \longrightarrow L\mathfrak{g} \longrightarrow 0$$

of Lie algebras with the Lie bracket given by

$$[A \otimes f(t), B \otimes g(t)] = [A, B] \otimes f(t)g(t) - (\operatorname*{Res}_{t=0} f dg)\,(A, B)K, \quad [K, -] = 0.$$

Here the bilinear form $(-,-)$ on $\mathfrak{g}$ is the normalized Killing form given by

$$(A, B) := \frac{1}{2h^\vee}(A, B)_K = \frac{1}{2h^\vee} \operatorname{tr}_{\mathfrak{g}}(\operatorname{ad}(A)\operatorname{ad}(B)) \tag{1.2.1}$$

with $h^\vee$ the dual Coxeter number of $\mathfrak{g}$.

The vacuum representation of level $k \in \mathbb{C}$ of $\widehat{\mathfrak{g}}$ is the induced representation

$$V_k(g) := U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[[t]] \otimes \mathbb{C}K)} \mathbb{C}_k$$

where $\mathbb{C}_k$ is the one-dimensional representation of $\mathfrak{g}[[t]] \otimes \mathbb{C}K$ on which $\mathfrak{g}[[t]]$ acts by 0 and K acts by k.

As a vector space, we have an isomorphism $V_k(\mathfrak{g}) \simeq U(\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}])$. Take an ordered basis $\{J^a\}_{a=1}^{\dim \mathfrak{g}}$ of $\mathfrak{g}$, and set $A_n := A \otimes t^n \in L\mathfrak{g}$ for $A \in \mathfrak{g}$. Denote by v_k the basis of $\mathbb{C}_k$ and the resulting element in V_k. Then $V_k(\mathfrak{g})$ has a PBW basis of monomials of the form

$$J^{a_1}_{n_1} \cdots J^{a_m}_{n_m} v_k, \quad n_1 \le n_2 \le \cdots \le n_m < 0, \text{ if } n_i = n_{i+1} \text{ then } a_i \le a_{i+1}.$$

Now $V_k(\mathfrak{g})$ has a structure of vertex algebra with

- $|0\rangle = v_k$.
- T is determiend by $Tv_k = 0$ and $[T, J^a_n] = -nJ^a_{n-1}$.
- Y is determiend by

$$Y(J^a_{-1}v_k, z) = J^a(z) := \sum_{n \in \mathbb{Z}} J^a_n z^{-n-1}$$

$$Y(J^{a_1}_{n_1} \cdots J^{a_m}_{n_m} v_k, z) =$$

$$\frac{1}{(-n_1-1)!\cdots(-n_m-1)!} : \partial_z^{-n_1-1} J^{a_1}(z)\cdots\partial_z^{-n_m-1} J^{a_m}(z) : .$$

Here the symbol $:\,:$ denotes the normally ordered product given by

$$: J^a_n J^b_m := \begin{cases} J^b_m J^a_n & n>0 \text{ or } a>b \\ J^a_n J^b_m & \text{otherwise} \end{cases}$$

for two-length monomials and $: AX := : A(: X :) :$ for longer elements. We also assume the linearity over the space of formal series.

The locality axiom follows from the computation

$$[J^a(z), J^b(w)] = [J^a, J^b](w)\delta(z,w) + k(J^a, J^b)\partial_w\delta(z,w).$$

The resulting vertex algebra $V_k(\mathfrak{g})$ will be called the *affine vertex algebra* of $\mathfrak{g}$ with level k.

3. Let $\mathcal{K} := \mathbb{C}((t)) \supset \mathcal{O} := \mathbb{C}[[t]]$ be the structure sheaf of the punctured disc $D^\times$ and the formal disc D. The space $\operatorname{Der}\mathcal{K}$ of derivations of $\mathcal{K}$ (or vector fields on $D^\times$) has a natural structure of Lie algebra. The Virasoro Lie algebra is the central extension

$$0 \longrightarrow \mathbb{C}C \longrightarrow \mathcal{V}ir \longrightarrow \operatorname{Der}\mathcal{K} \longrightarrow 0.$$

The Lie bracket is given by

$$[f(t)\partial_t, g(t)\partial_t] = (fg' - f'g)\partial_t - \frac{1}{12}(\operatorname*{Res}_{t=0} fg'''dt)C, \quad [C,-]=0.$$

The standard generators and the relation are

$$L_n = -t^{n+1}\partial_t, \quad [L_n, L_m] = (n-m)L_{n+m} + \frac{1}{12}(n^3-n)\delta_{n,-m}C.$$

For $c \in \mathbb{C}$ consider the induced representation

$$\mathrm{Vir}_c := U(\mathcal{V}ir) \otimes_{U(\operatorname{Der}\mathcal{O}\oplus\mathbb{C}C)} \mathbb{C}_c, \tag{1.2.2}$$

where $\mathbb{C}_c = \mathbb{C}v_c$ is the one-dimensional vector space on which

$$\operatorname{Der}\mathcal{O} = \mathbb{C}[[t]]\partial_t = \langle L_n\,(n \geq -1)\rangle,$$

acts trivially and C acts by c. Vir_c has a PBW basis of the form

$$L_{j_1}\cdots L_{j_m}v_c, \quad j_1 \leq j_2 \leq \cdots \leq j_m \leq -2$$

Vir_c has a structure of vertex algebra with

- $|0\rangle = v_k$.
- $T := L_{-1}$.
- Y is determiend by

$$Y(L_{-2}v_k, z) = T(z) := \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$$

$$Y(L_{j_1}\cdots L_{j_m}v_k, z) = \frac{1}{(-j_1-2)!\cdots(-j_m-2)!} :\partial_z^{-j_1-2}T(z)\cdots\partial_z^{-j_m-2}T(z): .$$

The normally ordered product is defined similarly as before.

The locality axiom is checked by the computation

$$[T(z), T(w)] = \frac{c}{12}\partial_w^3\delta(z,w) + 2T(w)\partial_w\delta(z,w) + \partial_w T(w)\cdot\delta(z,w). \quad (1.2.3)$$

The resulting vertex algebra Vir_c will be called the *Virasoro vertex algebra* of central charge c.

It is known that both $\pi_{k,0}$ and $V_k(\mathfrak{g})$ have a Virasoro element ω, namely the elements

$$\omega_\lambda := \frac{1}{k}\Big(\frac{1}{2}b_{-1}^2 + \lambda b_{-2}\Big)v_0 \in \pi_{k,0},$$

$$\omega_{\mathfrak{g},k} := \frac{1}{2(k+h^\vee)}\sum_{a=1}^{\dim\mathfrak{g}} J_{-a}^a J_{a,-1}v_k \in V_k(\mathfrak{g})$$

yield $T(z) := Y(\omega, z)$ satisfying the equation (1.2.3) with the central charges given respectively by

$$c_\lambda = 1 - 12\lambda^2, \quad c(\mathfrak{g},k) = \frac{k\dim\mathfrak{g}}{k+h^\vee}.$$

1.3. Quasi-conformal vertex algebras

The existence of Virasoro element ensures an action of the Lie algebra $\mathcal{V}ir$ on the vertex algebra considered. Let us introduce a weaker condition to ensure an action of a certain Lie subalgebra of $\mathcal{V}ir$. To state that, we consider the following Lie subalgebras of $\mathcal{V}ir$.

$$\operatorname{Der}\mathcal{O} = \mathbb{C}[[t]]\partial_t = \langle L_n\ (n\ge -1)\rangle$$
$$\supset \operatorname{Der}_+\mathcal{O} := z^2\mathbb{C}[[z]]\partial_z = \langle L_n\ (n\ge 1)\rangle.$$

Definition. A vertex algebra is called *quasi-conformal* if it carries an action of $\operatorname{Der}\mathcal{O}$ such that

- $L_{-1} = -\partial_z$ acts as T,
- $L_0 = -z\partial_z$ acts semi-simply with integral eigenvalues,
- Let $\{v_{-1}, v_0, v_1, \ldots\}$ be a set of infinite variables, and set $v(z) := \sum_{n\geq -1} v_n z^{n+1}$ and $\widetilde{v} := \sum_{n\geq -1} v_n L_n$. Then the following formula holds for any $A \in V$.
$$[\widetilde{v}, Y(A,w)] = -\sum_{m\geq -1} \frac{1}{(m+1)!}\partial_w^{m+1} v(w)\cdot Y(L_m A, w).$$
- The Lie subalgebra $\operatorname{Der}_+\mathcal{O}$ of $\operatorname{Der}\mathcal{O}$ acts locally nilpotently.

A typical example is the *conformal vertex algebra*, which is defined to be a vertex algebra with a Virasoro element and a $\mathbb{Z}$-gradation bounded below. Vertex algebras in Example 1.1 are conformal, and therefore quasi-conformal.

The nilpotency axiom of a quasi-conformal algebra V enables us to exponentiate the action of $\operatorname{Der}\mathcal{O}$ on V to the action of $\operatorname{Aut}\mathcal{O} = \operatorname{Aut}\mathbb{C}[[z]]$, the group of topological automorphisms of $\mathcal{O}$. In order to see it, we remark

Lemma 1.2 ([**8**, 6.2.1 Lemma]). Consider the following Lie groups

$$\operatorname{Aut}_+\mathcal{O} := \{z + a_2 z^2 + \cdots\} \subset \operatorname{Aut}\mathcal{O} = \{a_1 z + a_2 z^2 + \cdots \mid a_1 \neq 0\}$$

and Lie algebras

$$\operatorname{Der}_+\mathcal{O} := z^2\mathbb{C}[[z]]\partial_z \subset \operatorname{Der}_0\mathcal{O} := z\mathbb{C}[[z]]\partial_z \subset \operatorname{Der}\mathcal{O} = \mathbb{C}[[z]]\partial_z \tag{1.3.1}$$

1. $\operatorname{Aut}\mathcal{O}$ is a semi-direct product of the one-dimensional multiplicative group $\mathbb{G}_m$ and the group $\operatorname{Aut}_+\mathcal{O}$.
2. $\operatorname{Aut}_+\mathcal{O}$ has a structure of pro-unipotent pro-algebraic group.
3. $\operatorname{Lie}(\operatorname{Aut}\mathcal{O}) = \operatorname{Der}_0\mathcal{O}$ and $\operatorname{Lie}(\operatorname{Aut}_+\mathcal{O}) = \operatorname{Der}_+\mathcal{O}$.
4. The exponential map $\exp : \operatorname{Der}_+\mathcal{O} \to \operatorname{Aut}_+\mathcal{O}$ is an isomorphism.

1.4. Vertex algebra bundles

Let X be a smooth complex curve. For a point $x \in X$, denote by $\mathcal{O}_x$ the completion of the local ring at x, $\mathcal{K}_x$ the field of fractions of $\mathcal{O}_x$. We also set $D_x := \operatorname{Spec}\mathcal{O}_x$, the disc with x the center, and $S_x^{\times} := \operatorname{Spec} shK_x$, the punctured disc at x.

A choice t_x of formal coordinate at x corresponds to a choice of isomorphism $\mathcal{O}_x \xrightarrow{\sim} \mathcal{O} = \mathbb{C}[[z]]$. Let us denote by $\mathcal{A}ut_x$ the set of all formal coordinates at x.

The group $\operatorname{Aut}\mathcal{O}$ acts on $\mathcal{A}ut_x$ so that $\mathcal{A}ut_x$ is an $\operatorname{Aut}\mathcal{O}$-torsor (principal bundle of infinite rank). On the curve X, these torsors form an $\operatorname{Aut}\mathcal{O}$-torsor which we denote by Aut_X.

Now let V be a quasi-conformal vertex algebra. By the argument in the last subsection, V has an $\operatorname{Aut}\mathcal{O}$-action. Let

$$\mathcal{V}_X := \mathcal{A}ut_X \times_{\operatorname{Aut}\mathcal{O}} V.$$

Precisely speaking, V is of infinite dimension so that we need to take a completion with respect to a filtration on V. Such a filtration comes from the action of the operator $L_0 = -z\partial_z \in \operatorname{Der}\mathcal{O}$ on V, since the semisimplicity yields a $\mathbb{Z}$-gradation $V = \oplus_n V_n$. Let us call the obtained torsor $\mathcal{V}_X$ the *vertex algebra bundle*.

The state-field correspondence Y gives the following important property of the vertex algebra bundle. Hereafter we sometimes suppress the subscript X and denote $\mathcal{V}$ if confusion may not occur. Denote by $\mathcal{O}$ the sheaf of differentials on X.

Fact 1.3 ([**8**, §6.3]). Let $U \subset X$ be an open subset and z a coordinate on U. Define

$$\nabla : \mathcal{V} \longrightarrow \mathcal{V} \otimes \Omega, \quad \nabla_{\partial_z} := \partial_z + L_{-1}.$$

Then ∇ is a well-defined connection on $\mathcal{V}$ and independent of the choice of z.

Thus every quasi-conformal vertex algebra V gives rise a left $\mathcal{D}$-module $\mathcal{V}$ on a smooth curve X.

2. Chiral algebras and factorization algebras

We quickly review the notions of chiral algebras and factorization algebras which are sheaf theoretic reformulations of vertex algebras. These notions are introduced by Beilinson and Drinfeld in [**3**]. We also cite [**9**] and [**8**, Chap. 19, 20] for nice previews of the theory of chiral algebras.

2.1. Recollection on $\mathcal{D}$-modules

Since we need to use the language of $\mathcal{D}$-modules extensively, let us give a brief recollection. For a detailed presentation, see [**11**, Chap. 1] for example.

Let Z be a smooth algebraic variety over $\mathbb{C}$ (or a field of characteristic 0). Denote by $\mathcal{M}_{\mathcal{O}}(Z)$ the category of $\mathcal{O}$-modules on Z (recall §0.3). Also denote by $\mathcal{M}^{\ell}(Z)$ and $\mathcal{M}(Z)$ the categories of left and right $\mathcal{D}$-modules on Z respectively. (In this note the smooth case is enough. For general case, we need the Kashiwara lemma.)

Recall that the canonical sheaf

$$\omega_Z := \wedge^{\dim Z}\Omega_Z$$

has a natural right $\mathcal{D}$-module structure determined by

$$\nu \cdot \tau := -\mathcal{L}ie_\tau(\nu), \quad \nu \in \omega_Z,\ \tau \in \Theta_Z \subset \mathcal{D}_Z$$

where $\mathcal{L}ie_\tau$ is the Lie derivative with respect to τ. Then the two categories $\mathcal{M}(Z)$ and $\mathcal{M}^\ell(Z)$ are equivalent under the functor

$$\mathcal{M}^\ell(Z) \longrightarrow \mathcal{M}(Z), \quad \mathcal{L} \longmapsto \mathcal{L}^r := \omega_Z \otimes_{\mathcal{O}_Z} \mathcal{L},$$

where the structure of right $\mathcal{D}$-module on $\mathcal{L}^r$ is determined by

$$(\nu \otimes l) \cdot \tau := (\nu \cdot \tau) \otimes l - \nu \otimes (\tau \cdot l).$$

The inverse functor is given by

$$\mathcal{M} \longmapsto \mathcal{M}^\ell := \omega_Z^{-1} \otimes_{\mathcal{O}_Z} \mathcal{M}. \tag{2.1.1}$$

For a morphism $f : Z_1 \to Z_2$ of smooth varieties, the inverse image functor f^* of $\mathcal{D}$-module as follows. As an $\mathcal{O}$-module we set

$$f^* : \mathcal{M}^\ell(Z_2) \longrightarrow \mathcal{M}^\ell(Z_1), \quad f^*\mathcal{L} := \mathcal{O}_{Z_1} \otimes_{f^\cdot \mathcal{O}_{Z_2}} f^\cdot \mathcal{L}.$$

Recall that we have a morphism

$$\Theta_{Z_1} \longrightarrow f^*\Theta_{Z_2} = \mathcal{O}_{Z_1} \otimes_{f^\cdot \mathcal{O}_{Z_2}} f^\cdot \Theta_{Z_2}, \quad \tau \longmapsto \widehat{\tau}$$

of $\mathcal{O}_{Z_1}$-modules, defined by taking the $\mathcal{O}_{Z_1}$-dual of the morphism $\mathcal{O}_{Z_1} \otimes_{f^\cdot \mathcal{O}_{Z_2}} f^\cdot \Omega_{Z_2} \to \Omega_{Z_1}$. Then the left $\mathcal{D}$-module structure on $f^*\mathcal{L}$, is determined by

$$\tau(s \otimes l) := \tau(s) \otimes l + s\widehat{\tau}(s), \quad \tau \in \Theta_{Z_1}, \quad s \in \mathcal{O}_{Z_1}, l \in \mathcal{L}.$$

The direct image functor f_* is naturally defined in the derived category, so we make a detour. Let us introduce the $\mathcal{D}_{Z_1}$-$f^\cdot\mathcal{D}_{Z_2}$-bimodule

$$\mathcal{D}_{Z_1 \to Z_2} := \mathcal{O}_{Z_1} \otimes_{f^\cdot \mathcal{O}_{Z_2}} f^\cdot \mathcal{D}_{Z_2}$$

for a morphism $f : Z_1 \to Z_2$ of smooth varieties. Then we have an isomorphism

$$f^*\mathcal{L} \simeq \mathcal{D}_{Z_1 \to Z_2} \otimes_{f^\cdot \mathcal{D}_{Z_2}} f^\cdot \mathcal{L}$$

of left $\mathcal{D}_{Z_1}$-modules.

The abelian category $\mathcal{M}^\ell(Z)$ has the following nice property.

FACT. Assume Z is quasi-projective. Then any $\mathcal{L} \in \mathcal{M}^\ell(Z)$ is a quotient of a locally free left $\mathcal{D}_Z$-module, so that it has a resolution by locally free $\mathcal{D}_Z$-modules.

$\mathcal{M}(Z)$ also has a similar property.

Denote by $D^b\mathcal{M}^\ell(Z)$ the bounded derived category of left $\mathcal{D}$-modules on a smooth quasi-projective variety Z. By the above fact any object of $D^b\mathcal{M}^\ell(Z)$ is represented by a bounded complex of locally free $\mathcal{D}_Z$-modules. We also denote by $D^b\mathcal{M}(Z)$ the bounded derived category of right $\mathcal{D}$-modules. We have an equivalence

$$D^b\mathcal{M}^\ell(Z) \xrightarrow{\sim} D^b\mathcal{M}(Z), \quad \mathcal{L} \longmapsto \mathcal{L}^r[\dim Z]. \tag{2.1.2}$$

Therefore $D^b\mathcal{M}(Z)$ has two t-structures given by $\mathcal{M}(Z)$ and $\mathcal{M}^\ell(Z)$.

Now let $f : Z_1 \to Z_2$ be a morphism of smooth quasi-projective varieties. Define a functor Lf^* by

$$Lf^*(\mathcal{L}^\bullet) := \mathcal{D}_{Z_1 \to Z_2} \otimes^L_{f^\cdot \mathcal{D}_{Z_2}} f^\cdot \mathcal{L}^\bullet$$

by using a locally free resolution of $\mathcal{L}^\bullet \in D^b\mathcal{M}^\ell(Z_1)$. Then we have

FACT. Lf^* sends $D^b\mathcal{M}^\ell(Z_2)$ to $D^b\mathcal{M}^\ell(Z_1)$.

Lf^* is called the *derived inverse image functor*. Let us also recall the *shifted inverse image functor*

$$f^! := Lf^*[\dim Z_1 - \dim Z_2] : \ D^b\mathcal{M}^\ell(Z_2) \longrightarrow D^b\mathcal{M}^\ell(Z_1).$$

We denote by the same symbol $f^! : D^b\mathcal{M}(Z_2) \longrightarrow D^b\mathcal{M}(Z_1)$ induced by the equivalence (2.1.2).

Now we can treat the definition of the direct image functor. Let $f : Z_1 \to Z_2$ be a morphism of smooth algebraic varieties. Consider the direct image functor $f_\cdot$ for $\mathcal{O}$-modules. We have the derived functor

$$Rf_\cdot : D^b\mathcal{M}_\mathcal{O}(Z_1) \longrightarrow D^b\mathcal{M}_\mathcal{O}(Z_2).$$

Assuming that Z_1 and Z_2 are quasi-projective, we define the derived direct image functor f_* by

$$f_*(\mathcal{M}^\bullet) := Rf_\cdot(\mathcal{M}^\bullet \otimes^L_{\mathcal{D}_X} \mathcal{D}_{Z_1 \to Z_2})$$

for $\mathcal{M}^\bullet \in D^b\mathcal{M}(Z_1)$. Then we have

FACT. f_* sends $D^b\mathcal{M}(Z_1)$ to $D^b\mathcal{M}(Z_2)$.

A non-trivial result is

Fact. If f is a closed embedding, then for $\mathcal{M} \in \mathcal{M}^{\ell}(Z_1)$ we have

$$H^k(f_* \mathcal{M}) = 0 \quad (k \neq 0).$$

The 0th cohomology gives an exact functor

$$H^0 f_* : \mathcal{M}(Z_1) \longrightarrow \mathcal{M}(Z_2).$$

We will simply write $f_! := H^0 f_*$.

Thus we have two functors

$$f_* : D^b\mathcal{M}(Z_1) \leftrightarrows D^b\mathcal{M}(Z_2) : f^!.$$

We also note that

Fact. Consider the t-structure of $D^b\mathcal{M}(Z)$ given by $\mathcal{M}(Z)$.

1. If f is quasi-finite, then f_* is left exact.
2. If f is affine, then f_* is right exact.

Hereafter the term "a $\mathcal{D}$-module" means a right $\mathcal{D}$-module.

2.2. Chiral algebra

The notion of chiral algebra is introduced by Beilinson and Drinfeld in [**3**] to formulate the vertex algebra in terms of $\mathcal{D}$-modules and operads. We will not explain the formalism of operads here, and give a review of chiral algebras following the presentation in [**8**, Chap. 19]. For a more detailed review including operads, see [**17**] for example.

Let X be a smooth complex curve as in the previous §1.4. Denote by $\mathcal{D} = \mathcal{D}_X$ the sheaf of differential operators and by $\Omega = \Omega_X$ the sheaf of differential forms on X. We also denote by

$$\Delta : X \hookrightarrow X^2, \quad j : X^2 \setminus \Delta \hookrightarrow X^2 \tag{2.2.1}$$

the embeddings of the diagonal divisor and its complement.

Note that for $\mathcal{O}$-modules $\mathcal{M}$ and $\mathcal{N}$ on X we can identify

$$j_\cdot j^\cdot(\mathcal{M} \otimes \mathcal{N}) \simeq (\mathcal{M} \otimes \mathcal{N})(\infty\Delta)$$

on X^2. Recall also the direct image functor $\Delta_!$ of $\mathcal{D}$-modules, which is expressed as

$$\Delta_! \mathcal{M} = (\Omega \boxtimes \mathcal{M})(\infty\Delta)/\Omega \boxtimes \mathcal{M}.$$

On the other hand, we have

$$\Omega^{\boxtimes 2} \xrightarrow{\sim} \omega_{X^2}, \quad dz \boxtimes dw \longmapsto dz \wedge dw,$$

where ω denotes the canonical sheaf. We also have $\Delta_! \Omega \simeq \omega_{X^2}(\infty\Delta)/\omega_{X^2}$. Let us define the composition of morphisms

$$\mu_\Omega : j_* j^* \Omega \xrightarrow{\sim} \omega_{X^2}(\infty\Delta) \twoheadrightarrow \omega_{X^2}(\infty\Delta)/\omega_{X^2} \xrightarrow{\sim} \Delta_! \Omega \tag{2.2.2}$$

where the second morphism is the natural projection.

Definition. A *chiral algebra* on X is a right $\mathcal{D}$-module $\mathcal{A}$ together with a $\mathcal{D}_{X^2}$-module morphism $\mu : \mathcal{A}^{\boxtimes 2}(\infty\Delta) \to \Delta_! \mathcal{A}$ called the *chiral multiplication* and an embedding $\Omega \hookrightarrow \Delta_! \mathcal{A}$ called the unit, satisfying the following axioms.

- The skew-symmetry $\mu = -\sigma_{12} \circ \mu \circ \sigma_{12}$, where the permutation acts on the factors of X^2.
- The Jacobi identity for μ.
- The unit map is compatible with the morphism μ_Ω in (2.2.2).

Roughly speaking, the first and second conditions on the chiral product μ indicate that a chiral algebra is a Lie object in a "tensor category" of right $\mathcal{D}$-modules with the non-standard tensor structure. See [**3**, Chap. 2,3] for the precise definition of this non-standard tensor structure which is called the chiral pseudo-tensor structure.

In order to relate chiral algebras and vertex algebras, we need to restate the state-field correspondence Y in the language of $\mathcal{D}$-modules. Recall that the canonical sheaf ω_X has a natural right $\mathcal{D}$-module structure determined by

$$\nu\tau := -\mathcal{L}ie_\tau(\nu), \quad \nu \in \omega_X, \ \tau \in \Theta_X \subset \mathcal{D}_X$$

with $\mathcal{L}ie_\tau$ the Lie derivative with respect to τ. For a left $\mathcal{D}$-module $\mathcal{M}$ on X, set

$$\mathcal{M}^r := \omega_X \otimes \mathcal{M}$$

which is naturally a right $\mathcal{D}$-module by

$$(\nu \otimes m)\tau := \nu\tau \otimes m - \nu \otimes \tau m.$$

For a quasi-conformal vertex algebra V, we have the vertex algebra bundle $\mathcal{V}$ which is a left $\mathcal{D}$-module by Fact 1.3. Then a local section of $\mathcal{V}^r$ on D_x with coordinate z can be expressed as $f(z)Adz$ with $f(z) \in \mathcal{O}_X(D_x)$ and $A \in V$. Recall the state-field correspondence Y for V. Then we have

Fact ([**8**, §19.2.1–19.2.2.6]). Define a morphism

$$(\mathcal{Y}^2)^r_x : j_* j^* (\mathcal{V}^{\boxtimes 2})^r \longrightarrow \Delta_! \mathcal{V}^r$$

of $\mathcal{O}$-modules on D^2_x by

$$(\mathcal{Y}^2)^r_x (f(z,w) A dz \boxtimes B dw) := f(z,w) Y(A, z-w) B dz \wedge dw \mod V[[z,w]]$$

Then $(\mathcal{Y}^2)^r_x$ is well defined and independent of the choice of z. Moreover the morphisms $(\mathcal{Y}^2)^r_x$ for $x \in X$ form a morphism

$$(\mathcal{Y}^2)^r : j_* j^* (\mathcal{V}^{\boxtimes 2})^r \longrightarrow \Delta_! \mathcal{V}^r$$

of $\mathcal{D}$-modules on X^2.

Now we can state the relation of chiral algebras and vertex algebras.

Fact ([**8**, §19.3.3], [**3**, §3.3]). For a quasi-conformal vertex algebra V and a smooth curve X, the right $\mathcal{D}$-module $\mathcal{V}^r$ carries the structure of a chiral algebra with the chiral multiplication $\mu = (\mathcal{Y}^2)^r$.

Definition 2.1. The chiral algebras $\mathcal{V}^r$ associated to the vertex algebras $V = \pi_{k,0}$, $V_k(\mathfrak{g})$, Vir_c in Example 1.1 will be called the *Heisenberg*, *affine*, and *Virasoro chiral algebra* respectively.

2.3. Factorization algebra

The notion of factorization algebra is introduced in [**3**, §3.4] as an equivalent notion of chiral algebra. Its origin goes back to the geometric Langlands correspondence, but we will not touch this topic.

We will repeatedly use the following category of sets.

Definition 2.2. Let $\mathcal{S}$ be the category of finite non-empty sets and surjections. For $\pi : J \twoheadrightarrow I$ in $\mathcal{S}$ and $i \in I$ we set $J_i := \pi^{-1}(i) \subset J$.

Let X be a smooth complex curve as before. For $\pi : J \twoheadrightarrow I$ in $\mathcal{S}$, denote by

$$\Delta^{(\pi)} \equiv \Delta^{(J/I)} : X^I \hookrightarrow X^J \tag{2.3.1}$$

an embedding of the locus such that $x_j = x_{j'}$ if $\pi(j) = \pi(j')$ for $j, j' \in J$. Also set

$$U^{(\pi)} \equiv U^{(J/I)} := \left\{ (x_j)_{j \in J} \in X^J \mid x_j \neq x_{j'} \text{ if } \pi(j) \neq \pi(j') \right\}.$$

One can also see that $U^{(\pi)}$ is the complement of the diagonals that are transversal to $\Delta^{(\pi)} : X^I \hookrightarrow X^J$. Denote the open embedding $U^{(\pi)} \subset X^J$ by

$$j^{(\pi)} \equiv j^{(J/I)} : U^{(\pi)} \hookrightarrow X^J.$$

If the surjection π is $\{1,2\} \twoheadrightarrow \{1\}$, then these notations coincide with (2.2.1).

Definition 2.3. A *factorization algebra* on X is the data consisting of

- for each $i \in \mathcal{S}$, a quasi-coherent $\mathcal{O}$-modules $\mathcal{V}_I$ over X^I
- for any surjection $J \twoheadrightarrow I$ in $\mathcal{S}$, a functorial isomorphism
$$\Delta^{(J/I)\cdot}\mathcal{V}_J \xrightarrow{\sim} \mathcal{V}_I$$
of $\mathcal{O}$-modules on X^I
- for any surjection $J \twoheadrightarrow I$ in $\mathcal{S}$, a functorial isomorphism
$$j^{(J/I)\cdot}\mathcal{V}_J \xrightarrow{\sim} j^{(J/I)\cdot}\big(\boxtimes_{i\in I}\mathcal{V}_{J_i}\big)$$
of $\mathcal{O}$-modules over $U^{(J/I)}$, called the *factorization isomorphism*
- a global section $1 \in \mathcal{V}(X)$ called the *unit*

satisfying the following conditions.

1. $\mathcal{V}_I$ should have no non-zero local sections supported on the union of all partial diagonals.
2. Set $\mathcal{V} := \mathcal{V}_{\{1\}}$ and $\mathcal{V}_2 := \mathcal{V}_{\{1,2\}}$. For every local section $f \in \mathcal{V}(U)$ with $U \subset X$ open, the section
$$1 \boxtimes f \in \mathcal{V}_2(U^2 \setminus \Delta)$$
defined by the factorization isomorphism extends across the diagonal and restricts to
$$f \in \mathcal{V} \simeq \Delta^{\cdot}\mathcal{V}_2 = \mathcal{V}_2|_{\Delta}\,.$$

One may worry that factorization algebras lack $\mathcal{D}$-module structures which appear in the definitions of chiral algebras and vertex algebra bundles. Actually the axiom of factorization algebra ensures left $\mathcal{D}$-module structures for all $\mathcal{V}_I$.

The key is the crystalline point of view. Recall that a connection on a $\mathcal{O}$-module $\mathcal{M}$ on a scheme X is equivalent to the data of a morphism
$$p_1^{\cdot}\mathcal{M}|_{N^1(\Delta)} \xrightarrow{\sim} p_2^{\cdot}\mathcal{M}|_{N^1(\Delta)}$$
of $\mathcal{O}$-modules on the first-order neighborhood $N^1(\Delta)$ of the diagonal divisor $\Delta \in X^2$. Here $p_i : X^2 \to X$ is the i-th projection.

Given a factorization algebra $\{\mathcal{V}_I\}$ on X, set $\mathcal{V} := \mathcal{V}_{\{1\}}$ and $\mathcal{V}_2 := \mathcal{V}_{\{1,2\}}$. Then we have two morphisms

$$p_1^{\cdot}\mathcal{V} = \mathcal{V} \boxtimes \mathcal{O}_X \longrightarrow \mathcal{V}_2 \longleftarrow \mathcal{O}_X \boxtimes \mathcal{V} = p_2^{\cdot}\mathcal{V}$$

defined by the left and right multiplication by the unit. By the second condition, both are isomorphisms when restricted to Δ, so that they are on the formal neighborhood of Δ. The composition of these isomorphisms gives the desired connection on $\mathcal{V}$. The same argument can be applied to $\mathcal{V}_I$ for any I.

Next we discuss the Lie property hidden in the factorization algebra. We keep the notations $\mathcal{V}$ and $\mathcal{V}_2$. The second condition says that $\mathcal{V}_2$ has no sections on the diagonal. Together with the factorization isomorphism applied to $\pi = \mathrm{id}_{\{1,2\}}$, we have an injective morphism $\mathcal{V}_2 \hookrightarrow j_* j^* \mathcal{V}^{\boxtimes 2}$ with $j = j^{(\pi)} : X^2 \setminus \Delta \hookrightarrow X^2$. Now set

$$\mathcal{Y}^2 : j_* j^* \mathcal{V}^{\boxtimes 2} \twoheadrightarrow \mathcal{V}^{\boxtimes 2}/\mathcal{V}_2.$$

By the construction the image of $\mathcal{Y}^2$ is supported on the diagonal, and we can identify

$$\mathcal{V}^{\boxtimes 2}/\mathcal{V}_2 \simeq j_* j^*(\mathcal{O} \boxtimes \mathcal{V})/(\mathcal{O} \boxtimes \mathcal{V}).$$

Also note that the factorization isomorphism applied to the permutation $\{1,2\} \to \{2,1\}$ implies that the sheaf $\mathcal{V}_2$, hence the morphism $\mathcal{Y}^2$, is symmetric under the permutation of factors of X^2. Therefore, on the right $\mathcal{D}$-module

$$\mathcal{A} := \mathcal{V}^r = \mathcal{V} \otimes \Omega_X,$$

we have an anti-symmetric morphism

$$\mu := (\mathcal{Y}^2)^r : j_* j^* \mathcal{A}^{\boxtimes 2} \longrightarrow j_* j^*(\Omega \boxtimes \mathcal{A})/(\Omega \boxtimes \mathcal{A}) = \Delta_! \mathcal{A}. \tag{2.3.2}$$

We skip the check of the Jacobi identity of μ, which comes from the Cousin complex of ω on X^3. See [**8**, §19.3.4, §20.2.2] for the detail. The unit of $\{\mathcal{V}_I\}_I$ yields

$$u : \Omega \longrightarrow \mathcal{A} = \mathcal{V}^r \tag{2.3.3}$$

which satisfies the conditions of the unit of chiral algebra. Now the conclusion of this subsection is

Fact 2.4 ([**3**, §3.4]). Factorization algebras and chiral algebras on a smooth curve X are equivalent under the assignment

$$\{\mathcal{V}_I\}_I \longmapsto \{\mathcal{A} = \mathcal{V}^r, \mu, u\}$$

with μ given by (2.3.2) and u given by the construction (2.3.3)

3. Factorization space

We follow [**8**, §20.4.1], [**9**, Chap. 5] and [**12**].

3.1. Ran space

Factorization algebras are defined in terms of the category $\mathcal{S}$ of finite sets with surjections (recall Definition 2.2). One may reformulate factorization algebras by considering the system $\{X^I\}$ of schemes parametrized by $I \in \mathcal{S}$ together with some morphisms between them. This is the origin of the Ran space named in [**3**] after the work of Z. Ran.

Actually, Ran introduced the limit object of this system in [**14**] for constructing a universal deformation theory. See also [**7, 10**] for related studies. Let us recall the original definition of the Ran space here.

Definition. For a topological space X, denote by $\mathcal{R}(X)$ the *Ran space* which is the set of all non-empty finite subsets in X with the strongest topology such that the following obvious map is continuous for any finite index set I.

$$r_I : X^I \longrightarrow \mathcal{R}(X)$$

The point of $\mathcal{R}(X)$ associated to a finite subset $S \subset X$ is denoted by $[S]$.

Recall the diagonal map $\Delta^{(J/I)} : X^I \hookrightarrow X^J$ in (2.3.1). For any surjection $J \twoheadrightarrow I$ we have $r_J \Delta^{(J/I)} = r_I$, and $\mathcal{R}(X)$ is the inductive limit of the spaces X^I with respect to these embeddings $\Delta^{(J/I)}$.

For $n \in \mathbb{Z}$, denote by $\mathcal{R}(X)_n$ the subspace of $\mathcal{R}(X)$ consisting of $[S]$ such that $|S| \le n$. Then we have

$$r_n = r_{\{1,\dots,n\}} : X^n \twoheadrightarrow \mathcal{R}(X)_n = X^n/\sim,$$

where $(x_i)_{i=1}^n \sim (x'_i)_{i=1}^n$ if and only if $\{x_i\} = \{x'_i\}$.

$\mathcal{R}(X)$ is a commutative semi-group under the continuous map

$$u : \mathcal{R}(X) \times \mathcal{R}(X) \longrightarrow \mathcal{R}(X), \quad ([S],[T]) \longmapsto [S \cup T].$$

Denoting by $u_{m,n}$ the restriction of u to $\mathcal{R}(X)_m \times \mathcal{R}(X)_n$, we have the relation

$$r_{m+n} = u_{m,n} \circ (r_m \times r_n).$$

The subspaces $\mathcal{R}(X)_n$ form an increasing filtration

$$\begin{aligned} \mathcal{R}(X)_0 = \varnothing \subset \mathcal{R}(X)_1 = X \subset \mathcal{R}(X)_2 = \mathrm{Sym}^2(X) \subset \mathcal{R}(X)_3 \subset \cdots \\ \cdots \subset \mathcal{R}(X)_\infty := \mathcal{R}(X). \end{aligned}$$

Here $\mathrm{Sym}^n(X) := X^n/\mathfrak{S}_n$ is the usual symmetric product. Denoting $\mathcal{R}(X)_n^\circ := \mathcal{R}(X)_n \setminus \mathcal{R}(X)_{n-1}$, we have $\mathcal{R}(X)_n^\circ = U^{(n)}/\mathfrak{S}_n$ with $U^{(n)} := X^n \setminus \cup$ (partial diagonals). Thus $\mathcal{R}(X)_n^\circ$ is nothing but the configuration space of n points in X.

3.2. Factorization algebra as sheaves on the Ran space

Beilinson and Drinfeld used the Ran space to rebuild the theory of chiral algebras and factorization algebras. The point is that these notions can be seen as sheaves on the Ran space. Let us briefly explain this reformulation following [**3**, §3.4.1].

In this subsection X denotes a scheme with finite cohomological dimension. Sheaves on schemes mean the ones in the étale topology. Let $\mathcal{S}$ be the category of finite non-empty sets and surjections.

Definition. An $\mathcal{O}$*-module on* $\mathcal{R}(X)$ is a rule F assigning to each $I \in \mathcal{S}$ an $\mathcal{O}_{X^I}$-module F_I and to each $\pi : J \twoheadrightarrow I$ in $\mathcal{S}$ an isomorphism

$$\nu_F^{(\pi)} : \Delta^{(\pi)\cdot} F_J \xrightarrow{\sim} F_I$$

of $\mathcal{O}_{X^I}$-modules compatible with the composition of surjections, namely for any $\rho : K \twoheadrightarrow J$ and $\pi : J \twoheadrightarrow I$ we have

$$\nu_F^{(\pi)} \circ \Delta^{(\pi)\cdot}(\nu_F^{(\rho)}) = \nu_F^{(\pi\circ\rho)},$$

and also have $\nu_F^{(\mathrm{id})} = \mathrm{id}_{F_I}$. Denote by $\mathcal{M}_{\mathcal{O}}(\mathcal{R}(X))$ the category of $\mathcal{O}$-modules on $\mathcal{R}(X)$.

Then a factorization algebra (without unit) can be reformulated as follows.

Definition 3.1. A factorization structure on $F \in \mathcal{M}_{\mathcal{O}}(\mathcal{R}(X))$ is the set of isomorphisms

$$c^{(\pi)} \equiv c^{(J/I)} : j^{(J/I)\cdot}\big(\boxtimes_{i\in I} F_{J_i}\big) \xrightarrow{\sim} j^{(J/I)\cdot} F_J$$

of $\mathcal{O}$-modules on $U^{(J/I)}$ for every $\pi : J \twoheadrightarrow I$ in $\mathcal{S}$, satisfying the following conditions.

1. For every $\rho : K \twoheadrightarrow J$ and $\pi : J \twoheadrightarrow I$ in $\mathcal{S}$,

$$c^{(K/J)} = c^{(K/I)}\big(\boxtimes_{i\in I} c^{(K_i/J_i)}\big).$$

   Here we set $c^{(K/I)} := c^{(\sigma)}$ with $\sigma := \pi\circ\rho : K \twoheadrightarrow I$, and $c^{(K_i/J_i)} := c^{(\rho_i)}$ with $\rho_i := \rho|_{\sigma^{-1}(i)} : \sigma^{-1}(i) \twoheadrightarrow \pi^{-1}(i)$.

2. For every $K \twoheadrightarrow J$ and $J \twoheadrightarrow I$,

$$\nu_F^{(K/J)} \Delta^{(K/J)\cdot}(c^{(K/I)}) = c^{(J/I)}\big(\boxtimes_{i\in I} \nu_F^{(K_i/J_i)}\big).$$

We immediately have

Proposition 3.2. A factorization algebra in the sense of Definition 2.3 is equivalent to an object $F \in \mathcal{M}_{\mathcal{O}}(\mathcal{R}(X))$ with factorization structure in the sense of Definition 3.1.

3.3. Recollection on ind-schemes

Before introducing the factorization spaces, we need to recall the notion of ind-schemes, since otherwise we have no good examples. We follow [**12**, §1.1] for the presentation.

For any category $\mathcal{C}$, an ind-object of $\mathcal{C}$ is a filtering inductive system over $\mathcal{C}$. Ind-objects form a category where a morphism is a collection of morphisms between the objects in the inductive systems satisfying some compatibility conditions. An ind-object will be represented by the symbol "$\varinjlim_i$"C_i with $C_i \in \mathcal{C}$.

Denote by $\mathcal{S}ch$ the category of separated schemes over $\mathbb{C}$. An *ind-scheme* is an ind-object of $\mathcal{S}ch$ represented by an inductive system of schemes. For example, formal schemes are ind-schemes, like

$$\operatorname{Spf} \mathbb{C}[[t]] = \text{“}\varinjlim_n\text{”} \operatorname{Spec} \mathbb{C}[t]/(t^{n+1}).$$

A *strict ind-scheme* is an ind-scheme with an inductive system given by closed embeddings of quasi-compact schemes.

Recall that a scheme S is equivalent to the functor of points $F_s : T \mapsto \operatorname{Hom}_{\mathcal{S}ch}(T, S)$. This functor is a sheaf of sets on $\mathcal{S}ch$ which is seen as the Zariski site. Let us call such a sheaf a $\mathbb{C}$-space. An ind-scheme is a $\mathbb{C}$-space represented by an inductive system of schemes.

One can define an ind-scheme over a scheme Z similarly by replacing the category $\mathcal{S}ch$ by the category $\mathcal{S}ch_Z$ of schemes over Z. For a morphism $f : Z_1 \to Z_2$ of schemes, denote by f_* and f^* the push-forward and the pull-back of ind-schemes over Z_i's. Although these notations are the same as those for functors of $\mathcal{D}$-modules, let us use them for the simplicity of symbols.

3.4. Definition of factorization space

Now we turn to our main object, factorization space. It is a non-linear counterpart of factorization algebra, which can be seen as sheaves on the Ran space. Recall once again the category $\mathcal{S}$ in Definition 2.2.

Definition 3.3. Let X be a scheme. A *factorization space* $\mathcal{G}$ on X consists of the following data.

- A formally smooth ind-scheme $\mathcal{G}_I$ over X^I for each $I \in \mathcal{S}$.

- An isomorphism

$$\nu^{(\pi)} \equiv \nu^{(J/I)} : \Delta^{(\pi)*}\mathcal{G}_J \xrightarrow{\sim} \mathcal{G}_I$$

of ind-schemes over X^I for each $\pi : J \twoheadrightarrow I$ in $\mathcal{S}$.

- An isomorphism

$$\kappa^{(\pi)} \equiv \kappa^{(J/I)} : j^{(\pi)*}\big(\prod_{i\in I} \mathcal{G}_{J_i}\big) \xrightarrow{\sim} j^{(\pi)*}\mathcal{G}_J$$

of ind-schemes over $U^{(\pi)}$ for each $\pi : J \twoheadrightarrow I$ in $\mathcal{S}$, called the *factorization isomorphism*.

These should satisfy the following compatibility conditions.

1. $\nu^{(\pi)}$'s are compatible with compositions of surjections π.
2. For any $\pi : J \twoheadrightarrow I$ and $\rho : K \twoheadrightarrow J$, we should have

$$\kappa^{(K/J)} = \kappa^{(K/I)}\big(\boxtimes_{i\in I}\kappa^{(K_i/J_i)}\big).$$

3. For any $J \twoheadrightarrow I$ and $K \twoheadrightarrow J$, we should have

$$\nu^{(K/J)}\Delta^{(K/J)*}\big(\kappa^{(K/I)}\big) = \kappa^{(J/I)}\big(\boxtimes_{i\in I}\nu^{(K_i/J_i)}\big).$$

A factorization space $\mathcal{G}$ is attached with the structure morphism $r^{(I)} : \mathcal{G}_I \to X^I$ of ind-schemes. Thus a factorization space can be considered as an ind-scheme over the Ran space $\mathcal{R}(X)$.

3.5. Units and connections

Let us also introduce the object corresponding to the unit of factorization algebra.

Definition. A *unit* of a factorization space $\mathcal{G}$ on a scheme X is the collection of morphisms

$$u^{(I)} : X^I \longrightarrow \mathcal{G}_I$$

of ind-schemes for each $I \in \mathcal{S}$ such that for any morphism $f : U \to \mathcal{G}_{\{1\}}$ with open $U \subset X$, $u^{(\{1\})} \boxtimes f$, which can be seen as a morphism $U^2 \setminus \Delta \to \mathcal{G}_{\{1,2\}}$ by $\kappa^{(\pi:\{1,2\}\twoheadrightarrow\{1\})}$, extends to a morphism $U^2 \to \mathcal{G}_{\{1,2\}}$, and $\Delta^{(\pi)*}(u^{(\{1\})} \boxtimes f) = f$.

Recall the argument in §2.3 of the connection on a factorization algebra. One can make a similar argument for a factorization space. The result is

Proposition 3.4. A factorization space $\mathcal{G}$ on X together with a unit has a connection along X.

Let us explain precisely what a connection on $\mathcal{G}$ along X is. Assume that we are given

- a local Artin scheme T of length 1,
- a morphism $f : T \times S \to X$ of schemes with a scheme S,
- $g_0 : T_0 \times S \to \mathcal{G}_{\{1\}}$ of ind-schemes with $T_0 := T_{\mathrm{red}} \simeq \mathrm{Spec}(\mathbb{C})$ the reduced scheme of T

such that $r^{(1)} \circ g_0 : T_0 \times S \to \mathcal{G}_{\{1\}} \to X$ coincides with the composition $T_0 \times S \to T \times S \to X$. A connection on $\mathcal{G}$ along X is equivalent to the property that for given (T, f, g_0) there is a map $g : T \times S \to \mathcal{G}$ such that $r \circ g = f$ extending g_0.

3.6. Linearization of factorization spaces

Let $\mathcal{G}$ be a factorization space over a smooth scheme X with a unit. Recall the morphisms $r^{(I)} : \mathcal{G}_I \to X^I$ and $u^{(I)} : X^I \to \mathcal{G}_I$. Consider the $\mathcal{O}$-module

$$\mathcal{A}_{\mathcal{G},I} := r^{(I)}_* u^{(I)}_! \omega_{X^I}.$$

This sheaf can be considered as the space of delta functions on $\mathcal{G}_I$ along the section $u^{(I)}(X^I)$.

The connection on $\mathcal{G}$ along X given in Proposition 3.4 defines a right $\mathcal{D}$-module structure on $\mathcal{A}_{\mathcal{G},I}$ and the section $u^{(I)}$ defines an embedding $\omega_{X^I} \hookrightarrow \mathcal{A}_{\mathcal{G},I}$. Then the axiom of factorization space implies

Proposition 3.5. If X is a smooth curve, then the collection $\{\mathcal{A}^{\ell}_{\mathcal{G},I}\}_{I \in \mathcal{S}}$ has a structure of factorization algebra on X, and hence $\mathcal{A}_{\mathcal{G},\{1\}}$ has a structure of chiral algebra on X.

Here we used the notation (2.1.1). As a corollary, $\mathcal{A}_{\mathcal{G},\{1\}}$ has a sturcutre of chiral algebra.

Definition. We call the obtained chiral algebra $\mathcal{A}_{\mathcal{G},\{1\}}$ the *chiral algebra associated to* $\mathcal{G}$.

3.7. Twisted version

One can make a twist on this construction. We start with

Definition. Let $\mathcal{G}$ be a factorization space over a smooth scheme X. A *factorization line bundle* $\mathcal{L}$ over $\mathcal{G}$ is a collection of line bundles $\mathcal{L}_I$ on $\mathcal{G}_I$ together with isomorphisms

$$j^{(J/I)\cdot}\mathcal{L}_I \xrightarrow{\sim} j^{(J/I)\cdot}\big(\otimes_{i \in I}\mathcal{L}_{J_i}\big)$$

over $U^{(J/I)}$ which should satisfy the factorization property (similar as the conditions in Definition 3.3).

Now assume X is a curve and consider the $\mathcal{O}$-modules $\mathcal{A}_{\mathcal{G},I}$. We can twist these sheaves by $\mathcal{L}_I$, and obtain the collection of sheaves

$$\mathcal{A}^{\mathcal{L}}_{\mathcal{G},I} := r^{(I)}_{*}\big(\mathcal{L}_I \otimes_{\mathcal{G}_I} u^{(I)}_{!}(\omega_{X^I})\big).$$

The construction in Proposition 3.5 can be applied to this $\mathcal{L}$-twisted sheaf.

Proposition 3.6. Assume that $\mathcal{G}$ is a factorization space with a unit over a smooth curve X, and that a factorization linear bundle $\mathcal{L}$ over $\mathcal{G}$ is given. Then $\mathcal{A}^{\mathcal{L}}_{\mathcal{G},\{1\}}$ has a structure of chiral algebra on X.

The chiral algebra $\mathcal{A}^{\mathcal{L}}_{\mathcal{G},\{1\}}$ is called the *$\mathcal{L}$-twisted chiral algebra associated to* $\mathcal{G}$.

4. The Beilinson-Drinfeld Grassmannian

In this section we explain the first example of factorization space, namely the Beilinson-Drinfeld Grassmannian $\mathcal{G}r_{G,X}$. Here G is a reductive group and X is a smooth algebraic curve. $\mathcal{G}r_{G,X} = \{\mathcal{G}r_{G,X,I}\}_{I\in\mathcal{S}}$ is a collection of moduli spaces of G-torsors on X with a trivialization away from finite points $\{x_i\}_{i\in I} \subset X$.

$\mathcal{G}r_{G,X}$ was introduced in the study of geometric Langlands correspondence, but we skip that topic. We only explain that the associated chiral algebra coincides with the affine chiral algebra (recall Definition 2.1). Let us also comment that Beilinson and Drinfeld started with the study of $\mathcal{G}r_{G,X}$, and then they reached the notion of factorization space.

Before introducing $\mathcal{G}r_{G,X}$, we recall some facts on the moduli space of G-bundles on an algebraic curve. We denote by $\mathcal{M}_G(X)$ the category of G-bundles on X.

4.1. Affine Grassmannian and moduli space of G-bundles on curve

Let G be a reductive algebraic group. Recall that the *affine Grassmannian*

$$G(\mathcal{K})/G(\mathcal{O}) = G\big(\mathbb{C}((z))\big)/G\big(\mathbb{C}[[z]]\big)$$

can be considered as the moduli space of G-bundles on the disc $D = \operatorname{Spec}\mathcal{O}$ together with a trivialization on $D^\times = \operatorname{Spec}\mathcal{K}$. Strictly speaking, $G(\mathcal{K})$ is an ind-scheme, and $G(\mathcal{K})/G(\mathcal{O})$ is a formally smooth strict ind-scheme. Recall also the following fact.

Fact 4.1 ([**2, 5**]). Let X be a smooth algebraic curve and $x \in X$ be a point.

1. A choice of local coordinate z at x gives an identification

$$\mathcal{G}r_{X,G,x} := \{(\mathcal{P},\varphi) \mid \mathcal{P} \in \mathcal{M}_G(X),\ \varphi : \text{trivialization of } \mathcal{P}|_{X\setminus\{x\}}\} \qquad (4.1.1)$$

$$\xrightarrow{\sim} G(\mathcal{K})/G(\mathcal{O}).$$

2. If G is semi-simple, then any G-bundle on $X \setminus \{x\}$ is trivial.

Let $\mathfrak{M}_G(X)$ be the moduli stack of G-bundles on a smooth projective curve X. We have a natural morphism $\mathcal{G}r_{X,G,x} \to \mathfrak{M}_G(X)$ by forgetting the trivialization φ. If G is semisimple, then Fact 4.1 implies the following adelic description of $\mathfrak{M}_G(X)$.

$$\mathfrak{M}_G(X) \simeq G(\mathcal{K}_x)_{\mathrm{out}} \backslash G(\mathcal{K}_x)/G(\mathcal{O}_x).$$

Here and hereafter we simplify the symbols $\mathcal{O}_x := \mathcal{O}_{X,x}$ and $\mathcal{K}_x := \mathcal{K}_{X,x}$. Also $G(\mathcal{K}_x)_{\mathrm{out}}$ denotes the space of regular functions $X \setminus \{x\} \to G$, which is naturally a subgroup of $G(\mathcal{K}_x)$.

4.2. Definition of Beilinson-Drinfeld Grassmannian

Let X be a smooth algebraic curve as before. Denote by $\mathcal{S}ch$ the category of schemes over $\mathbb{C}$.

For $I \in \mathcal{S}$, consider the functor which maps $S \in \mathcal{S}ch$ to the data $(f^I, \mathcal{P}, \varphi)$ consisting of

- a morphism $f^I : S \to X^I$ of schemes
- a G-torsor $\mathcal{P}$ on $S \times X$
- a trivialization φ of $\mathcal{P}$ on $S \times X \setminus \{\Gamma_{f_i^I}\}_{i \in I}$, where $f_i^I : S \to X$ is the composition of f^I with the i-th projection $X^I \to X$, and $\Gamma_s \subset S \times X$ is the graph scheme of the morphism $s : S \to X$.

By Fact 4.1, this functor can be represented by an ind-scheme $\mathcal{G}r_{X,G,I}$. For $S = \operatorname{Spec}\mathbb{C}$, the image of the functor is the collection of the data

$$\Big\{ \big(\mathcal{P}, \{x_i\}_{i \in I}, \varphi\big) \,\big|\, \mathcal{P} \in \mathcal{M}_G(X),\ x_i \in X,\ \varphi : \text{trivialization of } \ \mathcal{P}|_{X \setminus \{x_i\}_{i \in I}} \Big\}.$$

Thus the space $\mathcal{G}r_{X,G,x}$ in (4.1.1) is the special case of $|I| = 1$. We have a natural morphism

$$r^{(I)} : \mathcal{G}r_{X,G,I} \longrightarrow X^I, \quad \big(\mathcal{P}, \{x_i\}_{i \in I}, \varphi\big) \longmapsto \{x_i\}_{i \in I}.$$

It is known that $\mathcal{G}r_{X,G,I}$ is a formally smooth ind-scheme over X^I.

THEOREM 4.2. The collection

$$\mathcal{G}r_{X,G} := \{\mathcal{G}r_{X,G,I}\}_{I \in \mathcal{S}}$$

has a structure of factorization space on X.

Here we only check the axioms for $I = \{1, 2\}$. The general case can be checked similarly. Let us denote by $\mathcal{G}r_{\{x_i\}_{i\in I}}$ the fiber of $r^{(I)}$ over $\{x_i\}_{i\in I} \subset X^I$. By Fact 4.1, in the case $|I| = 1$, we have

$$\mathcal{G}r_{\{x\}} \simeq G(\mathcal{K})/G(\mathcal{O}).$$

If $x_1 \neq x_2$, then $\{x_1, x_2\} \in X^2 \setminus \Delta$, and we have a morphism $\mathcal{G}r_{\{x_1,x_2\}} \to \mathcal{G}r_{\{x_1\}} \times \mathcal{G}r_{\{x_2\}}$ by restricting the data to $X \setminus \{x_1\}$ and $X \setminus \{x_2\}$. We also have the other direction map $\mathcal{G}r_{\{x_1\}} \times \mathcal{G}r_{\{x_2\}} \to \mathcal{G}r_{\{x_1,x_2\}}$ by gluing the G-bundles over $X \setminus \{x_1, x_2\}$. If $x_1 = x_2$, then $\{x_1, x_2\} \in \Delta$, and we have an identification $\mathcal{G}r_{\{x_1,x_2\}} \xrightarrow{\sim} \mathcal{G}r_{\{x_1\}}$.

Definition 4.3. The resulting factorization space $\mathcal{G}r_{X,G}$ will be called the *Beilinson-Drinfeld Grassmannian.*

We immediately have

Lemma 4.4. $\mathcal{G}r_{X,G}$ is equipped with a unit determined by the trivial G-bundle.

In other words, define $u^{(I)} : X^I \to \mathcal{G}r_{X,G,I}$ by setting $u^{(I)}\big(\{x_i\}_{i\in I}\big)$ to be the trivial G-bundle with the obvious trivialization away from x_i's, then we have a unit of $\mathcal{G}r_{X,G}$.

4.3. Relation to afffine vertex algebra

By Lemma 4.4 and Proposition 3.5 we have the chiral algebra $\mathcal{A}_{X.G}$ associated to $\mathcal{G}r_{X,G}$. The motivation of the introduction of $\mathcal{G}r_{X.G}$ is the following theorem.

Theorem 4.5 ([**9**, Theorem 16.1]). If G is semi-simple, then the chiral algebra $\mathcal{A}_{X,G}$ is isomorphic to the affine chiral algebra of Lie(G) of level 0.

4.4. Twist construction and level of affine vertex algebra

$\mathcal{G}r_{X,G}$ has a natural factorization line bundle, so that the twisted construction (Proposition 3.6) can be applied. Assume G is semi-simple, and let $\mathfrak{g} = \mathrm{Lie}(G)$ be the corresponding Lie algebra. Recall the normalized invariant inner product (1.2.1) for $\mathfrak{g}$. It yields a central extension

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \widehat{G} \longrightarrow G(\mathcal{K}) \longrightarrow 1$$

of algebraic group ind-schemes, and hence a $\mathbb{G}_m$-torsor

$$\widehat{G}(\mathcal{K})/G(\mathcal{O}) \longrightarrow \widehat{G}(\mathcal{K})/G(\mathcal{O})$$

over the affine Grassmannian. It defines a factorization line bundle $\mathcal{L}_G$ over $\mathcal{G}r$.

Now Proposition 3.6 yields an $\mathcal{L}_G$-twisted chiral algebra. We can also construct an $\mathcal{L}_G^{\otimes k}$-twisted chiral algebra for any $k \in \mathbb{Z}$. One can prove

THEOREM 4.6 ([**8**, Proposition 20.4.3]). If G is semi-simple, then the $\mathcal{L}_G^{\otimes k}$-twisted chiral algebra associated to $\mathcal{G}r_{X,G}$ is isomorphic to the affine chiral algebra of $\mathrm{Lie}(G)$ of level k.

4.5. Abelian case

Continuing the situation in Definition 4.3, let us set $G = \mathrm{GL}_1$ and consider the Beilinson-Drinfeld Grassmaniann $\mathcal{G}r_{X,\mathrm{GL}_1}$. Still Proposition 3.5 can be applied.

We also have a factorization line bundle on $\mathcal{G}r_{X,\mathrm{GL}_1}$ by choosing an even integral bilinear form on $\mathrm{Lie}(G) = \mathfrak{gl}_1$. Such a form is determined by the choice of $N \in 2\mathbb{Z}$, so that the coweight lattice is identified with the lattice $\sqrt{N}\mathbb{Z}$. Denote by $\mathcal{L}_N$ the resulting factorization line bundle. Now we have

THEOREM 4.7 ([**9**, Chap. 6]). The $\mathcal{L}_N$-twisted chiral algebra associated to $\mathcal{G}r_{X,\mathrm{GL}_1}$ coincides with the Heisenberg chiral algebra $\pi_{\sqrt{N},0}$.

5. Moduli space of pointed curves as factorization space

We saw in the previous §4 that moduli stacks of G-torsors on an algebraic curve X give rise to the Beilinson-Drinfeld Grassmannian, and that one can construct the affine or Heisenberg vertex algebra as the associated chiral algebras. Let us now consider the Virasoro vertex algebra. Does it have a corresponding factorization space? The answer is yes, and the factorization space is related to the moduli spaces of curves as we now start to explain.

5.1. Recollection of moduli spaces of pointed curves

Let $\mathfrak{M}_{g,n}$ denote the moduli space of smooth projective curves of genus g with n distinct points. We assume the stability condition, namely

$$(g > 1, n \geq 0) \text{ or } (g = 1, n \geq 1) \text{ or } (g = 0, n \geq 3). \tag{5.1.1}$$

$\mathfrak{M}_{g,n}$ is a Deligne-Mumford stack.

Denote by $\widehat{\mathfrak{M}}_{g,n}$ the moduli space of collections $(X, \{x_i, z_i\}_{i=1}^n)$ of smooth projective curves X with distinct points $\{x_i\}$ and formal coordinates z_i at x_i. It has a natural map

$$\widehat{\mathfrak{M}}_{g,n} \longrightarrow \mathfrak{M}_{g,n}, \quad (X, \{x_i, z_i\}_{i=1}^n) \longmapsto (X, \{x_i\}_{i=1}^n). \tag{5.1.2}$$

Recalling the group scheme $\mathrm{Aut}\,\mathcal{O} = \mathrm{Aut}\,\mathbb{C}[[z]]$ of automorphisms of formal coordinates, one can find that $(\mathrm{Aut}\,\mathcal{O})^n$ acts on the fiber of this map (5.1.2) simply transitively. Thus $\widehat{\mathfrak{M}}_{g,n}$ is an $(\mathrm{Aut}\,\mathcal{O})^n$-torsor over $\mathfrak{M}_{g,n}$.

The above observation tells us that the Lie algebra $\mathrm{Lie}(\mathrm{Aut}\,\mathcal{O}) \simeq \mathrm{Der}_0\,\mathcal{O}$ (recall Lemma 1.2 and (1.3.1)) acts on the fiber of $\widehat{\mathfrak{M}}_{g,1} \to \mathfrak{M}_{g,1}$. The following statement says that $\widehat{\mathfrak{M}}_{g,1}$ has an action of the much larger Lie algebra $\operatorname{Der}\mathcal{K}$.

FACT 5.1 ([**1, 5, 13, 16**]). There is an action of the Lie algebra $(\operatorname{Der}\mathcal{K})^n$ on $\widehat{\mathfrak{M}}_{g,n}$ which is compatible with the $(\operatorname{Aut}\mathcal{O})^n$-action along the fibers of the map (5.1.2).

PROOF. Let us explain an outline of the proof in the case $n=1$. It is enough to construct a right action of the corresponding ind-group $\operatorname{Aut}\mathcal{K}$ on $\widehat{\mathfrak{M}}_{g,1}$. Let us take a $\mathbb{C}$-algebra R and consider R-points of $\operatorname{Aut}\mathcal{K}$ and $\widehat{\mathfrak{M}}_{g,1}$ (otherwise $(\operatorname{Aut}\mathcal{K})(\mathbb{C}) = (\operatorname{Aut}\mathcal{O})(\mathbb{C})$ so that we can do nothing).

Let $(X,x,z) \in \widehat{\mathfrak{M}}_{g,1}(R)$ and $\rho \in (\operatorname{Aut}\mathcal{K})(R)$. We want to define a new R-point (X_ρ, x_ρ, z_ρ). Define X_ρ to be the scheme with the same topological space as X but with the structure sheaf $\mathcal{O}_{X_\rho}$ such that $\mathcal{O}_{X_\rho}(U)$ is the subring of $\mathcal{O}_X(U\setminus\{x\})$ given by

$$\mathcal{O}_{X_\rho}(U) := \{f \in \mathcal{O}_X(U\setminus\{x\}) \mid f_x(\rho^{-1}(z)) \in R[[z]]\}.$$

Here $f_x(z) \in R((z))$ is the expansion of f at x in the coordinate z. Then X_ρ is an algebraic curve over R.

Next define $x_\rho \in X_\rho$ by the ideal of $\mathcal{O}_{X_\rho}(U)$ given by the intersection of $zR[[z]]$ with the image of $\mathcal{O}_{X_\rho}(U) \hookrightarrow R[[z]]$. Here the embedding is defined by

$$\mathcal{O}_{X_\rho}(U\setminus\{x\}) \longhookrightarrow R((z)), \quad f \longmapsto f_x\big(\rho^{-1}(z)\big).$$

By the definition of $\mathcal{O}_{X_\rho}$, it extends to an embedding $\mathcal{O}_{X_\rho}(U) \hookrightarrow R[[z]]$. The triple (X_ρ, x_ρ, z) is an R-point of $\widehat{\mathfrak{M}}_{g,1}$.

The correspondence $(X,x,z) \mapsto (X_\rho, z_\rho, z)$ defines a right action of $\operatorname{Aut}\mathcal{K}$ on $\widehat{\mathfrak{M}}_{g,1}$, extending the $\operatorname{Aut}\mathcal{O}$-action by changing the coordinate z.

The transitivity of the corresponding action of $\operatorname{Der}\mathcal{K}$ follows from the statement that for any pointed curve (X',x') over $R := \mathbb{C}[[t]]/(t^2)$ which is an infinitesimal deformation of (X,x) can be obtained by the above construction for R. However it follows from the fact that $X'\setminus\{x'\} \simeq (X \ \setminus\{x\}) \times \operatorname{Spec} R$ and the formal neighborhood of x' in X' is isomorphic to $D_x \times \operatorname{Spec} R$, which is the consequence of the Kodaira-Spencer isomorphism. □

As a corollary, we can identify the tangent spaces as

$$\begin{aligned} T_{(X,x,z)}\widehat{\mathfrak{M}}_{g,1} &\simeq \operatorname{Vect}(X\setminus\{x\})\backslash \operatorname{Der}\mathcal{K}_x/\operatorname{Der}_0\mathcal{O}_x, \\ T_{(X,x)}\mathfrak{M}_{g,1} &\simeq \operatorname{Vect}(X\setminus\{x\})\backslash \operatorname{Der}\mathcal{K}_x. \end{aligned} \tag{5.1.3}$$

Here $\operatorname{Vect}(Z)$ denotes the space of vector fields regular on Z, which is naturally a Lie algebra.

5.2. Factorization space associated to pointed curves

Mimicking the discussion on the Beilinson-Drinfeld Grassmannian in the previous section, we can construct a factorization space associated to the moduli space $\widehat{\mathfrak{M}}_{g,n}$.

Let X be a smooth projective curve of genus g and $\mathcal{S}ch$ the category of schemes over $\mathbb{C}$. In this section we only consider $I \in \mathcal{S}$ such that $(g, |I|)$ satisfies the stability condition (5.1.1).

Fix an $I \in \mathcal{S}$. Consider the functor which maps $S \in \mathcal{S}ch$ to the data $(f^I, \mathcal{X}, \{s_i\}_{i \in I}, \varphi)$ consisting of

- a morphism $f^I : S \to X^I$ of schemes
- a family $\mathcal{X} \to S$ of smooth projective curves over S
- sections $s_i : S \to \mathcal{X}$ of $\mathcal{X} \to S$ for $i \in I$
- an isomorphism

$$\varphi : \mathcal{X} \setminus \left\{\Gamma_{s_i}\right\}_{i \in I} \xrightarrow{\sim} S \times X \setminus \left\{\Gamma_{f_i^I}\right\}_{i \in I},$$

  where f_i^I is the composition of f^I with the i-th projection $X^I \to X$, and Γ_s is the graph scheme of $s : S \to X$.

As in the case of the Beilinson-Drinfeld Grassmannian in §4.2, one can write down the image of $S = \operatorname{Spec}(\mathbb{C})$ under this functor. Simply stating, it is the collection of the data $(\{x_i\}_{i \in I}, X', \{x'_i\}_{i \in I}, \varphi)$ consisting of a finite set $\{x_i\}_{i \in I}$ of points in X, an algebraic curve X' and an isomorphism

$$\varphi : X \setminus \{x_i\}_{i \in I} \xrightarrow{\sim} X' \setminus \{x'_i\}_{i \in I}$$

away from finitely many points $\{x'_i\}_{i \in I} \subset X'$. This functor is also represented by a formally smooth ind-scheme $\mathcal{G}_{X,I}$. We have a natural map $r^{(I)} : \mathcal{G}_{X.I} \longrightarrow X^I$. Similar arguments in §4.2 yields

Fact 5.2 ([**8**, §17.3]). The collection

$$\mathcal{G}_X := \{\mathcal{G}_{X,I}\}_{I \in \mathcal{S}}$$

has a structure of factorization space. It has a unit associated to the trivial family $S \times X$ and the identity morphism on X.

5.3. Relation to Virasoro vertex algebra

Now we can state the main statement in this section.

Theorem 5.3 ([**8**, §17.3]). *The chiral algebra $\mathcal{A}_X$ associated to the factorization space $\mathcal{G}_X$ coincides with the Virasoro chiral algebra of central charge 0.*

Let us explain an outline of the proof. The map $r^{(1)} : \mathcal{G}_{X,\{1\}} \to X$ enables us to consider the fiber $\mathcal{A}_{X,x}$ of the chiral algebra $\mathcal{A}_X$ at a point $x \in X$. The action of $\operatorname{Der}\mathcal{K}$ on $\widehat{\mathfrak{M}}_{g,1}$ given in Fact 5.1 induces an action

$$a_x : \operatorname{Der}\mathcal{K} \otimes \mathcal{A}_{X,x} \longrightarrow \mathcal{A}_{X,x}.$$

On the other hand, we have a map $\mathcal{G}_{X,\{1\}} \to \mathfrak{M}_{g,1}$. Also recall the description (5.1.3) of the tangent space of $\mathfrak{M}_{g,1}$. We then find that there is a natural isomorphism

$$\mathcal{A}_{X,x} \xrightarrow{\sim} U(\operatorname{Der}\mathcal{K}) \otimes_{U(\operatorname{Der}_0 \mathcal{O})} \mathbb{C}_0 = \mathrm{Vir}_0$$

(see (1.2.2)). Let us denote by $\mathcal{V}$ the vertex algebra bundle associated to Vir_0. $\mathcal{V}^r$ is the associated right $\mathcal{D}$-module. Now the key point of the proof is that we have a morphism

$$j_* j^* (\mathcal{V}^r \boxtimes \mathcal{A}_X) \longrightarrow \Delta_! \mathcal{A}_X$$

such that $\mathcal{V}^r$ acts by derivations of the chiral algebra structure and that the induced map

$$H^0(\operatorname{Spec}(\mathcal{K}_x), \mathcal{V}^r/\mathcal{V}^r\Theta_X) \otimes \mathcal{A}_{X,x} \longrightarrow \mathcal{A}_{X,x}$$

coincides with a_x. Assuming the existence of such a morphism, the result follows from a formal argument on chiral algebras (see [**9**, Chap. 6]).

5.4. Determinant line bundle on the moduli space as a factorization line bundle

The moduli space $\mathfrak{M}_{g,n}$ has a universal family $\pi : \mathfrak{X}_{g,n} \to \mathfrak{M}_{g,n}$. Let ω be the relative canonical sheaf on $\mathfrak{X}_{g,n}$ with respect to π whose fiber over $X \in \mathfrak{M}_{g,n}$ is the canonical bundle ω_X. For $\mu \in \mathbb{Z}$, the determinant line bundle $\mathcal{D}et_\mu$ on $\mathfrak{M}_{g,n}$ is defined to be

$$\mathcal{D}et_\mu := \det R\pi_\cdot \omega^{\otimes \mu}.$$

Now we have

Fact 5.4 ([**8**, §17.3]). The determinant line bundle $\mathcal{D}et_\mu$ induces a factorization line bundle $\mathcal{L}_\mu$ on $\mathcal{G}_X$ such that the $\mathcal{L}_\mu$-twisted chiral algebra associated to $\mathcal{G}_X$ is isomorphic to the Virasoro chiral algebra of central charge

$$c(\mu) = -2(6\mu^2 - 6\mu + 1).$$

Let us remark that this central charge formula was given in [**5, 13**].

6. Deformation theory and factorization space

6.1. Universal deformation

Let us briefly recall the universal deformation theory by Ziv Ran [**14, 15**]. See also [**18**].

Assume that $\mathfrak{g}$ is a sheaf of Lie algebras on a scheme X. Then we can consider the Chevalley complex $C(\mathfrak{g})$ of $\mathfrak{g}$. As an $\mathcal{O}_X$-module it is given by the symmetric algebra

$$C(\mathfrak{g}) := \mathrm{Sym}^{\bullet}_{\mathcal{O}_X}(\mathfrak{g}[1]).$$

The standard Chevalley-Eilenberg construction gives a dg Lie algebra structure on $C(\mathfrak{g})$.

Let M be an $\mathcal{O}_X$-module with $\mathcal{O}_X$-linear $\mathfrak{g}$-action. Then we have a complex

$$C(\mathfrak{g}, M) := C(\mathfrak{g}) \otimes_{\mathcal{O}_X} M$$

with a $C(\mathfrak{g})$-action.

The standard deformation theory tells us that the deformations of $\mathfrak{g}$ and M are controlled by the dg Lie algebras $C(\mathfrak{g})$ and $C(\mathfrak{g}, M)$ respectively. Namely on the schemes $\operatorname{Spec} H^0(C(\mathfrak{g}))$ and $\operatorname{Spec} H^0(C(\mathfrak{g}, M))$ we have universal deformation families of $\mathfrak{g}$ and M respectively.

6.2. Factorization spaces from deformation problems

Let us state a general construction of factorization spaces from deformation problems. We have in mind such deformation problems having fine moduli schemes, but the following assumptions are enough.

Assume that we are given

- a scheme X and an algebraic stack $\mathfrak{M}$
- an $\mathcal{O}_X$-module $\mathfrak{g}$ which is a sheaf of Lie algebras over $\mathcal{O}_X$
- a sheaf $\mathcal{E}$ on $\mathfrak{M} \times X$ which is an $\mathcal{O}_X$-module and a $\mathfrak{g}$-module

such that for each point $m \in \mathfrak{M}$, setting $E := \mathcal{E}|_{m \times X} = \mathcal{E} \otimes k(m)$ with $k(m)$ the residue field of $\mathfrak{M}$ at m, the formal completion $\widehat{\mathcal{E}}$ of $\mathcal{E}$ along $m \times X$ is isomorphic to the universal $\mathfrak{g}$-deformation of E.

Then for $I \in \mathcal{S}$ let us consider a triple $(E, \{x_i\}_{i \in I}, \varphi)$ consisting of

- $\{x_i\}_{i \in I} \subset X$
- $E = \mathcal{E}|_{m \times X}$ with some $m \in \mathfrak{M}$
- trivialization of E away from $\{x_i\}_{i \in I}$

As in §4.2 and §5.2, we have a corresponding functor $F_{\mathfrak{M},I}$ from $\mathcal{S}ch$ whose value on $\mathrm{Spec}(\mathbb{C})$ is the collection of the triples above.

Now one can show

PROPOSITION 6.1. There exists a strict ind-scheme $\mathcal{G}_{\mathfrak{M},I}$ representing the functor $F_{\mathfrak{M},I}$. If $\mathfrak{M}$ is formally smooth, then so is $\mathcal{G}_{\mathfrak{M},I}$.

The construction of factorization spaces considered in the previous sections can be applied in the present situation.

THEOREM 6.2. The collection $\{\mathcal{G}_{\mathfrak{M},I}\}_{I\in\mathcal{S}}$ has a structure of factorization space with the unit induced by the sheaf $\mathcal{E}$ having trivial $\mathfrak{g}$-module structure.

Therefore if X is a curve, then by Proposition 3.5 we obtain a chiral algebra. If we take $\mathfrak{M}$ to be the moduli stack $\mathfrak{M}_G(X)$ of G-bundles on X, $\mathfrak{g}$ to be the sheaf $\mathrm{Lie}(G)\otimes\mathcal{O}_X$ and $\mathcal{E}$ to be the tautological $\mathfrak{g}$-module associated to $\mathfrak{M}_G(X)$, then we obtain the Beilinson-Drinfeld Grassmannian. A similar consideration for the moduli space $\widehat{\mathfrak{M}}_{g,n}$ of pointed curves gives the factorization space given in §5.2.

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References

[1] Arbarello, E., DeConcini, C., Kac, V., Procesi, C., *Moduli spaces of curves and representation theory*, Comm. Math. Phys., **117** (1988), 1–36.

[2] Beauville, A., Laszlo, Y., *Un lemme de descente*, C.R. Acad. Sci. Paris, Ser. I Math., **320** (1995), 335–340.

[3] Beilinson, A., Drinfeld, V., *Chiral algebras*, American Mathematical Society Colloquium Publications, **51**, American Mathematical Society, Providence, RI, 2004.

[4] Beilinson, A., Ginzburg, V., *Infinitesimal structure of moduli spaces of G-bundles*, Internat. Math. Res. Notices, **1992** (1992), no. 4, 63–74.

[5] Beilinson, A., Schechtman, V., *Determinant bundles and Virasoro algebras*, Comm. Math. Phys., **118** (1988), 651–701.

[6] Drinfeld, V., Simpson, C., *B-structures on G-bundles and local triviality*, Math. Res. Lett. **2** (1995), 823–829.

[7] Esnault, H. Viehweg, E., *Higher Kodaira-Spencer classes*. Math. Ann., **299** (1994), no. 3, 491–527.

[8] Frenkel, E., Ben-Zvi, D., *Vertex algebras and algebraic curves*, second edition, Mathematical Surveys and Monographs, **88**. American Mathematical Society, Providence, RI, 2004.

[9] Gaitsgory, D., *Notes on 2D conformal field theory and string theory. Quantum fields and strings*, in *a course for mathematicians, Vol. 1, 2* (Princeton, NJ, 1996/1997), 1017–1089, Amer. Math. Soc., Providence, RI, 1999.

[10] Hinich, V., Schechtman, V., *Deformation theory and Lie algebra homology. I, II*, Algebra Colloq., **4** (1997), no. 2, 213–240; no. 3, 291–316.

[11] Hotta, R., Takeuchi, K., Tanisaki, T., *D-Modules, Perverse Sheaves, and Representation Theory*, Progress in Mathematics, **236**, Birkhäuser, 2008.

[12] Kapranov, M., Vasserot, E., *Vertex algebras and the formal loop space*, Publ. Math. Inst. Hautes Études Sci., No. **100** (2004), 209–269.

[13] Kontsevich, M., *The Virasoro algebra and Teichmüller spaces*, Funct. Anal. Appl., **21** (1987), no. 2, 156–157.

[14] Ran, Z., *Canonical infinitesimal deformations*, J. Algebraic Geom., 9 (2000), no. 1, 43–69.

[15] Ran, Z., *Jacobi cohomology, local geometry of moduli spaces, and Hitchin connections*, Proc. London Math. Soc., (3) **92** (2006), 545–580.

[16] Tsuchiya, A., Ueno, K., Yamada, Y., *Conformal field theory on universal family of stable curves with gauge symmetries*, in *Integrable systems in quantum field theory and statistical mechanics*, 459–566, Adv. Stud. Pure Math., **19**, Academic Press, Boston, MA, 1989.

[17] Yanagida, S., *Deformation quantization of vertex Poisson algebras*, preprint (2016), arXiv:1607.02068.

[18] Yanagida, S., *Jacobi complexes on the Ran space*, preprint (2016), arXiv:1608.07472.

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