Josai Mathematical Monographs
vol. 10 (2017), pp. 139 – 160

Tau functions and Hamiltonians of isomonodromic deformations

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Abstract. We describe the isomonodromy equations of Jimbo–Miwa–Ueno as completely integrable non-autonomous Hamiltonian systems using the isomonodromy tau functions.

1. Introduction

In [**8**] Jimbo, Miwa and Ueno established a general theory of isomonodromic deformation for a system of first order linear ordinary differential equations with rational coefficients

$$\frac{dY}{dx} = A(x)Y, \quad A(x) \in M_n(\mathbb{C}(x)). \tag{1}$$

In particular they found the *isomonodromy equations*, the systems of non-linear differential equations which govern the isomonodromic deformations. The purpose of this article is to describe the isomonodromy equations as completely integrable non-autonomous Hamiltonian systems using the (*isomonodromy*) *tau functions* introduced in [**8**].

It is known that the isomonodromy equations of Jimbo et al. can be described as Hamiltonian systems, see e.g. [**3, 7, 11, 12**]. However the explicit relationship between those descriptions and tau functions is not clear for us. Since well-known classical examples of isomonodromy equations, especially Schlesinger equations and Painlevé equations, can be described as non-autonomous Hamiltonian systems with Hamiltonians being the logarithmic derivatives of tau functions, it seems natural to expect that the tau functions always give Hamiltonians.

Another motivation comes from the quantization problem. In [**9**] Reshetikhin showed that the Knizhnik–Zamolodchikov equations are obtained from Schlesinger equations by quantization (see also [**5**]). It is thus interesting for us to find the quantization of general isomonodromy equations.

2010 *Mathematics subject classification.* Primary 34M56; Secondary 70H06.
Key words and phrases. Isomonodromic deformation, tau function, non-autonomous Hamiltonian system.

The organization of this article is as follows. Section 2 is a review on isomonodromy equations. In Section 3 we calculate the Hamiltonian vector fields associated to the logarithmic derivatives of tau functions. Results similar to Theorem 3.1 and Proposition 3.3 were announced in Harnad's talk [**6**] in 2004, so they are essentially not new. In Section 4 we examine the difference of those Hamiltonian vector fields and the infinitesimal isomonodromic deformations, and show that it gives a complete flat symplectic Ehresmann connection on our extended phase space (which is a symplectic fiber bundle over the time space) for isomonodromy equation. Finally in Section 5, we state and prove the main theorem of this article (Theorem 5.1).

Throughout this article we fix the following data:

- a standard coordinate x on $\mathbb{P}^1$,
- a positive integer n (the size of systems),
- a non-negative integer m (the number of poles in $\mathbb{C} = \mathbb{P}^1 \setminus \{\infty\}$),
- a non-negative integer $m_0 \leq m$ (the number of multiple poles in $\mathbb{C}$),
- $(r_i)_{i=0}^{m_0} \in \mathbb{Z}_{>0}^{m_0+1}$ (the order minus one of each multiple pole).

Set $G = \mathrm{GL}_n(\mathbb{C})$, $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$. Let $H \subset G$ be the standard maximal torus and $\mathfrak{h} \subset \mathfrak{g}$ its Lie algebra.

2. Isomonodromy equations

In this section let us recall the isomonodromy equations.

First, we introduce the independent variables for isomonodromy equations.

Definition 2.1. *The space of* deformation parameters *is the product* $\mathbb{D} = \mathbb{D}_{\mathrm{pl}} \times \mathbb{D}_{\mathrm{irr}}$, *where*

- *the set* $\mathbb{D}_{\mathrm{pl}}$ *consists of all tuples* $\mathbf{a} = (a_i)_{i=0}^{m}$ *of distinct points on* $\mathbb{P}^1$ *with* $a_0 = \infty$,
- *the set* $\mathbb{D}_{\mathrm{irr}}$ *consists of all tuples*
$$\mathbf{T} = (T_j^{(i)} \mid i = 0, 1, \ldots, m_0, j = 1, 2, \ldots, r_i) \in \mathfrak{h}^{\sum_{i=0}^{m_0} r_i}$$
of diagonal matrices such that each $T_{r_i}^{(i)}$ *has distinct diagonal entries.*

We use the symbols $x_0, x_1, \ldots, x_m$ as indeterminates and also as meromorphic functions on $\mathbb{P}^1 \times \mathbb{D}$ in the following way:

$$x_i = \begin{cases} 1/x & (i = 0), \\ x - a_i & (i > 0). \end{cases}$$

Also we use $\mathfrak{h}$-valued meromorphic functions

$$T_i = \frac{T_1^{(i)}}{x_i} + \frac{T_2^{(i)}}{x_i^2} + \cdots + \frac{T_{r_i}^{(i)}}{x_i^{r_i}} \quad (i = 0, 1, \ldots, m_0)$$

on $\mathbb{P}^1 \times \mathbb{D}$. Set $T_i = 0$ for $i = m_0 + 1, \ldots, m$ and write

$$T_i' = \frac{\partial T_i}{\partial x_i} = -\sum_{j=1}^{r_i} j T_j^{(i)} x_i^{-j-1} \quad (i = 0, 1, \ldots, m).$$

For $\mathbf{a} \in \mathbb{D}_{\mathrm{pl}}$, let $\mathfrak{g}(*\mathbf{a})$ be the Lie algebra of $\mathfrak{g}$-valued rational functions $A(x)$ holomorphic on $\mathbb{P}^1 \setminus \{ a_0, a_1, \ldots, a_m \}$. Then the partial fraction decomposition yields a vector space isomorphism

$$\mathfrak{g}(*\mathbf{a}) \simeq \bigoplus_{i=0}^{m} \mathcal{L}_i^-, \quad \mathcal{L}_i^- := \begin{cases} \mathfrak{g}[x_0^{-1}] & (i = 0), \\ x_i^{-1}\mathfrak{g}[x_i^{-1}] & (i > 0). \end{cases}$$

The right hand side is contained in the Lie algebra $\mathcal{L} = \bigoplus_{i=0}^{m} \mathfrak{g}(\!(x_i)\!)$ and complementary to the Lie subalgebra

$$\mathcal{L}^+ = \bigoplus_{i=0}^{m} \mathcal{L}_i^+, \quad \mathcal{L}_i^+ := \begin{cases} x_0\mathfrak{g}[\![x_0]\!] & (i = 0), \\ \mathfrak{g}[\![x_i]\!] & (i > 0). \end{cases}$$

The pairing

$$((X_i), (Y_i)) \mapsto -\operatorname*{res}_{x_0=0} \operatorname{tr}(x_0^{-2} X_0 Y_0) + \sum_{i=1}^{m} \operatorname*{res}_{x_i=0} \operatorname{tr}(X_i Y_i)$$

on $\mathcal{L}$ enables us to identify $\mathfrak{g}(*\mathbf{a})$ as (a dense subset of) the dual of $\mathcal{L}^+$ and hence induces a Poisson structure on $\mathfrak{g}(*\mathbf{a})$. The symplectic leaves of $\mathfrak{g}(*\mathbf{a})$ are finite-dimensional and given by the coadjoint orbits of the group

$$\widetilde{G}^+ := G(\mathbb{C}[\![x_0]\!])_1 \times \prod_{i=1}^{m} G(\mathbb{C}[\![x_i]\!]),$$

where $G(\mathbb{C}[\![x_i]\!]) := \mathrm{GL}_n(\mathbb{C}[\![x_i]\!])$ and $G(\mathbb{C}[\![x_0]\!])_1 \subset G(\mathbb{C}[\![x_0]\!])$ is the kernel of the evaluation at $x_0 = 0$. The action of $\widetilde{G}^+$ on $\mathfrak{g}(*\mathbf{a}) = \bigoplus \mathcal{L}_i^-$ is explicitly given by

$$\widehat{g} = (\widehat{g}_i)_{i=0}^{m} \colon (A_i) \mapsto ([\widehat{g}_i A_i \widehat{g}_i^{-1}]_{i,-}),$$

where $[\ \ ]_{i,-}$ means applying the projection $\mathfrak{g}(\!(x_i)\!) \to \mathcal{L}_i^-$.

Let $\mathbf{L} = (L_i)_{i=1}^{m}$ be a tuple of diagonal matrices $L_i = \operatorname{diag}(\ell_{i,\alpha})_{\alpha=1}^{n} \in \mathfrak{h}$ satisfying

$$\ell_{i,\alpha} - \ell_{i,\beta} \notin \mathbb{Z} \quad (i > m_0,\ \alpha \neq \beta).$$

For $(\mathbf{a}, \mathbf{T}) \in \mathbb{D}$, we then define

$$\Lambda_i(x_i) = \begin{cases} T_0' & (i = 0), \\ T_i' + L_i x_i^{-1} & (i = 1, 2, \ldots, m), \end{cases}$$

and let $\mathcal{M}(\mathbf{a}, \mathbf{T}; \mathbf{L})$ be the $\widetilde{G}^{+}$-coadjoint orbit through the element

$$\Lambda(x) := -x_0^2 \Lambda_0 + \sum_{i=1}^{m} \Lambda_i. \tag{2}$$

We use the symplectic fiber bundle

$$\mathcal{M}(\mathbf{L}) := \bigcup_{(\mathbf{a},\mathbf{T}) \in \mathbb{D}} \mathcal{M}(\mathbf{a}, \mathbf{T}; \mathbf{L}) \to \mathbb{D}$$

as the (extended) phase space for isomonodromy equations.

The proof of the following lemma is an easy exercise:

LEMMA 2.2. *An element $\widehat{g} = (\widehat{g}_i)_{i=0}^m \in \widetilde{G}^{+}$, $\widehat{g}_i(x_i) = \sum_{j=0}^{\infty} g_j^{(i)} x_i^j$ stabilizes Λ if and only if*

- $g_j^{(0)}$, $j = 1, 2, \ldots, r_0 - 1$ *are diagonal,*
- $g_j^{(i)}$, $j = 0, 1, \ldots, r_i$ *are diagonal for $i > 0$.*

To introduce the isomonodromy equations we need the following classical fact:

PROPOSITION 2.3 ([**10**]). *For any $A \in \mathcal{M}(\mathbf{a}, \mathbf{T}; \mathbf{L})$, there exists $\widehat{u} = (\widehat{u}_i)_{i=0}^m \in \widetilde{G}^{+}$ such that*

$$\widehat{u}_i^{-1} \circ (d_{\mathbb{P}^1} - A\, dx) \circ \widehat{u}_i = d_{\mathbb{P}^1} - d_{\mathbb{P}^1} T_i - L_i \frac{d_{\mathbb{P}^1} x_i}{x_i} \quad (i = 0, 1, \ldots, m)$$

for some unique $L_0 \in \mathfrak{h}$ (in particular, $A = \widehat{u} \cdot \Lambda$), where $d_{\mathbb{P}^1}$ is the exterior derivation in the $\mathbb{P}^1$-direction.

In other words, for any $A \in \mathcal{M}(\mathbf{a}, \mathbf{T}; \mathbf{L})$ there exists a unique $L_0 \in \mathfrak{h}$ such that the system $dY/dx = AY$ has the (formal) fundamental solutions of the form

$$Y_i = \widehat{u}_i e^{T_i} x_i^{L_i}, \quad i = 0, 1, \ldots, m$$

with $\widehat{u} = (\widehat{u}_i) \in \widetilde{G}^{+}$.

Remark 2.4. *Such $\widehat{u}$ is unique up to multiplication $\widehat{u}_i \mapsto \widehat{u}_i h_i$, $h_i \in H$ $(i > 0)$ and the map $\mathcal{M}(\mathbf{a}, \mathbf{T}; \mathbf{L}) \to \mathfrak{h}$, $A \mapsto -L_0$ is known to be a moment map generating the conjugation action of H (see* [**2**]*).*

Put

$$\Xi_i = \begin{cases} d_{\mathbb{D}} T_0 & (i = 0), \\ d_{\mathbb{D}} T_i + L_i d_{\mathbb{D}} \log x_i & (i = 1, 2 \dots, m), \end{cases}$$

where $d_{\mathbb{D}}$ is the exterior derivation in the $\mathbb{D}$-direction. Note that $x_i = x - a_i$ depends on a_i and so for instance $d_{\mathbb{D}} \log x_i = -da_i/x_i$. For $A \in \mathcal{M}(\mathbf{a}, \mathbf{T}; \mathbf{L})$, take $\widehat{g} = (\widehat{g}_i)_{i=0}^m \in \widetilde{G}^+$ so that $A = \widehat{g} \cdot \Lambda$ and define a $\mathfrak{g}$-valued meromorphic one-form Ω on $\mathbb{P}^1 \times \mathbb{D}$ by

$$\Omega = \sum_{i=0}^m \Omega_i, \qquad \Omega_i = \left[\widehat{g}_i \Xi_i \widehat{g}_i^{-1}\right]_{i,-}.$$

Lemma 2.2 shows that multiplying $\widehat{g}$ by an element stabilizing Λ from the right has no effect on Ω; hence Ω only depends on A.

Definition 2.5. *The* isomonodromy equation *is the differential equation on $\mathcal{M}(\mathbf{L})$ defined by*

$$d_{\mathbb{D}} A = \frac{\partial \Omega}{\partial x} + [\Omega, A].$$

Remark 2.6. *In* [**8**] *Jimbo et al. uses more larger space as the phase space for isomonodromy equation. Our space $\mathcal{M}(\mathbf{L})$ is obtained from their phase space by taking the Hamiltonian reduction. See* [**2**] *for more detail.*

3. Isomonodromy tau functions and Hamiltonians

For $A \in \mathcal{M}(\mathbf{a}, \mathbf{T}; \mathbf{L})$, we take $\widehat{u} = (\widehat{u}_i)_{i=0}^m \in \widetilde{G}^+$ as in Proposition 2.3 and define

$$\varpi = \sum_{i=0}^m \varpi_i, \quad \varpi_i = -\operatorname*{res}_{x_i=0} \operatorname{tr}\left(\widehat{u}_i^{-1} \frac{\partial \widehat{u}_i}{\partial x_i} \Xi_i\right),$$

which is viewed as a horizontal one-form on the fiber bundle $\mathcal{M}(\mathbf{L})$. We call it the *Jimbo–Miwa–Ueno one-form* (*JMU one-form* for short). In [**8**] Jimbo et al. showed that for any solution $s\colon \Delta \to \mathcal{M}(\mathbf{L})$ (where $\Delta \subset \mathbb{D}$ is an open subset) of the isomonodromy equation, the pull-back $s^*\varpi$ is closed. Hence if Δ is simply-connected there is a function τ on Δ such that $d \log \tau = s^*\varpi$. This is called the *isomonodromy tau function.*

This section is devoted to calculate the Hamiltonian vector fields of the JMU one-form. The goal is to prove the following:

THEOREM 3.1. *For any (local) vector field v on $\mathbb{D}$, the Hamiltonian vector field X_f associated to the function $f = \langle \varpi, v\rangle$ is given by $X_f(A) = [\langle \Omega, v\rangle, A]$.*

Put

$$B_0(x_0) = -x_0^{-2}A, \quad B_i(x_i) = u_i^{-1}Au_i \ (i > 0),$$

where $u_i := \widehat{u}_i(0)$. Note that B_i, $i > 0$ depend on the choice of $\widehat{u}$. We write

$$T_i = \operatorname{diag}(t_{i,\alpha})_{\alpha=1}^n, \quad T_i' = \operatorname{diag}(t'_{i,\alpha})_{\alpha=1}^n, \quad \Lambda_i = \operatorname{diag}(\lambda_{i,\alpha})_{\alpha=1}^n.$$

LEMMA 3.2. *Let $X = (x_{\alpha\beta})_{\alpha,\beta=1}^n$ be an n by n matrix of indeterminates $x_{\alpha\beta}$. Then for each $i = 0, 1, \ldots, m$ and $\gamma = 1, 2, \ldots, n$, the substitution $X = (B_i - \Lambda_i)(y - \Lambda_i)^{-1}$ gives a well-defined map*

$$\mathbb{C}[\![X]\!] = \mathbb{C}[\![x_{\alpha\beta}; \alpha, \beta = 1, 2, \ldots, n]\!] \to \mathbb{C}(\!(x_i)\!)[\![y_{i,\gamma}, y_{i,\gamma}^{-1}]\!],$$

where $y_{i,\gamma} := y - \lambda_{i,\gamma}$ and we regard $(y - \Lambda_i)^{-1}$ as an element of $\mathfrak{g}(\!(x_i)\!)(\!(y_{i,\gamma})\!)$.

PROOF. The formal Laurent series $B_i - \Lambda_i$ in x_i has order at least $-r_i$. Also, for any $j \in \mathbb{Z}$ and $\alpha_1, \alpha_2, \ldots, \alpha_k \in \{1, 2, \ldots, n\}$, the degree j coefficient of the formal Laurent series

$$\frac{1}{(y - \lambda_{i,\alpha_1})(y - \lambda_{i,\alpha_2}) \cdots (y - \lambda_{i,\alpha_k})} \in \mathbb{C}(\!(x_i)\!)(\!(y_{i,\gamma})\!)$$

in $y_{i,\gamma}$ has, as an element of $\mathbb{C}(\!(x_i)\!)$, order at least $(j + k)(r_i + 1)$. Hence the substitution $X = (B_i - \Lambda_i)(y - \Lambda_i)^{-1}$ transforms any degree k homogeneous polynomial in $x_{\alpha\beta}$ into an element of $\mathbb{C}(\!(x_i)\!)(\!(y_{i,\gamma})\!)$ whose degree j coefficient has order at least

$$-kr_i + (j + k)(r_i + 1) = j(r_i + 1) + k.$$

Hence for any $f(X) \in \mathbb{C}[\![X]\!]$, the substitution $X = (B_i - \Lambda_i)(y - \Lambda_i)^{-1}$ gives an element of $\mathbb{C}(\!(x_i)\!)[\![y_{i,\gamma}, y_{i,\gamma}^{-1}]\!]$. □

We apply the above lemma to the formal power series

$$\operatorname{tr}\log(1 - X) = \sum_{k=1}^{\infty} \frac{1}{k} \operatorname{tr} X^k,$$

which is equal to

$$\log\det(1 - X) = \sum_{k=1}^{\infty} \frac{1}{k}(1 - \det(1 - X))^k.$$

Since

$$1 - (B_i - \Lambda_i)(y - \Lambda_i)^{-1} = (y - B_i)(y - \Lambda_i)^{-1},$$

we have the equality

$$\sum_{k=1}^{\infty} \frac{1}{k} \operatorname{tr} \left[(B_i - \Lambda_i)(y - \Lambda_i)^{-1}\right]^k = \sum_{k=1}^{\infty} \frac{1}{k} \left(1 - \frac{\det(y - B_i)}{\det(y - \Lambda_i)}\right)^k \tag{3}$$

in $\mathbb{C}(\!(x_i)\!)[\![y_{i,\alpha}, y_{i,\alpha}^{-1}]\!]$. Note that the right hand side is invariant under the conjugation $B_i \mapsto \widehat{g}_i B_i \widehat{g}_i^{-1}$ by $\widehat{g}_i \in G(\mathbb{C}(\!(x_i)\!))$.

We write

$$\Xi_i = \operatorname{diag}(\xi_{i,\alpha})_{\alpha=1}^n \quad (i = 0, 1, \dots, m).$$

Proposition 3.3. *For $i = 0, 1, \dots, m$, the following equality holds:*

$$\varpi_i = -\sum_{\alpha=1}^{n} \sum_{k=1}^{\infty} \frac{1}{k} \operatorname*{res}_{x_i=0} \operatorname*{res}_{y_{i,\alpha}=0} \operatorname{tr} \left[(B_i - \Lambda_i)(y - \Lambda_i)^{-1}\right]^k \xi_{i,\alpha}.$$

Proof. Assume $i > 0$. Observe that the equality

$$\widehat{u}_i^{-1} u_i B_i u_i^{-1} \widehat{u}_i = \Lambda_i + \widehat{u}_i^{-1} \widehat{u}_i'$$

holds, where $\widehat{u}_i' := \partial \widehat{u}_i / \partial x_i$. By $G(\mathbb{C}(\!(x_i)\!))$-invariance we have

$$\sum_{k=1}^{\infty} \frac{1}{k} \left(1 - \frac{\det(y - B_i)}{\det(y - \Lambda_i)}\right)^k = \sum_{k=1}^{\infty} \frac{1}{k} \left(1 - \frac{\det(y - \Lambda_i - \widehat{u}_i^{-1} \widehat{u}_i')}{\det(y - \Lambda_i)}\right)^k.$$

By order counting we can check that the substitution $X = \widehat{u}_i^{-1} \widehat{u}_i' (y - \Lambda_i)^{-1}$ gives a map $\mathbb{C}[\![X]\!] \to \mathbb{C}(\!(x)\!)[\![y_{i,\gamma}, y_{i,\gamma}^{-1}]\!]$. Thus we obtain

$$\sum_{k=1}^{\infty} \frac{1}{k} \operatorname{tr} \left[(B_i - \Lambda_i)(y - \Lambda_i)^{-1}\right]^k = \sum_{k=1}^{\infty} \frac{1}{k} \operatorname{tr} X^k, \quad X := \widehat{u}_i^{-1} \widehat{u}_i' (y - \Lambda_i)^{-1}.$$

The order counting shows

$$\operatorname{ord} \left(\operatorname*{res}_{y_{i,\alpha}=0} \operatorname{tr} X^k \xi_{i,\alpha}\right) \geq (k-1)(r_i + 1) - (r_i + 1) = (k-2)(r_i + 1)$$

for $\alpha = 1, 2, \dots, n$. In particular, $\operatorname{res}_{y_{i,\alpha}=0} \operatorname{tr} X^k \xi_{i,\alpha}$ has no residue at $x_i = 0$ if

$k \geq 2$. Thus we obtain

$$\sum_{k=1}^{\infty} \frac{1}{k} \operatorname*{res}_{x_i=0} \operatorname*{res}_{y_{i,\alpha}=0} \operatorname{tr}\left[(B_i - \Lambda_i)(y-\Lambda_i)^{-1}\right]^k \xi_{i,\alpha}$$
$$= \operatorname*{res}_{x_i=0} \operatorname*{res}_{y_{i,\alpha}=0} \operatorname{tr}\left[\widehat{u}_i^{-1}\widehat{u}_i'(y-\Lambda_i)^{-1}\right] \xi_{i,\alpha}$$
$$= \operatorname*{res}_{x_i=0} \operatorname{tr}(\widehat{u}_i^{-1}\widehat{u}_i' E_\alpha)\xi_{i,\alpha} = \operatorname*{res}_{x_i=0} \operatorname{tr}\left(\widehat{u}_i^{-1}\widehat{u}_i'\Xi_i E_\alpha\right),$$

where E_α is the diagonal matrix with 1 on the α-th diagonal entry and zero elsewhere. In the case of $i=0$, we have

$$\sum_{k=1}^{\infty} \frac{1}{k} \operatorname{tr}\left[(B_0 - \Lambda_0)(y-\Lambda_0)^{-1}\right]^k = \sum_{k=1}^{\infty} \frac{1}{k} \operatorname{tr} X^k, \quad X := \left(\frac{L_0}{x_0} + \widehat{u}_0^{-1}\widehat{u}_0'\right)(y-\Lambda_0)^{-1}.$$

Since

$$\operatorname{ord}\left(\operatorname*{res}_{y_{0,\alpha}=0} \operatorname{tr} X^k \xi_{0,\alpha}\right) \geq -k + (k-1)(r_0+1) - r_0 = (k-2)r_0 - 1,$$

the formal Laurent series $\operatorname{res}_{y_{0,\alpha}=0} \operatorname{tr} X^k \xi_{0,\alpha}$ has no residue if $k \geq 3$. It is also true in the case of $k=2$ because

$$\operatorname*{res}_{x_0=0} \operatorname*{res}_{y_{0,\alpha}=0} \operatorname{tr}\left[\widehat{u}_0^{-1}\widehat{u}_0'(y-\Lambda_0)^{-1}\right]^2 \xi_{0,\alpha} = 0$$

by order counting and

$$\operatorname*{res}_{y_{0,\alpha}=0} \operatorname{tr}\left[L_0 x_0^{-1}(y-\Lambda_0)^{-1}\widehat{u}_0^{-1}\widehat{u}_0'(y-\Lambda_0)^{-1}\right]$$
$$= x_0^{-1} \operatorname*{res}_{y_{0,\alpha}=0} \operatorname{tr}\left[(y-\Lambda_0)^{-2} L_0 \widehat{u}_0^{-1}\widehat{u}_0'\right] = 0,$$
$$\operatorname*{res}_{y_{0,\alpha}=0} \operatorname{tr}\left[L_0 x_0^{-1}(y-\Lambda_0)^{-1} L_0 x_0^{-1}(y-\Lambda_0)^{-1}\right]$$
$$= \operatorname*{res}_{y_{0,\alpha}=0} \operatorname{tr}\left[x_0^{-2} L_0^2 (y-\Lambda_0)^{-2}\right] = 0.$$

Thus we obtain

$$\sum_{k=1}^{\infty} \frac{1}{k} \operatorname*{res}_{x_0=0} \operatorname*{res}_{y_{0,\alpha}=0} \operatorname{tr}\left[(B_0 - \Lambda_0)(y-\Lambda_0)^{-1}\right]^k \xi_{0,\alpha}$$
$$= \operatorname*{res}_{x_0=0} \operatorname*{res}_{y_{0,\alpha}=0} \operatorname{tr}\left[(L_0 x_0^{-1} + \widehat{u}_0^{-1}\widehat{u}_0')(y-\Lambda_0)^{-1}\right] \xi_{0,\alpha}$$
$$= \operatorname*{res}_{x_0=0} \operatorname{tr}\left[(L_0 x_0^{-1} + \widehat{u}_0^{-1}\widehat{u}_0')E_\alpha\right] \xi_{0,\alpha}$$
$$= \operatorname*{res}_{x_0=0} \operatorname{tr}(L_0 E_\alpha) x_0^{-1} \xi_{0,\alpha} + \operatorname*{res}_{x_0=0} \operatorname{tr}(\widehat{u}_0^{-1}\widehat{u}_0' E_\alpha)\xi_{0,\alpha}$$

$$= \operatorname*{res}_{x_0=0} \operatorname{tr}(\widehat{u}_0^{-1}\widehat{u}_0' E_\alpha)\xi_{0,\alpha} = \operatorname*{res}_{x_0=0} \operatorname{tr}\left(\widehat{u}_0^{-1}\widehat{u}_0'\Xi_0 E_\alpha\right).$$

□

Remark 3.4. *For $i > m_0$, the above formula implies the following well-known expression:*

$$\varpi_i = \frac{1}{2}\operatorname*{res}_{x_i=0} \operatorname{tr} A^2\, da_i.$$

Using the above proposition we can calculate the Hamiltonian vector fields associated to the JMU one-form as follows:

LEMMA 3.5. *For each (local) vector field v on $\mathbb{D}$ the Hamiltonian vector field X_f associated to the function $f := \langle\varpi, v\rangle$ is given by $X_f(A) = [\langle\overline{\Omega}, v\rangle, A]$, where*

$$\overline{\Omega} = \sum_{i=0}^{m} \overline{\Omega}_i,$$
$$\overline{\Omega}_i := \sum_{\alpha=1}^{n}\sum_{k=1}^{\infty} u_i \left(\operatorname*{res}_{y_{i,\alpha}=0}(y-\Lambda_i)^{-1}\left[(B_i-\Lambda_i)(y-\Lambda_i)^{-1}\right]^{k-1}\xi_{i,\alpha}\right)_{i,-} u_i^{-1}.$$

PROOF. For each $i = 0, 1, \ldots, m$ the direct sum decomposition $\mathfrak{g}(\!(x_i)\!) = \mathcal{L}_i^+ \oplus \mathcal{L}_i^-$ induces a new Lie bracket $[X,Y]_R = [X_{i,+}, Y_{i,+}] - [X_{i,-}, Y_{i,-}]$, $X, Y \in \mathfrak{g}(\!(x_i)\!)$ and hence a Poisson structure on $\mathfrak{g}(\!(x_i)\!)$ via the pairing $(X,Y) \mapsto \operatorname{res}_{x_i=0} \operatorname{tr} XY\, dx$. For $A \in \mathcal{M}(\mathbf{a}, \mathbf{T}; \mathbf{L})$, let $\iota_0(A) \in \mathfrak{g}(\!(x_0)\!)$ be the Laurent expansion of $-x_0^{-2}A$ and for $i > 0$ let $\iota_i(A) \in \mathfrak{g}(\!(x_i)\!)$ be the Laurent expansion of A. We first show that the maps $\iota_i \colon \mathcal{M}(\mathbf{a}, \mathbf{T}; \mathbf{L}) \to \mathfrak{g}(\!(x_i)\!)$ are Poisson. By a direct calculation we see that the transpose $\iota_i^* \colon \mathfrak{g}(\!(x_i)\!) \to \bigoplus_{j=0}^m \mathcal{L}_j^+$ is given by $\iota_i^*(X) = (\iota_{i,j}^*(X))_{j=0}^m$, where $\iota_{i,i}^*(X) := X_{i,+}$ and $\iota_{i,j}^*(X) \in \mathcal{L}_j^+$ ($j \neq i$) is the Laurent expansion of $-X_{i,-}$ in x_j. For $X, Y \in \mathfrak{g}(\!(x_i)\!)$, we have

$$\iota_{i,i}^*([X,Y]_R) = [X_{i,+}, Y_{i,+}] = [\iota_{i,i}^*(X), \iota_{i,i}^*(Y)],$$
$$\iota_{i,j}^*([X,Y]_R) = [X_{i,-}, Y_{i,-}] = [-X_{i,-}, -Y_{i,-}] = [\iota_{i,j}^*(X), \iota_{i,j}^*(Y)] \quad (j \neq i).$$

Hence ι_i^* is a Lie algebra homomorphism and hence ι_i is Poisson.

Now set $f_i = \langle\varpi_i, v\rangle$. To show $X_{f_i}(A) = [\langle\overline{\Omega}_i, v\rangle, A]$, we calculate the differential df_i of f_i. Since the assertion is local, we may assume that each u_i is chosen so that it depends analytically on A. Then a tangent vector $\delta A \in T_A\mathcal{M}(\mathbf{a}, \mathbf{T}; \mathbf{L})$ and the corresponding vector $\delta B_i \in T_{B_i}\mathfrak{g}(\!(x_i)\!) = \mathfrak{g}(\!(x_i)\!)$ are related by

$$\delta B_i = \begin{cases} -x_0^{-2}\delta A & (i = 0), \\ u_i^{-1}\delta A u_i + [B_i, u_i^{-1}\delta u_i] & (i > 0), \end{cases}$$

where we embeds $T_A\mathcal{M}(\mathbf{a},\mathbf{T};\mathbf{L})$ into $\mathfrak{g}(\!(x_i)\!)$ using ι_i and $\delta u_i := (u_i)_*(\delta A) \in T_{u_i}G = M_n(\mathbb{C})$. Taking the degree $-(r_i+1)$ part of the above equality for $i>0$, we obtain

$$u_i^{-1}\delta A^{(i)}_{r_i} u_i = \begin{cases} r_i[T^{(i)}_{r_i}, u_i^{-1}\delta u_i] & (i=1,2,\dots,m_0), \\ [L_i, u_i^{-1}\delta u_i] & (i>m_0) \end{cases}$$

(where $A=\sum_{j\in\mathbb{Z}} A^{(i)}_j x_i^{-j-1}$), which determines the off-diagonal part $(u_i^{-1}\delta u_i)_{\mathrm{OD}}$ of the matrix $u_i^{-1}\delta u_i$:

$$(u_i^{-1}\delta u_i)_{\mathrm{OD}} = \begin{cases} r_i^{-1}\operatorname{ad}^{-1}_{T^{(i)}_{r_i}}(u_i^{-1}\delta A^{(i)}_{r_i} u_i) & (i=1,2,\dots,m_0), \\ \operatorname{ad}^{-1}_{L_i}(u_i^{-1}\delta A^{(i)}_{r_i} u_i) & (i>m_0), \end{cases}$$

where for a diagonal matrix T with distinct eigenvalues and an off-diagonal matrix X, we denote by $\operatorname{ad}_T^{-1} X$ a unique off-diagonal matrix Y satisfying $[T,Y]=X$. The differential $(df_i)_A \in T^*_A\mathcal{M}(\mathbf{a},\mathbf{T};\mathbf{L})$ of f_i is given by

$$(df_i)_A(\delta A) = -\sum_{\alpha=1}^{n}\sum_{k=1}^{\infty} \operatorname*{res}_{x_i=0}\operatorname*{res}_{y_{i,\alpha}=0} \operatorname{tr}\left(\delta B_i X^{(i)}_{\alpha,k}\right),$$

where

$$X^{(i)}_{\alpha,k} := (y-\Lambda_i)^{-1}\left[(B_i-\Lambda_i)(y-\Lambda_i)^{-1}\right]^{k-1}\langle \xi_{i,\alpha}, v\rangle.$$

Note that f_i is invariant under the conjugation $B_i \mapsto hB_ih^{-1}$ by $h\in H$, which implies

$$\sum_{\alpha=1}^{n}\sum_{k=1}^{\infty} \operatorname*{res}_{x_i=0}\operatorname*{res}_{y_{i,\alpha}=0} \operatorname{tr}\left([B_i,R]X^{(i)}_{\alpha,k}\right) = 0 \quad (R\in\mathfrak{h}).$$

For $i=1,2,\dots,m_0$, we thus have

$$\begin{aligned}
&\sum_{\alpha=1}^{n}\sum_{k=1}^{\infty} \operatorname*{res}_{x_i=0}\operatorname*{res}_{y_{i,\alpha}=0} \operatorname{tr}\left([B_i,u_i^{-1}\delta u_i]X^{(i)}_{\alpha,k}\right) \\
&\qquad = \sum_{\alpha=1}^{n}\sum_{k=1}^{\infty} \operatorname*{res}_{x_i=0}\operatorname*{res}_{y_{i,\alpha}=0} \operatorname{tr}\left([B_i,(u_i^{-1}\delta u_i)_{\mathrm{OD}}]X^{(i)}_{\alpha,k}\right) \\
&\qquad = \sum_{\alpha=1}^{n}\sum_{k=1}^{\infty} \operatorname*{res}_{x_i=0}\operatorname*{res}_{y_{i,\alpha}=0} \operatorname{tr}\left((u_i^{-1}\delta u_i)_{\mathrm{OD}}[X^{(i)}_{\alpha,k},B_i]\right) \\
&\qquad = \frac{1}{r_i}\sum_{\alpha=1}^{n}\sum_{k=1}^{\infty} \operatorname*{res}_{x_i=0}\operatorname*{res}_{y_{i,\alpha}=0} \operatorname{tr}\left(\operatorname{ad}^{-1}_{T^{(i)}_{r_i}}(u_i^{-1}\delta A^{(i)}_{r_i}u_i)[X^{(i)}_{\alpha,k},B_i]\right)
\end{aligned}$$

$$= \frac{1}{r_i} \sum_{\alpha=1}^{n} \sum_{k=1}^{\infty} \operatorname*{res}_{x_i=0} \operatorname*{res}_{y_{i,\alpha}=0} \operatorname{tr} \left(u_i^{-1} \delta A_{r_i}^{(i)} u_i \operatorname{ad}_{T_{r_i}^{(i)}}^{-1} [X_{\alpha,k}^{(i)}, B_i]_{\mathrm{OD}} \right).$$

Hence

$$(df_i)_A(\delta A) = -\sum_{\alpha=1}^{n} \sum_{k=1}^{\infty} \operatorname*{res}_{x_i=0} \operatorname*{res}_{y_{i,\alpha}=0} \operatorname{tr} \left(\delta A \cdot u_i X_{\alpha,k}^{(i)} u_i^{-1} \right) - \frac{1}{r_i} \sum_{\alpha=1}^{n} \sum_{k=1}^{\infty} \operatorname*{res}_{x_i=0} \operatorname*{res}_{y_{i,\alpha}=0} \operatorname{tr} \left[\delta A_{r_i}^{(i)} \cdot u_i \operatorname{ad}_{T_{r_i}^{(i)}}^{-1} \left([X_{\alpha,k}^{(i)}, B_i]_{\mathrm{OD}} \right) u_i^{-1} \right],$$

in other words, the differential $(df_i)_A \in T_A^*\mathcal{M}(\mathbf{a}, \mathbf{T}; \mathbf{L})$ is obtained by taking the projection $(\iota_i)_A^* \colon \mathfrak{g}(\!(x_i)\!) \to T_A^*\mathcal{M}(\mathbf{a}, \mathbf{T}; \mathbf{L})$ of

$$-\sum_{\alpha=1}^{n} \sum_{k=1}^{\infty} u_i \left[\operatorname*{res}_{y_{i,\alpha}=0} X_{\alpha,k}^{(i)} + \frac{1}{r_i} x_i^{r_i} \operatorname*{res}_{x_i=0} \operatorname*{res}_{y_{i,\alpha}=0} \operatorname{ad}_{T_{r_i}^{(i)}}^{-1} \left([X_{\alpha,k}^{(i)}, B_i]_{\mathrm{OD}} \right) \right] u_i^{-1} \in \mathfrak{g}(\!(x_i)\!).$$

In a similar way, we see that the differential $(df_i)_A \in T_A^*\mathcal{M}(\mathbf{a}, \mathbf{T}; \mathbf{L})$ for $i > m_0$ is obtained by taking the projection $(\iota_i)_A^*$ of

$$-\sum_{\alpha=1}^{n} \sum_{k=1}^{\infty} u_i \left[\operatorname*{res}_{y_{i,\alpha}=0} X_{\alpha,k}^{(i)} + \operatorname*{res}_{x_i=0} \operatorname*{res}_{y_{i,\alpha}=0} \operatorname{ad}_{L_i}^{-1} \left([X_{\alpha,k}^{(i)}, B_i]_{\mathrm{OD}} \right) \right] u_i^{-1} \in \mathfrak{g}(\!(x_i)\!).$$

The minus of the $\mathcal{L}_i^-$-part of the above elements are

$$\sum_{\alpha=1}^{n} \sum_{k=1}^{\infty} u_i \left(\operatorname*{res}_{y_{i,\alpha}=0} X_{\alpha,k}^{(i)} \right)_{i,-} u_i^{-1} = \overline{\Omega}_i \quad (i = 1, 2, \ldots, m).$$

Therefore the theorem of Adler–Kostant–Symes (see e.g. [**1**]) implies $X_{f_i}(A) = [\langle \overline{\Omega}_i, v \rangle, A]$ for $i > 0$. On the other hand, the differential $(df_0)_A$ is given by

$$(df_0)_A(\delta A) = \sum_{\alpha=1}^{n} \sum_{k=1}^{\infty} \operatorname*{res}_{x_0=0} \operatorname*{res}_{y_{i,\alpha}=0} \operatorname{tr} \left(x_0^{-2} \delta A X_{\alpha,k}^{(0)} \right),$$

i.e., $(df_0)_A$ is the projection of

$$-\sum_{\alpha=1}^{n} \sum_{k=1}^{\infty} \operatorname*{res}_{y_{i,\alpha}=0} X_{\alpha,k}^{(0)} \in \mathfrak{g}(\!(x_0)\!).$$

The theorem of Adler–Kostant–Symes again implies $X_{f_0}(A) = [\langle \overline{\Omega}_0, v \rangle, A]$. □

Now Theorem 3.1 follows from the following lemma:

LEMMA 3.6. *We have* $\overline{\Omega} = \Omega$.

Proof. Set $\mathcal{R}_i = \mathbb{C}[x_i]/(x_i^{r_i+1})$ and

$$\widetilde{B}_i = x_i^{r_i+1} B_i, \quad \widetilde{\Lambda}_i = x_i^{r_i+1}\Lambda_i, \quad \tilde{y}_i = x_i^{r_i+1} y, \quad \tilde{y}_{i,\alpha} = x_i^{r_i+1} y_{i,\alpha}.$$

We regard $\widetilde{B}_i, \widetilde{\Lambda}_i$ as elements of $\mathfrak{g}\otimes\mathcal{R}_i \simeq M_n(\mathcal{R}_i)$ by taking projection $\mathbb{C}[\![x_i]\!] \to \mathcal{R}_i$. Also we regard $\tilde{y}_i - \widetilde{\Lambda}_i$ as an element of $M_n(\mathcal{R}_i \otimes \mathbb{C}(\tilde{y}_{i,\alpha}))$. Consider first the case of $i > 0$. By order counting we see that $\overline{\Omega}_i$ has order at least $(k-1)-(r_i+1) \geq -(r_i+1)$ and does not depend on the $\mathcal{L}_i^+$-part of B_i. Thus $x_i^{r_i+1}\overline{\Omega}_i$ may be viewed as a $M_n(\mathcal{R}_i)$-valued one-form and we have

$$u_i^{-1} x_i^{r_i+1}\overline{\Omega}_i u_i = \sum_{\alpha=1}^{n}\sum_{k=1}^{\infty} \operatorname*{res}_{\tilde{y}_{i,\alpha}=0} (\tilde{y}_i - \widetilde{\Lambda}_i)^{-1}\left[(\widetilde{B}_i - \widetilde{\Lambda}_i)(\tilde{y}_i - \widetilde{\Lambda}_i)^{-1}\right]^{k-1}(x_i^{r_i+1}\xi_{i,\alpha}),$$

where $x_i^{r_i+1}\xi_{i,\alpha}$ is also regarded as a $M_n(\mathcal{R}_i)$-valued one-form. Since $\widetilde{B}_i(0) = \widetilde{\Lambda}_i(0)$, the matrix $\widetilde{B}_i - \widetilde{\Lambda}_i \in M_n(\mathcal{R}_i)$ is nilpotent. Thus we have

$$\begin{aligned}\sum_{k=1}^{\infty}\left[(\widetilde{B}_i - \widetilde{\Lambda}_i)(\tilde{y}_i - \widetilde{\Lambda}_i)^{-1}\right]^{k-1} &= \left[1 - (\widetilde{B}_i - \widetilde{\Lambda}_i)(\tilde{y}_i - \widetilde{\Lambda}_i)^{-1}\right]^{-1} \\ &= (\tilde{y}_i - \widetilde{\Lambda}_i)(\tilde{y}_i - \widetilde{B}_i)^{-1}.\end{aligned}$$

Hence

$$x_i^{r_i+1}\overline{\Omega}_i = u_i \sum_{\alpha=1}^{n} \operatorname*{res}_{\tilde{y}_{i,\alpha}=0} (\tilde{y}_i - \widetilde{B}_i)^{-1}(x_i^{r_i+1}\xi_{i,\alpha}) u_i^{-1}.$$

Let $\tilde{u}_i \in \mathrm{GL}_n(\mathcal{R}_i)$ be the element obtained from $\widehat{u}_i$ by taking projection $\mathbb{C}[\![x_i]\!] \to \mathcal{R}_i$. Then

$$u_i \operatorname*{res}_{\tilde{y}_{i,\alpha}=0} (\tilde{y}_i - \widetilde{B}_i)^{-1} u_i^{-1} = \tilde{u}_i \left(\operatorname*{res}_{\tilde{y}_{i,\alpha}=0} (\tilde{y}_i - \widetilde{\Lambda}_i)^{-1}\right)\tilde{u}_i^{-1} = \tilde{u}_i E_\alpha \tilde{u}_i^{-1}.$$

Thus we obtain

$$\overline{\Omega}_i = \sum_{\alpha=1}^{n}\left[\widehat{u}_i E_\alpha \widehat{u}_i^{-1}\xi_{i,\alpha}\right]_{i,-} = \Omega_i.$$

In the case of $i = 0$, a similar argument shows that Ω_0 has order at least $-r_0$ and does not depend on the holomorphic part of B_0. Thus $x_0^{r_0}\overline{\Omega}_0$ may be viewed as a $M_n(\mathcal{R}_0)$-valued one-form and we have

$$x_0^{r_0}\overline{\Omega}_0 = \sum_{\alpha=1}^{n}\sum_{k=1}^{\infty} \operatorname*{res}_{\tilde{y}_{0,\alpha}=0} (\tilde{y}_0 - \widetilde{\Lambda}_0)^{-1}\left[(\widetilde{B}_0 - \widetilde{\Lambda}_0)(\tilde{y}_0 - \widetilde{\Lambda}_0)^{-1}\right]^{k-1}(x_0^{r_0}\xi_{0,\alpha}),$$

from which we deduce

$$x_0^{r_0}\overline{\Omega}_0 = \sum_{\alpha=1}^{n} \operatorname*{res}_{\tilde{y}_{0,\alpha}=0}(\tilde{y}_0 - \widetilde{B}_0)^{-1}(x_0^{r_0}\xi_{0,\alpha}).$$

Denoting by $\tilde{u}_0 \in \mathrm{GL}_n(\mathcal{R}_0)$ the projection of $\widehat{u}_0$, we have

$$\operatorname*{res}_{\tilde{y}_{0,\alpha}=0}(\tilde{y}_0 - \widetilde{B}_0)^{-1} = \tilde{u}_0 \left(\operatorname*{res}_{\tilde{y}_{0,\alpha}=0}(\tilde{y}_0 - \widetilde{\Lambda}_0 - L_0 x_0^{r_0})^{-1} \right) \tilde{u}_0^{-1} = \tilde{u}_0 E_\alpha \tilde{u}_0^{-1}.$$

Hence $\overline{\Omega}_0 = \Omega_0$. □

4. A flat connection on the phase space

In the previous section, we showed that the Hamiltonian vector fields associated to the JMU one-form ϖ is given by $[\Omega, A]$; so the associated differential equation is $d_{\mathbb{D}}A = [\Omega, A]$. In this section, we construct an Ehresmann connection on the symplectic fiber bundle $\mathcal{M}(\mathbf{L}) \to \mathbb{D}$ whose horizontal sections are characterized by the differential equation $d_{\mathbb{D}}A = \partial\Omega/\partial x$, and show that it is symplectic, flat and moreover complete.

Let Ω_{pl} (resp. Ω_{irr}) be the $\mathbb{D}_{\mathrm{pl}}$-part (resp. $\mathbb{D}_{\mathrm{irr}}$-part) of the one-form Ω:

$$\Omega_{\mathrm{pl}} = -\sum_{i=1}^{m} \left[\widehat{g}_i \Lambda_i \widehat{g}_i^{-1}\right]_{i,-} da_i, \quad \Omega_{\mathrm{irr}} = \sum_{i=0}^{m} \left[\widehat{g}_i (d_{\mathbb{D}_{\mathrm{irr}}} T_i) \widehat{g}_i^{-1}\right]_{i,-}.$$

Note that if we write $A = \sum_{i=0}^{m} A_i$, $A_i \in \mathcal{L}_i^-$, then $\Omega_{\mathrm{pl}} = -\sum_{i=1}^{m} A_i \, da_i$.

For each $\mathbf{T} \in \mathbb{D}_{\mathrm{irr}}$, the image of $\mathcal{M}(\mathbf{a}, \mathbf{T}; \mathbf{L})$ via the embedding $\mathfrak{g}(*\mathbf{a}) \to \bigoplus_{i=0}^{m} \mathfrak{g}(\!(x_i)\!)$ does not depend on $\mathbf{a}$; we denote it by $\mathcal{M}(\mathbf{T}; \mathbf{L})$. Then we obtain a partial trivialization $\mathcal{M}(\mathbf{L}) = \mathcal{M}(\mathbf{L})_{\mathrm{irr}} \times \mathbb{D}_{\mathrm{pl}}$, where $\mathcal{M}(\mathbf{L})_{\mathrm{irr}} := \bigcup_{\mathbf{T} \in \mathbb{D}_{\mathrm{irr}}} \mathcal{M}(\mathbf{T}; \mathbf{L})$. We define an Ehresmann connection on the fiber bundle $\mathcal{M}(\mathbf{L})_{\mathrm{irr}} \to \mathbb{D}_{\mathrm{irr}}$.

Identify each fiber $\mathcal{M}(\mathbf{T}; \mathbf{L})$ with the quotient $\widetilde{G}^+/\operatorname{Stab}(\Lambda)$, where $\operatorname{Stab}(\Lambda)$ is the stabilizer of the element Λ given in (2). Note that it does not depend on $\mathbf{T}$.

LEMMA 4.1. *There exists a connection on the fiber bundle $\widetilde{G}^+ \times \mathbb{D}_{\mathrm{irr}} \to \mathbb{D}_{\mathrm{irr}}$ such that the exterior derivation $\partial_{\mathbb{D}_{\mathrm{irr}}}$ on functions on $\widetilde{G}^+ \times \mathbb{D}_{\mathrm{irr}}$ in the horizontal direction satisfies the following conditions:*

$$\begin{gathered} \partial_{\mathbb{D}_{\mathrm{irr}}} g_0^{(i)} = 0, \\ [x_i^{2\delta_{0i}} \widehat{g}_i^{-1} \partial_{\mathbb{D}_{\mathrm{irr}}} \widehat{g}_i, T_i']_{i,-} = [x_i^{2\delta_{0i}} \widehat{g}_i^{-1} \widehat{g}_i', d_{\mathbb{D}_{\mathrm{irr}}} T_i]_{i,-} \quad (i = 0, 1, \ldots, m), \end{gathered} \tag{4}$$

where we write $\widehat{g}_i = \sum_{j=0}^{\infty} g_j^{(i)} x_i^j$, $\partial_{\mathbb{D}_{\mathrm{irr}}} \widehat{g}_i = \sum_{j=0}^{\infty} \partial_{\mathbb{D}_{\mathrm{irr}}} g_j^{(i)} \cdot x_i^j$.

PROOF. Write $\widehat{g}_i^{-1} = \sum_{j=0}^{\infty} \bar{g}_j^{(i)} x_i^j$. The above equalities hold if and only if the equality

$$-r_i[\widehat{g}_i(0)^{-1}\partial_{\mathbb{D}_{\rm irr}} g_k^{(i)}, T_{r_i}^{(i)}] = \sum_{j=1}^{k}\sum_{l=0}^{k-j} j[\bar{g}_{k-j-l}^{(i)} g_j^{(i)}, d_{\mathbb{D}} T_{r_i-l}^{(i)}] - \sum_{j=0}^{k-1}\sum_{l=0}^{k-j}(l-r_i)[\bar{g}_{k-j-l}^{(i)}\partial_{\mathbb{D}_{\rm irr}} g_j^{(i)}, T_{r_i-l}^{(i)}] \tag{5}$$

holds for all $i = 0, 1, \ldots, m$ and $k = 0, 1, \ldots, r_i - \delta_{0i}$ (when $k = 0$, the right hand side is understood to be zero). Note that each $\bar{g}_j^{(i)}$ is expressed as an algebraic combination of $g_l^{(i)}$, $l = 0, 1, \ldots, m$ and the expression on the right hand side of the above equality does not contain $\partial_{\mathbb{D}_{\rm irr}} g_j^{(i)}$, $j \geq k$. Thus, starting from the definition $\partial_{\mathbb{D}_{\rm irr}} g_0^{(i)} := 0$, we can recursively define $\partial_{\mathbb{D}_{\rm irr}} g_k^{(i)}$, $k = 0, 1, \ldots r_i - \delta_{0i}$ so that $\widehat{g}_i(0)^{-1}\partial_{\mathbb{D}_{\rm irr}} g_k^{(i)}$ is a unique off-diagonal matrix satisfying the above equality. For $k > r_i - \delta_{0i}$, we define $\partial_{\mathbb{D}_{\rm irr}} \widehat{g}_k^{(i)} = 0$. We extend the definition of $\partial_{\mathbb{D}_{\rm irr}} f$ for functions f on $\widetilde{G}^+ \times \mathbb{D}$ using the Leibniz rule and $\partial_{\mathbb{D}_{\rm irr}}(\mathbb{C}) = 0$. Then it satisfies conditions (4). □

Proposition 4.2. *The connection on $\widetilde{G}^+ \times \mathbb{D}_{\rm irr} \to \mathbb{D}_{\rm irr}$ given by $\partial_{\mathbb{D}_{\rm irr}}$ is equivariant with respect to the action of* $\mathrm{Stab}(\Lambda)$*, and hence descends to a connection on $\mathcal{M}(\mathbf{L})_{\rm irr} \to \mathbb{D}_{\rm irr}$. The horizontal sections wth respect to the induced connection on $\mathcal{M}(\mathbf{L})$ satisfy the differential equation $d_{\mathbb{D}} A = \partial\Omega/\partial x$.*

Proof. For $\widehat{g} \in \widetilde{G}^+$ and $\widehat{h} \in \mathrm{Stab}(\Lambda)$, we have

$$\begin{aligned}[x_i^{2\delta_{0i}}(\widehat{g}_i\widehat{h}_i)^{-1}(\widehat{g}_i\widehat{h}_i)', d_{\mathbb{D}_{\rm irr}} T_i]_{i,-} &= \left(\widehat{h}_i^{-1}[x_i^{2\delta_{0i}}\widehat{g}_i^{-1}\widehat{g}_i', d_{\mathbb{D}_{\rm irr}} T_i]\widehat{h}_i\right)_{i,-} \\ &= [x_i^{2\delta_{0i}}\widehat{h}_i^{-1}\widehat{g}_i^{-1}\partial_{\mathbb{D}_{\rm irr}}\widehat{g}_i \cdot \widehat{h}_i, T_i']_{i,-} \quad (i = 0, 1, \ldots, m),\end{aligned}$$

which implies that the connection is $\mathrm{Stab}(\Lambda)$-equivariant.

Let $\partial_{\mathbb{D}}$ be the exterior derivation in the horizontal direction with respect to the induced connection on $\mathcal{M}(\mathbf{L})$. Then its $\mathbb{D}_{\rm pl}$-part $\partial_{\mathbb{D}_{\rm pl}}$ satisfies

$$\partial_{\mathbb{D}_{\rm pl}} A = -\sum_{i=1}^{m}\frac{\partial A_i}{\partial x_i}\, da_i = \frac{\partial\Omega_{\rm pl}}{\partial x}.$$

Furthermore, conditions (4) imply

$$\begin{aligned}\partial_{\mathbb{D}_{\rm irr}} A &= -\partial_{\mathbb{D}_{\rm irr}}\left[\widehat{g}_0(x_0^2 T_0')\widehat{g}_0^{-1}\right]_{0,-} + \sum_{i=1}^{m}\partial_{\mathbb{D}_{\rm irr}}\left[\widehat{g}_i\Lambda_i\widehat{g}_i^{-1}\right]_{i,-} \\ &= -\left[\widehat{g}_0(x_0^2 d_{\mathbb{D}_{\rm irr}} T_0')\widehat{g}_0^{-1}\right]_{0,-} + \sum_{i=1}^{m}\left[\widehat{g}_i(d_{\mathbb{D}_{\rm irr}} T_i')\widehat{g}_i^{-1}\right]_{i,-}\end{aligned}$$

$$
\begin{aligned}
&\quad - \left(\widehat{g}_0[\widehat{g}_0^{-1}\partial_{\mathbb{D}_{\mathrm{irr}}}\widehat{g}_0, x_0^2 T_0']\widehat{g}_0^{-1}\right)_{0,-} + \sum_{i=1}^m \partial_{\mathbb{D}_{\mathrm{irr}}}\left(\widehat{g}_i[\widehat{g}_i^{-1}\partial_{\mathbb{D}_{\mathrm{irr}}}\widehat{g}_i, T_i']\widehat{g}_i^{-1}\right)_{i,-} \\
&= - \left[\widehat{g}_0(x_0^2 d_{\mathbb{D}_{\mathrm{irr}}} T_0')\widehat{g}_0^{-1}\right]_{0,-} + \sum_{i=1}^m \left[\widehat{g}_i(d_{\mathbb{D}_{\mathrm{irr}}} T_i')\widehat{g}_i^{-1}\right]_{i,-} \\
&\quad - \left(\widehat{g}_0[\widehat{g}_0^{-1}\widehat{g}_0', x_0^2 d_{\mathbb{D}_{\mathrm{irr}}} T_0]\widehat{g}_0^{-1}\right)_{0,-} + \sum_{i=1}^m \left(\widehat{g}_i[\widehat{g}_i^{-1}\widehat{g}_i', d_{\mathbb{D}_{\mathrm{irr}}} T_i]\widehat{g}_i^{-1}\right)_{i,-} = \frac{\partial \Omega_{\mathrm{irr}}}{\partial x}.
\end{aligned}
$$

The proof is completed. □

We have another way to construct the above connection. Recall that if $M \to X$ is a symplectic fiber bundle and ω is a two-form on M whose restriction to each fiber coincides with the symplectic form, then the orthogonal complement of the vertical subbundle of TM with respect to ω defines a connection on M. This connection is symplectic if and only if $\iota(v_1 \wedge v_2)d\omega = 0$ for every pair of vertical vector fields v_1, v_2; in particular, if ω is closed, then the associated connection is symplectic. Furthermore, if ω is closed, then the associated connection is flat if and only if for every pair of horizontal vector fields v_1, v_2, the function $\omega(v_1, v_2)$ is constant along each fiber. See [**4**] for those facts.

Proposition 4.3. *Define a two-form $\widetilde{\omega}$ on $\widetilde{G}^+ \times \mathbb{D}_{\mathrm{irr}}$ by*

$$
\widetilde{\omega} = \sum_{i=0}^m \widetilde{\omega}_i, \quad \widetilde{\omega}_i = d \operatorname*{res}_{x_i=0} \operatorname{tr}\left(\Lambda_i \widehat{g}_i^{-1} d\widehat{g}_i - dT_i \cdot \widehat{g}_i^{-1}\widehat{g}_i'\right).
$$

Then it descends to a closed two-form $\omega = \sum \omega_i$ on $\mathcal{M}(\mathbf{L})_{\mathrm{irr}}$ whose restriction to each fiber coincides with the Kirillov–Kostant–Souriau symplectic form. Furthermore, the associated connection on $\mathcal{M}(\mathbf{L})_{\mathrm{irr}}$ coincides with the one given in Proposition 4.2. In particular, the connection is symplectic.

Proof. For $i = 0, 1, \ldots, m$, we have

$$
\begin{aligned}
\widetilde{\omega}_i &= \operatorname*{res}_{x_i=0} \operatorname{tr}\left(d\Lambda_i \wedge \widehat{g}_i^{-1} d\widehat{g}_i - \Lambda_i \widehat{g}_i^{-1} d\widehat{g}_i \wedge \widehat{g}_i^{-1} d\widehat{g}_i\right) \\
&\quad - \operatorname*{res}_{x_i=0} \operatorname{tr}\left(dT_i \wedge \widehat{g}_i^{-1} d\widehat{g}_i \cdot \widehat{g}_i^{-1}\widehat{g}_i' - dT_i \wedge \widehat{g}_i^{-1} d\widehat{g}_i'\right) \\
&= \operatorname*{res}_{x_i=0} \operatorname{tr}(dT_i' \wedge \widehat{g}_i^{-1} d\widehat{g}_i) - \operatorname*{res}_{x_i=0} \operatorname{tr}\left(\Lambda_i \widehat{g}_i^{-1} d\widehat{g}_i \wedge \widehat{g}_i^{-1} d\widehat{g}_i\right) \\
&\quad - \operatorname*{res}_{x_i=0} \operatorname{tr}\left(dT_i \wedge \widehat{g}_i^{-1} d\widehat{g}_i \cdot \widehat{g}_i^{-1}\widehat{g}_i' - dT_i \wedge \widehat{g}_i^{-1} d\widehat{g}_i'\right).
\end{aligned}
$$

The first term on the most right hand side can be expressed as

$$
\operatorname*{res}_{x_i=0} \operatorname{tr}(dT_i' \wedge \widehat{g}_i^{-1} d\widehat{g}_i) = \operatorname*{res}_{x_i=0} \operatorname{tr}(dT_i \wedge \widehat{g}_i^{-1} d\widehat{g}_i)' - \operatorname*{res}_{x_i=0} \operatorname{tr}\left(dT_i \wedge (\widehat{g}_i^{-1} d\widehat{g}_i)'\right)
$$

$$= \operatorname*{res}_{x_i=0} \operatorname{tr}(dT_i \wedge \widehat{g}_i^{-1}\widehat{g}_i'\widehat{g}_i^{-1}d\widehat{g}_i) - \operatorname*{res}_{x_i=0} \operatorname{tr}(dT_i \wedge \widehat{g}_i^{-1}d\widehat{g}_i').$$

Thus we obtain

$$\widetilde{\omega}_i = -\operatorname*{res}_{x_i=0} \operatorname{tr}\left(\Lambda_i \widehat{g}_i^{-1}d\widehat{g}_i \wedge \widehat{g}_i^{-1}d\widehat{g}_i + [\widehat{g}_i^{-1}\widehat{g}_i', dT_i] \wedge \widehat{g}_i^{-1}d\widehat{g}_i\right). \tag{6}$$

Since $[x_i^{2\delta_{0i}}\widehat{h}_i^{-1}\widehat{h}_i', dT_i]_{i,-} = 0$ for all $\widehat{h} \in \operatorname{Stab}(\Lambda)$ and $i = 0, 1, \ldots, m$, we see that $\widetilde{\omega}$ is $\operatorname{Stab}(\Lambda)$-invariant. Also, since

$$[x_i^{2\delta_{0i}}\Lambda_i, \widehat{\xi}_i]_{i,-} = 0, \quad [x_i^{2\delta_{0i}}dT_i, \widehat{\xi}_i]_{i,-} = 0 \quad (i = 0, 1, \ldots, m)$$

for all elements $\widehat{\xi} = (\widehat{\xi}_i)$ of the Lie algebra of $\operatorname{Stab}(\Lambda)$, we see that $\iota(v)\widetilde{\omega} = 0$ for all infinitesimal action v of $\operatorname{Stab}(\Lambda)$. Hence $\widetilde{\omega}$ descends to a closed two-form ω on $\mathcal{M}(\mathbf{L})_{\mathrm{irr}}$. The restriction of ω to the vertical subbundle is given by

$$\omega_{\mathrm{vert}} = -\operatorname*{res}_{x_0=0} \operatorname{tr}\left(T_0'\widehat{g}_0^{-1}d\widehat{g}_0 \wedge \widehat{g}_0^{-1}d\widehat{g}_0\right) - \sum_{i=1}^{m} \operatorname*{res}_{x_i=0} \operatorname{tr}\left(\Lambda_i \widehat{g}_i^{-1}d\widehat{g}_i \wedge \widehat{g}_i^{-1}d\widehat{g}_i\right),$$

which is exactly the Kirillov–Kostant–Souriau form.

Let v be a vertical vector field and w a horizontal vector field. Then conditions (4) imply

$$\begin{aligned}
&[x_0^2\widehat{g}_0^{-1}\widehat{g}_0', dT_0(w)]_{0,-} = [\widehat{g}_0^{-1}d\widehat{g}_0(w), x_0^2T_0']_{0,-},\\
&\quad [\widehat{g}_i^{-1}\widehat{g}_i', dT_i(w)]_{i,-} = [\widehat{g}_i^{-1}d\widehat{g}_i(w), T_i']_{i,-} = [\widehat{g}_i^{-1}d\widehat{g}_i(w), \Lambda_i]_{i,-} \quad (i > 0).
\end{aligned}$$

For $i = 0, 1, \ldots, m$, we thus have

$$\begin{aligned}
&\operatorname*{res}_{x_i=0} \operatorname{tr}\left([\widehat{g}_i^{-1}\widehat{g}_i', dT_i] \wedge \widehat{g}_i^{-1}d\widehat{g}_i\right)(w, v)\\
&\quad = \operatorname*{res}_{x_i=0} \operatorname{tr}\left([\widehat{g}_i^{-1}\widehat{g}_i', dT_i(w)]\widehat{g}_i^{-1}d\widehat{g}_i(v)\right)\\
&\qquad - \operatorname*{res}_{x_i=0} \operatorname{tr}\left([\widehat{g}_i^{-1}\widehat{g}_i', dT_i(v)]\widehat{g}_i^{-1}d\widehat{g}_i(w)\right)\\
&\quad = \operatorname*{res}_{x_i=0} \operatorname{tr}\left([\widehat{g}_i^{-1}d\widehat{g}_i(w), T_i' + (1-\delta_{0i})L_i x_i^{-1}]\widehat{g}_i^{-1}d\widehat{g}_i(v)\right)\\
&\quad = -\operatorname*{res}_{x_i=0} \operatorname{tr}\left((T_i' + (1-\delta_{0i})L_i x_i^{-1})[\widehat{g}_i^{-1}d\widehat{g}_i(w), \widehat{g}_i^{-1}d\widehat{g}_i(v)]\right),
\end{aligned}$$

which implies $\omega(v, w) = 0$. □

Next we show that our connection is flat. The following lemma is useful:

Lemma 4.4.

$$[\widehat{g}_i^{-1}\partial_{\mathbb{D}_{\mathrm{irr}}}\widehat{g}_i, d_{\mathbb{D}_{\mathrm{irr}}}T_i] \in x_i^{1-\delta_{0i}}\mathfrak{g}[\![x_i]\!] \otimes \Omega^2(\mathbb{D}_{\mathrm{irr}}) \quad (i = 0, 1, \ldots, m).$$

Proof. Conditions (4) imply

$$[\widehat{g}_i^{-1}\partial_{\mathbb{D}_{\mathrm{irr}}}\widehat{g}_i, T_i'] - [\widehat{g}_i^{-1}\widehat{g}_i', d_{\mathbb{D}_{\mathrm{irr}}}T_i] \in x_i^{-\delta_{0i}}\mathfrak{g}[\![x_i]\!] \otimes \Omega^1(\mathbb{D}_{\mathrm{irr}}).$$

Hence the off-diagonal part of $\widehat{g}_i^{-1}\partial_{\mathbb{D}_{\mathrm{irr}}}\widehat{g}_i$ satisfies

$$\left(\widehat{g}_i^{-1}\partial_{\mathbb{D}_{\mathrm{irr}}}\widehat{g}_i\right)_{\mathrm{OD}} + \mathrm{ad}_{T_i'}^{-1}\left([\widehat{g}_i^{-1}\widehat{g}_i', d_{\mathbb{D}_{\mathrm{irr}}}T_i]\right) \in x_i^{r_i+1-\delta_{0i}}\mathfrak{g}[\![x_i]\!] \otimes \Omega^1(\mathbb{D}_{\mathrm{irr}}),$$

which implies

$$[\widehat{g}_i^{-1}\partial_{\mathbb{D}_{\mathrm{irr}}}\widehat{g}_i, d_{\mathbb{D}_{\mathrm{irr}}}T_i] + \left[\mathrm{ad}_{T_i'}^{-1}\left([\widehat{g}_i^{-1}\widehat{g}_i', d_{\mathbb{D}_{\mathrm{irr}}}T_i]\right), d_{\mathbb{D}_{\mathrm{irr}}}T_i\right] \in x_i^{1-\delta_{0i}}\mathfrak{g}[\![x_i]\!] \otimes \Omega^2(\mathbb{D}_{\mathrm{irr}}).$$

The assertion now follows from

$$\left[\mathrm{ad}_{T_i'}^{-1}\left([\widehat{g}_i^{-1}\widehat{g}_i', d_{\mathbb{D}_{\mathrm{irr}}}T_i]\right), d_{\mathbb{D}_{\mathrm{irr}}}T_i\right] = \mathrm{ad}_{T_i'}^{-1}\left([[\widehat{g}_i^{-1}\widehat{g}_i', d_{\mathbb{D}_{\mathrm{irr}}}T_i], d_{\mathbb{D}_{\mathrm{irr}}}T_i]\right) = 0.$$

□

Proposition 4.5. $\omega(v,w) = 0$ *for every pair of horizontal vector fields* v, w. *In particular, the associated connection is flat.*

Proof. For $i > 0$, equalities (5) and (6) show

$$\begin{aligned}\omega_i(v,w) &= -\operatorname*{res}_{x_i=0}\mathrm{tr}\left(\Lambda_i[\widehat{g}_i^{-1}d\widehat{g}_i(v), \widehat{g}_i^{-1}d\widehat{g}_i(w)]\right)\\ &\quad - \operatorname*{res}_{x_i=0}\mathrm{tr}\left([\widehat{g}_i^{-1}\widehat{g}_i', dT_i(v)]\widehat{g}_i^{-1}d\widehat{g}_i(w)\right)\\ &\quad + \operatorname*{res}_{x_i=0}\mathrm{tr}\left([\widehat{g}_i^{-1}\widehat{g}_i', dT_i(w)]\widehat{g}_i^{-1}d\widehat{g}_i(v)\right)\\ &= \operatorname*{res}_{x_i=0}\mathrm{tr}\left(\widehat{g}_i^{-1}\widehat{g}_i'[dT_i(w), \widehat{g}_i^{-1}d\widehat{g}_i(v)]\right),\end{aligned}$$

which is zero thanks to the previous lemma. □

Finally we show that our connection is complete.

Proposition 4.6. *The connection on* $\mathcal{M}(\mathbf{L})_{\mathrm{irr}}$ *is complete.*

Proof. Let $\gamma\colon [0,1] \to \mathbb{D}_{\mathrm{irr}}$ be an arbitrary smooth curve. To show that there exists a horizontal lift $\widetilde{\gamma}\colon [0,1] \to \widetilde{G}^{+} \times \mathbb{D}_{\mathrm{irr}}$ of γ with arbitrary initial value in

$\mathcal{M}(\gamma(0);\mathbf{L})$, we have to solve the system of ordinary differential equations

$$
\begin{aligned}
-r_i\left[(g_0^{(i)})^{-1}\frac{dg_k^{(i)}}{dt},\gamma^*T_{r_i}^{(i)}\right] &= \sum_{j=1}^{k}\sum_{l=0}^{k-j} j\left[\bar{g}_{k-j-l}^{(i)}g_j^{(i)},\frac{d(\gamma^*T_{r_i-l}^{(i)})}{dt}\right] \\
&\quad -\sum_{j=0}^{k-1}\sum_{l=0}^{k-j}(l-r_i)\left[\bar{g}_{k-j-l}^{(i)}\frac{dg_j^{(i)}}{dt},\gamma^*T_{r_i-l}^{(i)}\right] \quad (k\le r_i-\delta_{0i}), \\
\frac{dg_k^{(i)}}{dt} &= 0 \quad (k>r_i-\delta_{0i})
\end{aligned}
\tag{7}
$$

with $(g_0^{(i)})^{-1}g_k$ off-diagonal. We rewrite the first equation as

$$
-r_i\left[(g_0^{(i)})^{-1}\frac{dg_k^{(i)}}{dt},\gamma^*T_{r_i}^{(i)}\right] = k\left[(g_0^{(i)})^{-1}g_k^{(i)},\frac{d(\gamma^*T_{r_i}^{(i)})}{dt}\right]+\left\langle\widetilde{\gamma}^*\Gamma_{i,k},\frac{d}{dt}\right\rangle,
$$

where

$$
\Gamma_{i,k}:=\sum_{j=1}^{k-1}\sum_{l=0}^{k-j}\left(j[\bar{g}_{k-j-l}^{(i)}g_j^{(i)},d_{\mathbb{D}_{\mathrm{irr}}}T_{r_i-l}^{(i)}]-(l-r_i)[\bar{g}_{k-j-l}^{(i)}\partial_{\mathbb{D}_{\mathrm{irr}}}g_j^{(i)},T_{r_i-l}^{(i)}]\right).
$$

Note that it only depends on $g_j^{(i)}$, $j<k$ and $\mathbf{T}$. For $\alpha,\beta=1,2,\dots,n$ with $\alpha\neq\beta$, the (α,β)-entry of the above equation is expressed as

$$
\left((g_0^{(i)})^{-1}\frac{dg_k^{(i)}}{dt}\right)_{\alpha\beta} = -\frac{k}{r_i}\left((g_0^{(i)})^{-1}g_k^{(i)}\right)_{\alpha\beta}\frac{d}{dt}\log(t_{\alpha,r_i}^{(i)}-t_{\beta,r_i}^{(i)})+\frac{1}{r_i}\frac{(\Gamma_{i,k})_{\alpha\beta}}{t_{\alpha,r_i}^{(i)}-t_{\beta,r_i}^{(i)}},
$$

where we write $T_j^{(i)}=\mathrm{diag}(t_{\alpha,j}^{(i)})_{\alpha=1}^n$ and abbreviate $\gamma^*t_{\alpha,r_i}^{(i)}$ as $t_{\alpha,r_i}^{(i)}$. Assuming that $g_j^{(i)}$ for $j<k$ are known, this equation can be explicitly solved as

$$
\left((g_0^{(i)})^{-1}g_k^{(i)}\right)_{\alpha\beta} = \frac{1}{r_i}(t_{\alpha,r_i}^{(i)}-t_{\beta,r_i}^{(i)})^{-\frac{k}{r_i}}\int(t_{\alpha,r_i}^{(i)}-t_{\beta,r_i}^{(i)})^{\frac{k}{r_i}-1}(\Gamma_{i,k})_{\alpha\beta}.
$$

It follows that the connection is complete. □

5. Isomonodromy equations as completely integrable non-autonomous Hamiltonian systems

Let $\pi\colon\widetilde{\mathbb{D}}_{\mathrm{irr}}\to\mathbb{D}_{\mathrm{irr}}$ be a universal covering and choose a symplectic manifold $\mathcal{M}$ isomorphic to fibers of $\mathcal{M}(\mathbf{L})_{\mathrm{irr}}$. Then the results in the previous section shows that there exists a trivialization $\varphi\colon\pi^*\mathcal{M}(\mathbf{L})_{\mathrm{irr}}\xrightarrow{\simeq}\mathcal{M}\times\widetilde{\mathbb{D}}_{\mathrm{irr}}$ of symplectic fiber bundles under which the connection given by ω corresponds to the trivial connection on

$\mathcal{M} \times \widetilde{\mathbb{D}}_{\mathrm{irr}}$. Taking the base change via the projection $\mathbb{D} \to \mathbb{D}_{\mathrm{irr}}$, we thus obtain a covering $\tilde{\pi}\colon \widetilde{\mathbb{D}}_{\mathrm{irr}} \times \mathbb{D}_{\mathrm{pl}} \to \mathbb{D}$ and a trivialization $\tilde{\varphi}\colon \tilde{\pi}^*\mathcal{M}(\mathbf{L}) \xrightarrow{\simeq} \mathcal{M} \times (\widetilde{\mathbb{D}}_{\mathrm{irr}} \times \mathbb{D}_{\mathrm{pl}})$. Now our main theorem is as follows:

THEOREM 5.1. *Under the trivialization $\tilde{\varphi}$, the isomonodromy equation is described as the completely integrable non-autonomous Hamiltonian system with Hamiltonian one-form ϖ.*

PROOF. It remains to show that our non-autonomous Hamiltonian system is integrable. First, the theorem of Adler–Kostant–Symes shows that the Hamiltonians are Poisson commutative: $\{\varpi, \varpi\} = 0$. We show that ϖ is closed with respect to the exterior derivation in the horizontal direction: $\partial_{\mathbb{D}}\varpi = 0$. The proof is similar to [**8**, Theorem 5.1]. Since the assertion is local, we may assume that $\widehat{u}$ is chosen so that it depends analytically on A. Applying $\partial_{\mathbb{D}}$ to the equality

$$A = \widehat{u}_i \Lambda_i \widehat{u}_i^{-1} + \widehat{u}_i' \widehat{u}_i^{-1} \quad (i = 1, 2, \ldots, m),$$

we obtain

$$\begin{aligned} \frac{\partial \Omega}{\partial x} = \partial_{\mathbb{D}} A = [&\partial_{\mathbb{D}} \widehat{u}_i \cdot \widehat{u}_i^{-1}, \widehat{u}_i \Lambda_i \widehat{u}_i^{-1}] \\ &+ \widehat{u}_i d_{\mathbb{D}} \Lambda_i \widehat{u}_i^{-1} + \partial_{\mathbb{D}} \widehat{u}_i' \cdot \widehat{u}_i^{-1} - \widehat{u}_i' \widehat{u}_i^{-1} \partial_{\mathbb{D}} \widehat{u}_i \cdot \widehat{u}_i^{-1}, \end{aligned}$$

i.e.,

$$\widehat{u}_i^{-1} \frac{\partial \Omega}{\partial x} \widehat{u}_i = [\widehat{u}_i^{-1} \partial_{\mathbb{D}} \widehat{u}_i, \Lambda_i] + d_{\mathbb{D}} \Lambda_i + \widehat{u}_i^{-1} \partial_{\mathbb{D}} \widehat{u}_i' - \widehat{u}_i^{-1} \widehat{u}_i' \widehat{u}_i^{-1} \partial_{\mathbb{D}} \widehat{u}_i$$

for $i > 0$. We use it to calculate $\partial_{\mathbb{D}}\varpi_i$ for $i > 0$:

$$\begin{aligned} -\partial_{\mathbb{D}} \varpi_i &= \operatorname*{res}_{x_i=0} \operatorname{tr}\left(\partial_{\mathbb{D}}(\widehat{u}_i^{-1}\widehat{u}_i') \wedge \Xi_i\right) \\ &= \operatorname*{res}_{x_i=0} \operatorname{tr}\left(\widehat{u}_i^{-1} \partial_{\mathbb{D}} \widehat{u}_i' \wedge \Xi_i\right) - \operatorname*{res}_{x_i=0} \operatorname{tr}\left(\widehat{u}_i^{-1} \partial_{\mathbb{D}} \widehat{u}_i \cdot \widehat{u}_i^{-1} \widehat{u}_i' \wedge \Xi_i\right) \\ &= \operatorname*{res}_{x_i=0} \operatorname{tr}\left(\widehat{u}_i^{-1} \frac{\partial \Omega}{\partial x} \widehat{u}_i \wedge \Xi_i\right) - \operatorname*{res}_{x_i=0} \operatorname{tr}\left([\widehat{u}_i^{-1} \partial_{\mathbb{D}} \widehat{u}_i, \Lambda_i] \wedge \Xi_i\right) \\ &\quad - \operatorname*{res}_{x_i=0} \operatorname{tr}\left(d_{\mathbb{D}} \Lambda_i \wedge \Xi_i\right) + \operatorname*{res}_{x_i=0} \operatorname{tr}\left(\widehat{u}_i^{-1} \widehat{u}_i' [\widehat{u}_i^{-1} \partial_{\mathbb{D}} \widehat{u}_i, \Xi_i]\right). \end{aligned}$$

The second and third terms are zero because $[\widehat{u}_i^{-1}\partial_{\mathbb{D}}\widehat{u}_i, \Lambda_i]$ is off-diagonal and $d_{\mathbb{D}}\Lambda_i \wedge \Xi_i$ has no residue. We show that the fourth term is also zero. First, we have

$$\begin{aligned} &\operatorname*{res}_{x_i=0} \operatorname{tr}\left(\widehat{u}_i^{-1} \widehat{u}_i' [\widehat{u}_i^{-1} \partial_{\mathbb{D}_{\mathrm{pl}}} \widehat{u}_i, \Xi_i]\right) \\ &\qquad = -\operatorname*{res}_{x_i=0} \operatorname{tr}\left(\widehat{u}_i^{-1} \widehat{u}_i' [\widehat{u}_i^{-1} \widehat{u}_i' \, da_i, \Xi_i]\right) \\ &\qquad = -\operatorname*{res}_{x_i=0} \operatorname{tr}\left([\widehat{u}_i^{-1} \widehat{u}_i', \widehat{u}_i^{-1} \widehat{u}_i' \, da_i] \wedge \Xi_i\right) = 0. \end{aligned}$$

Also, since $\widehat{u} \cdot \Lambda = A$, we can decompose $\widehat{u}_i$ as $\widehat{u}_i = \widehat{g}_i \widehat{h}_i$, where $\widehat{h}_i \in \mathrm{Stab}(\Lambda)$ and $\widehat{g}_i$ satisfies equality (4). Therefore Lemma 4.4 implies

$$\begin{aligned}[\widehat{u}_i^{-1}\partial_{\mathbb{D}_{\mathrm{irr}}}\widehat{u}_i, \Xi_i]_{i,-} &= \left(\widehat{h}_i^{-1}[\widehat{g}_i^{-1}\partial_{\mathbb{D}_{\mathrm{irr}}}\widehat{g}_i, \Xi_i]\widehat{h}_i\right)_{i,-} + [\widehat{h}_i^{-1}\partial_{\mathbb{D}_{\mathrm{irr}}}\widehat{h}_i, \Xi_i]_{i,-} \\ &= -\left(\widehat{h}_i^{-1}[\widehat{g}_i^{-1}\partial_{\mathbb{D}_{\mathrm{irr}}}\widehat{g}_i, \Lambda_i]\widehat{h}_i\right)_{i,-} \wedge da_i \\ &= -\left(\widehat{h}_i^{-1}[\widehat{g}_i^{-1}\widehat{g}_i', d_{\mathbb{D}_{\mathrm{irr}}}T_i]\widehat{h}_i\right)_{i,-} \wedge da_i \\ &= -[\widehat{u}_i^{-1}\widehat{u}_i', d_{\mathbb{D}_{\mathrm{irr}}}T_i]_{i,-} \wedge da_i.\end{aligned}$$

Thus we obtain

$$\operatorname*{res}_{x_i=0} \mathrm{tr}\left(\widehat{u}_i^{-1}\widehat{u}_i'[\widehat{u}_i^{-1}\partial_{\mathbb{D}}\widehat{u}_i, \Xi_i]\right) = -\operatorname*{res}_{x_i=0} \mathrm{tr}\left([\widehat{u}_i^{-1}\widehat{u}_i', \widehat{u}_i^{-1}\widehat{u}_i']d_{\mathbb{D}_{\mathrm{irr}}}T_i \wedge da_i\right) = 0.$$

Hence

$$-\partial_{\mathbb{D}}\varpi_i = \operatorname*{res}_{x_i=0} \mathrm{tr}\left(\frac{\partial\Omega}{\partial x} \wedge \widehat{u}_i\Xi_i\widehat{u}_i^{-1}\right) \quad (i = 1, 2, \ldots, m).$$

A similar argument shows

$$-\partial_{\mathbb{D}}\varpi_0 = \operatorname*{res}_{x_0=0} \mathrm{tr}\left(\frac{\partial\Omega}{\partial x_0} \wedge \widehat{u}_i(d_{\mathbb{D}}T_0)\widehat{u}_i^{-1}\right).$$

Hence

$$\begin{aligned}-\partial_{\mathbb{D}}\varpi &= \sum_{i=0}^{m} \operatorname*{res}_{x_i=0} \mathrm{tr}\left(\frac{\partial\Omega}{\partial x_i} \wedge \widehat{u}_i\Xi_i\widehat{u}_i^{-1}\right) \\ &= \sum_{i=0}^{m} \operatorname*{res}_{x_i=0} \mathrm{tr}\left(\frac{\partial\Omega}{\partial x_i} \wedge \Omega_i\right) + \sum_{i=0}^{m} \operatorname*{res}_{x_i=0} \mathrm{tr}\left(\frac{\partial\Omega}{\partial x_i} \wedge R_i\right),\end{aligned}$$

where $R_i := \widehat{u}_i\Xi_i\widehat{u}_i^{-1} - \Omega_i \in \mathcal{L}_i^+$. Since

$$\mathrm{tr}\left(\frac{\partial}{\partial x_i}(\Omega_i + R_i) \wedge (\Omega_i + R_i)\right) = \mathrm{tr}\, \frac{\partial}{\partial x_i}\Xi_i \wedge \Xi_i$$

has no residue at $x_i = 0$, we have

$$\operatorname*{res}_{x_i=0} \mathrm{tr}\left(\frac{\partial\Omega}{\partial x_i} \wedge R_i\right) = \frac{1}{2} \operatorname*{res}_{x_i=0} \mathrm{tr}\left(\frac{\partial}{\partial x_i}(\Omega_i + R_i) \wedge (\Omega_i + R_i)\right) = 0.$$

Also, we have

$$\operatorname*{res}_{x_i=0} \mathrm{tr}\left(\frac{\partial\Omega}{\partial x_i} \wedge \Omega\right) = \sum_{j=0}^{m} \operatorname*{res}_{x_i=0} \mathrm{tr}\left(\frac{\partial\Omega}{\partial x_i} \wedge \Omega_j\right)$$

$$= \sum_{j \neq i} \operatorname*{res}_{x_i=0} \operatorname{tr}\left(\frac{\partial \Omega_i}{\partial x_i} \wedge \Omega_j\right) + \sum_{j \neq i} \operatorname*{res}_{x_i=0} \operatorname{tr}\left(\frac{\partial \Omega_j}{\partial x_i} \wedge \Omega_i\right)$$

$$= 2 \sum_{j \neq i} \operatorname*{res}_{x_i=0} \operatorname{tr}\left(\frac{\partial \Omega_j}{\partial x_i} \wedge \Omega_i\right) = 2 \operatorname*{res}_{x_i=0} \operatorname{tr}\left(\frac{\partial \Omega}{\partial x_i} \wedge \Omega_i\right).$$

Thus we obtain

$$-\partial_{\mathbb{D}} \varpi = \frac{1}{2} \sum_{i=0}^{m} \operatorname*{res}_{x_i=0} \operatorname{tr}\left(\frac{\partial \Omega}{\partial x_i} \wedge \Omega\right) = 0$$

by the residue theorem. □

Acknowledgments. This work was supported by JSPS KAKENHI Grant Number 15K17552.

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