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Ramified irregular singularities of meromorphic connections and plane curve singularities

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Abstract. In this paper we propose similarity between ramified irregular singularities of meromorphic connections on formal disk and plane curve singularities. First we relate Komatsu-Malgrange irregularities of meromorphic connections to intersection numbers and Milnor numbers of plane curve germs. Next we see that local Fourier transforms of connections can be seen as blow up of plane curves. Moreover a necessary and sufficient condition for an irreducible connection to have a resolution of the ramified singularity is determined as an analogy of the resolution of plane curve singularities. Finally, for meromorphic connections we define an analogue of Puiseux characteristics which are topological invariants of plane curve singularities and show that it can be seen as an invariant of Stokes structures of meromorphic connections.

Introduction

Our interest in this paper is the similarity between ramified irregular singularities of meromorphic connections on formal disk and plane curve singularities. Meromorphic connections can be seen as modules over a "non-commutative" ring, the ring of differential operators, and plane curve germs have "commutative" rings as their local rings, stalks of structure sheaves. Between these commutative and non-commutative ones, we shall find similarities of (i) invariants: intersection numbers and Milnor numbers of curves and Komatsu-Malgrange irregularities of connections, of (ii) transformations: the blow up of curves and the local Fourier transform of connections, and of (iii) topological structures: knots arising from curve singularities and Stokes structures of connections.

To state our main theorems, we recall some definitions which are explained in detail in the latter sections. Let K be an algebraically closed field of characteristic zero. For a positive integer q and $f \in K(\!(x^{\frac{1}{q}})\!)$ with $-p/q = \mathrm{ord}(f) < 0$, let us define $E_{f,q} = (V, \nabla)$, a connection over $K(\!(x)\!)$, as follows. Regard $V = K(\!(x^{\frac{1}{q}})\!)$ as

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a $K((x))$-vector space and define $\nabla(v) = (\frac{d}{dx} + x^{-1}f)v$ for $v \in V$. To an irreducible $E_{f,q}$, we associate a plane curve germ,

$$C_{f,q}(x,y) = \prod_{k=1}^{q}\left(y - \frac{1}{f_k(x^{\frac{1}{q}})}\right),$$

where $f_k(x^{\frac{1}{q}}) = f(\zeta_q^k x^{\frac{1}{q}})$ and ζ_q is a primitive q-th root of unity. Then the intersection numbers $I(\ ,\ )$ and Milnor numbers μ of curve germs can be written by the irregularities $\mathrm{Irr}(\ )$ of connections as follows.

Theorem 0.1 (Theorem 3.5). *Let $E_{f,q} = (V, \nabla)$, $E_{g,q'} = (W, \nabla')$ be irreducible $K((x))$-connections. Set $-p/q = \mathrm{ord}(f)$, $-p'/q' = \mathrm{ord}(g)$. If $E_{f,q} \not\cong E_{g,q'}$, then*

$$I\left(C_{f,q}, C_{g,q'}\right) = pq' + p'q - \mathrm{Irr}(\mathrm{Hom}_{K((x))}(V, W)).$$

Theorem 0.2 (Theorem 3.6). *Let $E_{f,q}$ be an irreducible $K((x))$-connection with* $\mathrm{ord}(f) = -p/q$. *Then the Milnor number μ of the associated curve $C_{f,q}$ is*

$$\mu = (2p-1)(q-1) - \mathrm{Irr}(\mathrm{End}_{K((x))}(V)).$$

The Milnor number is a geometric invariant of a curve defined as the first Betti number of a Milnor fiber, the dimension of the Jacobian ring and so on. Also Komatsu-Malgrange irregularity is an analytic invariant of a connection defined as an index of the differential operator, the dimension of the Malgrange-Shibya cohomology (see [**4**]), etc. The formula in the above theorem connects these geometric invariant and analytic one. Moreover this formula may evoke a similarity of global invariants, genera of curves and indices of rigidity of connections. Namely the topological genus of the normalization of an algebraic curve on $\mathbb{P}^2$ can be written as the sum of Milnor numbers of the singular points and the index of rigidity of a meromorphic connection on $\mathbb{P}^1$ is written as the sum of Komatsu-Malgrange irregularities of the singular points (see [**2**] for instance) as well.

Next we consider similarity between the local Fourier transform of connections and the blow up of plane curves, which is already pointed out by Sabbah who successfully uses the blowing up technique to calculate an explicit formula of local Fourier transforms in [**25**] after the work of Roucairol in [**24**].

The explicit formula of the local Fourier transforms is independently obtained by two authors, Fang in [**11**] and Sabbah in [**25**], from the different point of view. Fang uses an algebraic computation and Sabbah uses a technique of the blowing up of curves. After their works Graham-Squire gave an simple description of their formulas in [**13**]. We shall see that Graham-Squire's description leads us to a formula which may connect Fang's and Sabbah's different approaches. Namely, we show that the local Fourier transform of $E_{f,q}$ can be seen as the blowing up of the associated curve $C_{f,q}$, see Proposition 3.8.

The Fourier-Laplace transform plays important roles in the theory of linear ordinary differential equations on the Riemann sphere. The local analogy of the transform, say the local Fourier transform, is introduced by Laumon [**18**] in the l-adic setting, and by Bloch-Esnault [**7**] and García Lopéz [**12**] in the complex domain to study local structures of the image of Fourier transform of global differential equations. There are many applications of this transform to the analytic theory of differential equations, for example, see the works of Mochizuki [**23**], Sabbah [**26**] and Hien-Sabbah[**15**] in which the local Fourier transform is successfully applied to study the Stokes structure of differential equations. This realization of the local Fourier transform as the blow up may introduce a new algebraic and geometric tool for the study of this analytic transformation.

Furthermore as an application of this realization, we use the blowing up technique to show the following necessary and sufficient condition for the existence of a resolution of ramified irregular singularities of irreducible connections via local Fourier transforms.

To an irreducible $E_{f,q}$, we associate a sequence of integers as follows. Let us write $f(x^{\frac{1}{q}}) = a_n x^{\frac{n}{q}} + a_{n+1} x^{\frac{n+1}{q}} + \cdots$. Define

$$-\beta_1 = \min\{i \mid a_i \neq 0,\, q \nmid i\}, \quad e_1 = \gcd(q, \beta_1).$$

Also define

$$-\beta_k = \min\{i \mid a_i \neq 0,\, e_{k-1} \nmid i\}, \quad e_k = \gcd(e_{k-1}, \beta_k),$$

inductively till we reach g with $e_g = 1$. Then we call the sequence of the integers

$$(q, p; \beta_1, \ldots, \beta_g),$$

the dual Puiseux characteristic of $E_{f,q}$.

Theorem 0.3 (Theorem 3.10). *Let $E_{f,q}$ be an irreducible connection with the dual Puiseux characteristic $(q, p; \beta_1, \ldots, \beta_g)$. Then we can reduce $E_{f,q}$ to a rank 1 connection by a finite iteration of local Fourier transforms and additions if and only if we have*

$$e_{i-1} \equiv \pm e_i \pmod{\beta_i}$$

for all $i = 1, \ldots, g$. Here $e_0 = q$.

In the final section, we shall moreover consider the following problem. We take K as the field of complex numbers $\mathbb{C}$. Let us fix a dual Puiseux characteristic

$(q, p; \beta_1, \dots, \beta_g)$ and consider a family

$$\mathcal{E} = \{E_{f,q} \colon \text{irreducible connection} \mid E_{f,q} \text{ has the dual Puiseux characteristic } (q, p; \beta_1, \dots, \beta_g)\}.$$

We look for an invariance of this family whose elements are not isomorphic as connections in general. For instance, it is well known that if two plane curve germs have the same Puiseux characteristic, then the knot structures of these curves around the singular point are isotopic. Namely the Puiseux characteristic gives an topological invariant of plane curve germs. The aim of this section is to look for the analogy for connections.

Let us fix an element $E_{f,q} \in \mathcal{E}$ and define $\tilde{f}(x^{\frac{1}{q}}) = \sum_{i=1}^{g} a_{\beta_i} x^{-\frac{\beta_i}{q}}$ where we write $f(x^{\frac{1}{q}}) = a_p x^{-\frac{p}{q}} + a_{p-1} x^{-\frac{p-1}{q}} + \cdots$. Also define $\tilde{f}_i(x^{\frac{1}{q}}) = \tilde{f}(\zeta_q^i x^{\frac{1}{q}})$ for $i = 1, \dots, q$. If x moves in a small circle $S_\eta = \{z \in \mathbb{C} \mid |z| = \eta\}$, the order of sizes of $\mathrm{Re}(\tilde{f}_i(x^{\frac{1}{q}}))$ for $i = 0, \dots, q-1$ will change according to the argument of x. Namely, we have $\mathrm{Re}(\tilde{f}_i(x^{\frac{1}{q}})) < \mathrm{Re}(\tilde{f}_j(x^{\frac{1}{q}}))$ for an argument, $\mathrm{Re}(\tilde{f}_i(x^{\frac{1}{q}})) > \mathrm{Re}(\tilde{f}_j(x^{\frac{1}{q}}))$ for another argument and there also are some arguments for which these are incomparable. This is one of the reasons why the Stokes phenomenon happens. Thus to understand the Stokes phenomenon of the connections over $\mathbb{C}(\{x\})$ formally isomorphic to $E_{f,q}$, we study the closed curve

$$\mathrm{St} = \left\{ (x, y) \,\middle|\, x \in S_\eta,\, y = \mathrm{Re}(\tilde{f}(x^{\frac{1}{q}})) \right\}.$$

Then Theorem 4.7 and Corollary 4.8 show that the curve St has an invariance which depends only on the dual Puiseux characteristic and is independent of $E_{f,q} \in \mathcal{E}$. The invariance is obtained from the structure of iterated torus knot of the associated curve germ $C_{\tilde{f},q}(x, y)$.

Our theorem show that a structure of the space of the Stokes matrices of $\mathbb{C}(\{x\})$-connections which is formally isomorphic to $E_{f,q}$ is determined by the curve St (see Theorem 4.5 for instance). Thus we may say that our theorem also gives an 'topological' invariant of wild fundamental groupoid [**22**] and wild character varieties [**8**].

1. Singularities of plane curve germs

In this section we give basic definitions and facts on singularities of plane curve germs, which are found in standard references [**10, 14, 29**] for example. Let K be an algebraically closed field of characteristic zero. Let $K[x], K[\![x]\!]$ and $K(\!(x)\!)$ denote the polynomial ring, the ring of formal power series and the field of formal Laurent series. For $f(x) = a_n x^n + a_{n+1} x^{n+1} + \cdots \in K(\!(x)\!)$, we call the lowest exponent with nonzero coefficient the *order* of f and denote by $\mathrm{ord}_x(f)$, i.e.,

$$\mathrm{ord}_x(f) = \min\{i \mid a_i \neq 0\}.$$

Similarly the multi-variable analogue, $K[x,y], K[\![x,y]\!]$ are defined. We can decompose $f(x,y) \in K[\![x,y]\!]$ as the sum of homogeneous terms,

$$f(x,y) = \cdots + f_k(x,y) + f_{k+1}(x,y) + \cdots ,$$

where $f_k(x,y) \in K[x,y]$ are homogeneous polynomials of degree k. The least integer k_0 such that $f_k(x,y) \neq 0$ is called *multiplicity* of f.

Definition 1.1. A *plane curve germ* is the equivalence class of a non-invertible element f of $K[\![x,y]\!]\backslash\{0\}$. Here $f, g \in K[\![x,y]\!]$ are equivalent when there is a unit $u \in K[\![x,y]\!]$ such that $f = ug$. A plane curve germ of multiplicity one is called *regular*. When the multiplicity is greater than one, the curve is called *singular*.

1.1. Good parametrizations

Suppose that $f(x,y) \in K[\![x,y]\!]\backslash\{0\}$ is *regular of order* $m > 0$ with respect to y, i.e., $f(0,y) \in K[\![y]\!]$ has the order m. Then the Weierstrass preparation theorem says that there exists an unit $u \in K[\![x,y]\!]$ such that

$$f(x,y) = u\left(y^m + \sum_{r=0}^{m-1} a_r(x) y^r\right)$$

where $a_r(x) \in K[\![x]\!]$.

Then Puiseux's theorem tells us that f can be decomposed as

$$f(x,y) = u\prod_{j=1}^{m}\left(y - g_j(x^{\frac{1}{m_j}})\right),$$

where $g_j(t) \in K[\![t]\!]$.

Definition 1.2. Let $f(x,y)$ be an irreducible element in $K[\![x,y]\!]$ and regular of order $l > 0$ with respect of y. Then we see that the equation $f(x,y) = 0$ admits at least one solution of the form $y = \phi(x^{\frac{1}{m}})$ with $\phi(t) \in K[\![t]\!]$. Here we may assume

$$m = \min\left\{r \in \mathbb{N} \,\middle|\, \phi \in K(\!(x^{\frac{1}{r}})\!)\right\}.$$

Then the parametrization $x = t^m, y = \phi(t)$ of the curve germ is called the *good parametrization*.

Conversely a good parametrization defines an irreducible curve germ as follows.

Let $x = t^m, y = \phi(t)$ be a good parametrization and define

$$f(x,y) = \prod_{i=1}^{m} \left(y - \phi(\zeta_m^i x^{\frac{1}{m}}) \right),$$

where ζ_m is a primitive m-th root of unity.

Let $x = t^m, y = \sum_{i \geq n} a_i t^i$ $(a_n \neq 0)$ be a good parametrization. Here we may assume $n \geq m$ because if not, we can take another parametrization $y = u^n, x = \sum_{i \geq m} b_i u^i$ $(b_m \neq 0)$ by solving $u^n = \sum_{i \geq n} a_i t^i$. Define β_1 to be the first exponent of $\sum_{i \geq n} a_i t^i$ which is indivisible by m and e_1 to be the greatest common divisor of m and β_1, i.e.,

$$\beta_1 = \min\{i \mid a_i \neq 0,\, m \not| i\}, \quad e_1 = \gcd(m, \beta_1).$$

Inductively define

$$\beta_k = \min\{i \mid a_i \neq 0,\, e_{k-1} \not| i\}, \quad e_k = \gcd(e_{k-1}, \beta_k)$$

till we reach g with $e_g = 1$.

Definition 1.3. For the above good parametrization, the sequence of positive integers

$$(m; \beta_1, \dots, \beta_g)$$

is called the *Puiseux characteristic* of the curve germ.

1.2. Blowing up

Let us recall the blowing up of the affine space $\mathbb{A}^2(K)$.

Definition 1.4. Let us define a subspace of $\mathbb{A}^2(K) \times \mathbb{P}^1(K)$ by

$$T = \{(x, y, (\xi : \eta)) \mid x\eta = y\xi\},$$

where $(\xi : \eta)$ is the homogeneous coordinate of $\mathbb{P}^1(K)$. Then the natural projection $\pi : T \to \mathbb{A}^2(K)$ is called the *blowing up* of $\mathbb{A}^2(K)$ with the center $O = (0, 0)$.

The projective line $\mathbb{P}^1(K)$ is covered by two open sets $U_1 = \{(\xi : \eta) \mid \xi \neq 0\}$ and $U_2 = \{(\xi : \eta) \mid \eta \neq 0\}$ which are isomorphic to $\mathbb{A}(K)$. Hence we can cover T by $T_1 = T \cap (\mathbb{A}^2(K) \times U_1)$ and $T_2 = T \cap (\mathbb{A}^2(K) \times U_2)$. Both T_1 and T_2 can be seen as $\mathbb{A}^2(K)$ by

$$\begin{array}{cccc} \rho_1 : & T_1 & \longrightarrow & \mathbb{A}^2(K) \\ & (x, y, (\xi : \eta)) & \longmapsto & (X, Y) = (x, \frac{\eta}{\xi}) \end{array},$$

$$\begin{array}{rccc} \rho_2\colon & T_2 & \longrightarrow & \mathbb{A}^2(K) \\ & (x, y, (\xi : \eta)) & \longmapsto & (X, Y) = (\frac{\xi}{\eta}, y). \end{array}$$

Thus the restrictions of π on T_1 and T_2 give transformations in $\mathbb{A}^2(K)$, say $\sigma_1 = \pi \circ \rho_1^{-1}$ and $\sigma_2 = \pi \circ \rho_2^{-1}$.

Definition 1.5. Transformations in $\mathbb{A}^2(K)$ defined by

$$\begin{array}{rccc} \sigma_1\colon & \mathbb{A}^2(K) & \longrightarrow & \mathbb{A}^2(K) \\ & (x_1, y_1) & \longmapsto & (x, y) = (x_1, x_1 y_1) \end{array},$$

$$\begin{array}{rccc} \sigma_2\colon & \mathbb{A}^2(K) & \longrightarrow & \mathbb{A}^2(K) \\ & (x_1, y_1) & \longmapsto & (x, y) = (x_1 y_1, y_1) \end{array},$$

are called *quadratic transforms.* These induce homomorphisms

$$\begin{array}{rccc} \sigma_1\colon & K[\![x, y]\!] & \longrightarrow & K[\![x_1, y_1]\!] \\ & x & \longmapsto & x_1 \\ & y & \longmapsto & x_1 y_1 \end{array}, \qquad \begin{array}{rccc} \sigma_2\colon & K[\![x, y]\!] & \longrightarrow & K[\![x_1, y_1]\!] \\ & x & \longmapsto & x_1 y_1 \\ & y & \longmapsto & y_1 \end{array}.$$

These are called quadratic transforms as well.

If $f(x, y)$ has the multiplicity k, then $\sigma_1(f)(x_1, y_1)$ and $\sigma_2(f)(x_1, y_1)$ can be divided by x_1^k and y_1^k respectively.

Definition 1.6. Let $f(x, y)$ be a curve germ with the multiplicity k. Then $\sigma_1^*(f)(x_1, y_1) = \frac{1}{x_1^k} \sigma_1(f)(x_1, y_1)$ and $\sigma_2^*(f)(x_1, y_1) = \frac{1}{y_1^k} \sigma_2(f)(x_1, y_1)$ are called *strict transforms* of f.

Suppose that an irreducible curve germ f has a good parametrization $x = t^m$, $y = a_n t^n + a_{n+1} t^{n+1} + \cdots$ where $a_n \neq 0$ and $n \geq m$. Then the strict transform $\sigma_1^*(f)(x_1, y_1)$ has the good parametrization $x_1 = t^m$, $y_1 = a_n t^{n-m} + a_{n+1} t^{n-m+1} + \cdots$. Here we note that $\sigma_1(f)^*(0, 0) \neq 0$, i.e., σ_1^* is invertible if $n = m$. Thus we define

$$\sigma^*(f)(x_1, y_1) = \begin{cases} \sigma_1^*(f)(x_1, y_1) & \text{if } n > m, \\ \sigma_1^*(f)(x_1, y_1 - a_n) & \text{if } n = m, \end{cases}$$

and call $\sigma^*(f)(x_1, y_1)$ the *blowing up* of f. If f has a good parametrization $y = u^n$, $x = b_m u^m + b_{m+1} u^{m+1} + \cdots$ where $b_m \neq 0$ and $m \geq n$, then we define

$$\sigma^*(f)(x_1, y_1) = \begin{cases} \sigma_2^*(f)(x_1, y_1) & \text{if } m > n, \\ \sigma_2^*(f)(x_1 - b_m, y_1) & \text{if } m = n, \end{cases}$$

similarly.

Let us see how the blowing up changes Puiseux characteristics of curve germs.

Proposition 1.7 (see Theorem 3.5.5 in [**29**] for example). *For an irreducible curve germ $f(x,y)$ with the Puiseux characteristic $(m;\beta_1,\dots,\beta_g)$, we can compute the Puiseux characteristic of $\sigma^*(f)$ as follows.*

1. *If $\beta_1 > 2m$,*
$$(m;\beta_1-m,\dots,\beta_g-m).$$
2. *If $\beta_1 < 2m$ and $(\beta_1-m)\not| m$,*
$$(\beta_1-m;m,\beta_2-\beta_1+m,\dots,\beta_g-\beta_1+m).$$
3. *If $(\beta_1-m)\,|\,m$,*
$$(\beta_1-m;\beta_2-\beta_1+m,\dots,\beta_g-\beta_1+m).$$

1.3. Some invariants of curves

Definition 1.8. Let f,g be curve germs. Then the integer
$$I(f,g)=\dim_K K[\![x,y]\!]/\langle f,g\rangle$$
is called the *intersection number* of f and g. Here $\langle f,g\rangle$ is the ideal of $K[\![x,y]\!]$ generated by f,g.

Definition 1.9. Let $f(x,y)$ be a curve germ. Then the integer
$$\mu=I\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right)$$
is called the *Milnor number* of f.

These integers can be computed from good parametrizations and Puiseux characteristics as follows, see the section 4 in [**29**] and the section 7.4 in [**14**].

Lemma 1.10. *Let $f(x,y)$ be an irreducible curve germ with a good parametrization $x=t^m,\ y=\phi(t)=a_nt^n+a_{n+1}t^{n+1}+\cdots$ $(n\geq m)$, and the Puiseux characteristic $(m;\beta_1,\dots,\beta_g)$.*

1. *For a curve germ $g(x,y)\neq f(x,y)$, the intersection number $I(f,g)$ is equal to the order of $g(t^m,\phi(t))$.*

2. The Milnor number of f is

$$\mu = \sum_{i=1}^{g} (e_{i-1} - e_i)(\beta_i - 1).$$

Lemma 1.11 (see Corollary 7.16 and Theorem 7.18 in [**14**] for example). *Let $f(x,y)$ be an irreducible curve germ with $n = I(f,x)$ and $m = I(f,y)$. Then we have*

$$\mu = I\left(f, \frac{\partial f}{\partial x}\right) - m + 1, \qquad \mu = I\left(f, \frac{\partial f}{\partial y}\right) - n + 1.$$

2. Formal meromorphic connections on a disk

In this section we recall basic definitions and facts of formal meromorphic connections on a disk.

Definition 2.1. Let V be a finite dimensional vector space over $K(\!(x)\!)$. A *connection* on V is a K-linear map $\nabla\colon V \to V$ satisfying the Leibniz rule

$$\nabla(fv) = f\nabla(v) + \frac{df}{dx}\nabla(v)$$

for $f \in K(\!(x)\!)$ and $v \in V$. We call the pair (V, ∇) the $K(\!(x)\!)$-connection shortly. Sometimes we write (V, ∇_x) to emphasize the variable x.

The *rank* of (V, ∇) is the dimension of V as the $K(\!(x)\!)$-vector space. We say that (V, ∇) is *irreducible* if V has no proper nontrivial $K(\!(x)\!)$-subspace W such that $\nabla(W) \subset W$. Morphisms between connections (V_1, ∇_1) and (V_2, ∇_2) are $K(\!(x)\!)$-linear maps $\phi\colon V_1 \to V_2$ satisfying $\phi\nabla_1 = \nabla_2\phi$.

2.1. Indecomposable decompositions of connections

Let us give a quick review of indecomposable decompositions of connections based on the works of Hukuhara, Turrittin, Levelt, Balser-Jurkat-Lutz, Babbitt-Varadarajan and so on, [**16, 28, 19, 5, 3**]. We adopt the descriptions in [**13, 25**].

For a positive integer q and $f \in K(\!(x^{-\frac{1}{q}})\!)$, let us define $E_{f,q} = (V, \nabla)$, a connection over $K(\!(x)\!)$, as follows. Regard $V = K(\!(x^{\frac{1}{q}})\!)$ as a $K(\!(x)\!)$-vector space and define $\nabla(v) = (\frac{d}{dx} + x^{-1}f)v$ for $v \in V$. The irreducibility and isomorphic class of $E_{f,q}$ are determined as follows (see the section 3 in [**25**] for example). If $E_{f,q}$ and $E_{g,q}$ are isomorphic, then there exists an integer $0 \le r \le q-1$ such that

$$f(x^{\frac{1}{q}}) - g(\zeta_q^r x^{\frac{1}{q}}) \in R_q(x) = K(\!(x^{\frac{1}{q}})\!) / \left(x^{\frac{1}{q}} K[\![x^{\frac{1}{q}}]\!] + \frac{1}{q}\mathbb{Z} \right)$$

Also the converse is true. Let us define $R_q^o(x)$ as the set of $f \in R_q(x)$ that cannot be represented by elements of $K(\!(x^{\frac{1}{r}})\!)$ for any $0 < r < q$. Then the connection $E_{f,q}$ is irreducible if and only if the image of f in R^o_q.

PROPOSITION 2.2 (HUKUHARA-TURRITTIN-LEVELT DECOMPOSITION). *Every (V, ∇) decomposes as*

$$(V, \nabla) \cong \bigoplus_i (E_{f_i, q_i} \otimes J_{m_i})$$

where $f_i \in R^o_{q_i}(x)$ and $J_m = (\mathbb{C}(\!(x)\!)^{\oplus m}, \frac{d}{dx} + x^{-1} N_m)$ with the nilpotent Jordan block N_m of size m.

2.2. Local Fourier transforms

The local Fourier transform is introduced by Laumon, Bloch-Esnault and García López [**18, 7, 12**] to analyze formal local structures of the Fourier transform of meromorphic connections on $\mathbb{P}^1$. In this paper, we consider the local Fourier transform only for $E_{f,q}$ following Proposition 3.7, 3.9 and 3.12 in [**7**] and refer to original papers for general definitions and properties.

DEFINITION 2.3. Let $z, \hat{z}$ be indeterminates and set $\zeta = \frac{1}{z}, \hat{\zeta} = \frac{1}{\hat{z}}$.

1. Let $f \in R^o_q(z)$ and $f \neq 0$. Set $E_{f,q} = (V, \nabla_z)$. The connection $\mathcal{F}^{(0,\infty)}(E_{f,q}) = (V, \hat{\nabla}_{\hat{\zeta}})$ over $K(\!(\hat{\zeta})\!)$ is defined by K-linear operators on V,
$$\hat{\zeta} = -\nabla_z^{-1} \colon V \to V, \quad \hat{\nabla}_{\hat{\zeta}} = -\hat{\zeta}^{-2} z \colon V \to V.$$
2. Let $f \in R^o_q(\zeta)$, $\mathrm{ord}(f) = -p/q$, $f \neq 0$. Set $E_{f,q} = (V, \nabla_\zeta)$ and suppose that $p < q$. Then the connection $\mathcal{F}^{(\infty,0)}(E_{f,g}) = (V, \hat{\nabla}_{\hat{z}})$ over $K(\!(\hat{z})\!)$ is obtained by K-linear operators on V,
$$\hat{z} = \zeta^2 \nabla_\zeta \colon V \to V, \quad \hat{\nabla}_{\hat{z}} = -\zeta^{-1} \colon V \to V.$$
3. Let $f \in R^o_q(\zeta)$, $\mathrm{ord}(f) = -p/q$, $f \neq 0$. Set $E_{f,q} = (V, \nabla_\zeta)$ and suppose that $p > q$. Then the connection $\mathcal{F}^{(\infty,\infty)}(E_{f,g}) = (V, \hat{\nabla}_{\hat{\zeta}})$ over $K(\!(\hat{\zeta})\!)$ is obtained by K-linear operators on V,
$$\hat{z} = \zeta^2 \nabla_\zeta \colon V \to V, \quad \hat{\zeta}^2 \hat{\nabla}_{\hat{\zeta}} = -\zeta^{-1} \colon V \to V.$$

The following theorem for the explicit structures of local Fourier transforms $\mathcal{F}^{(*,*)}(E_{f,q})$ is due to Fang and Sabbah. We adopt the formulation given by Graham-Squire in [**13**] who gave a simple proof of the theorem.

THEOREM 2.4 (J. FANG [**11**] AND C. SABBAH [**25**]). *1. Let $f \in R^o_q(z)$,*

$\mathrm{ord}(f) = -p/q$ *and* $f \neq 0$. *Then*

$$\mathcal{F}^{(0,\infty)}(E_{f,q}) \cong E_{g,p+q},$$

where $g \in R^o_{p+q}(\hat{\zeta})$ *is determined by*

$$f = -z\hat{z}, \qquad g = f + \frac{p}{2(p+q)}.$$

2. *Let* $f \in R^o_q(\zeta)$, $\mathrm{ord}(f) = -p/q$ *and* $f \neq 0$. *Suppose that* $p < q$. *Then*

$$\mathcal{F}^{(\infty,0)}(E_{f,q}) \cong E_{g,q-p},$$

where $g \in R^o_{q-p}(\hat{z})$ *is determined by*

$$f = z\hat{z}, \qquad g = -f + \frac{p}{2(q-p)}.$$

3. *Let* $f \in R^o_q(\zeta)$, $\mathrm{ord}(f) = -p/q$ *and* $f \neq 0$. *Suppose that* $p > q$. *Then*

$$\mathcal{F}^{(\infty,\infty)}(E_{f,q}) \cong E_{g,p-q},$$

where $g \in R^o_{p-q}(\hat{\zeta})$ *is determined by*

$$f = z\hat{z}, \qquad g = -f + \frac{p}{2(p-q)}.$$

By the above theorem, we can define the inversion of local Fourier transforms as follows. Let $g \in R^o_q(\hat{\zeta})$, $\mathrm{ord}(g) = -p/q$ and $g \neq 0$. Suppose that $p < q$. Then we define $\left(\mathcal{F}^{(0,\infty)}\right)^{-1}(E_{g,q})$ as $E_{f,q-p}$ where $f \in R^o_{q-p}(z)$ is determined by

$$g - \frac{p}{2q} = -z\hat{z}, \qquad g = f + \frac{p}{2q}.$$

Then we have

$$\mathcal{F}^{(0,\infty)}\left(\left(\mathcal{F}^{(0,\infty)}\right)^{-1}(E_{g,q})\right) \cong E_{g,q}, \quad \left(\mathcal{F}^{(0,\infty)}\right)^{-1}\left(\mathcal{F}^{(0,\infty)}(E_{f,q})\right) \cong E_{f,q}.$$

Similarly we can define $(\mathcal{F}^{(\infty,0)})^{-1}(E_{g,q})$. If $p > q$, then $(\mathcal{F}^{(\infty,\infty)})^{-1}(E_{g,q})$ is defined as well.

2.3. Irregularity

For a $K(\!(x)\!)$-connection (V, ∇), let us fix a basis and identify $V \cong K(\!(x)\!)^{\oplus n}$. Then we can write $\nabla = \frac{d}{dx} + A(x)$ with $A(x) \in M(n, K(\!(x)\!))$. Moreover we can choose a suitable basis so that $A(x) \in M(n, K[x^{-1}])$, see [**5**] for example. We call $A(x) \in M(n, K[x^{-1}])$ the *normalized matrix* of (V, ∇).

Let us take K as the field of complex numbers $\mathbb{C}$ and $\mathbb{C}(\{x\})$ denote the field of meromorphic functions near 0. The irregularity defined below measures the difference between formal and convergent solutions of $\frac{d}{dx} + A(x)$.

Definition 2.5 (see H. Komatsu [**17**] and B. Malgrange [**21**]). Let (V, ∇) be a $\mathbb{C}((x))$-connection. The *irregularity* of (V, ∇) is

$$\mathrm{Irr}(V, \nabla) = \chi\left(\frac{d}{dx} + A(x),\, \mathbb{C}((x))^{\oplus n}\right) - \chi\left(\frac{d}{dx} + A(x),\, \mathbb{C}(\{x\})^{\oplus n}\right).$$

Here $\chi(\Phi, V)$ is the *index* of the $\mathbb{C}$-linear map $\Phi\colon V \to V$, i.e.,

$$\chi(\Phi, V) = \dim_{\mathbb{C}} \mathrm{Ker}\, \Phi - \dim_{\mathbb{C}} \mathrm{Coker}\Phi.$$

It is known that $\mathrm{Irr}(V, \nabla)$ is independent from choices of normalized matrices $A(x)$. Moreover if we decompose

$$(V, \nabla) \cong \bigoplus_i (E_{f_i, q_i} \otimes J_{m_i})$$

as Proposition 2.2, the irregularity can be written by

$$\mathrm{Irr}((V, \nabla)) = -\sum_i ord(f_i).$$

Thus not only for $\mathbb{C}$ but also general K, we can define the irregularity by the above formula.

3. Formal meromorphic connections and associated curves

In this section we shall define curve germs associated to irreducible formal meromorphic connections. Then intersection numbers and Milnor numbers of these curves will be written by the irregularities of the connections. Next we shall see the relationship between the local Fourier transforms of connections and the blowing up of the curve germs. Finally we shall determine a necessary and sufficient condition for an irreducible formal connection to have a resolution of ramified irregular singularities via local Fourier transforms.

3.1. Associated curves

Let us take $f \in K((x^{\frac{1}{q}}))$ so that the image is in $R_q^o \backslash \{0\}$. Then the curve germ associated to the irreducible $E_{f,q}$ is defined as follows.

Definition 3.1. The *associated curve germ* of an irreducible $K((x))$-

connection $E_{f,q}$ is

$$C_{f,q}(x,y) = \prod_{i=1}^{q} \left(y - \frac{1}{f_i(x^{\frac{1}{q}})} \right),$$

where $f_i(x^{\frac{1}{q}}) = f(\zeta_q^i x^{\frac{1}{q}})$.

To the above $E_{f,q}$, we associate a sequence of integers as an analogy of the Puiseux characteristic of curves. Let us write $f(x^{\frac{1}{q}}) = a_n x^{\frac{n}{q}} + a_{n+1} x^{\frac{n+1}{q}} + \cdots$. Define

$$-\beta_1 = \min\{i \mid a_i \neq 0,\ q \not| i\}, \quad e_1 = \gcd(q, \beta_1).$$

Also define

$$-\beta_k = \min\{i \mid a_i \neq 0,\ e_{k-1} \not| i\}, \quad e_k = \gcd(e_{k-1}, \beta_k),$$

inductively till we reach g with $e_g = 1$.

Definition 3.2. Let $E_{f,q}$ be as above with $-p/q = \mathrm{ord}(f)$. Then the sequence of the integers

$$(q, p; \beta_1, \ldots, \beta_g)$$

is called the *dual Puiseux characteristic* of $E_{f,q}$.

Let us compare the dual Puiseux characteristic of $E_{f,q}$ with the Puiseux characteristic of $C_{f,q}$.

Proposition 3.3. *Let $E_{f,q}$ and $C_{f,q}$ be as above and*

$$(q, p; \beta_1, \ldots, \beta_g)$$

the dual Puiseux characteristic of $E_{f,q}$. Then the Puiseux characteristic of $C_{f,q}$ is

$$(q; 2p - \beta_1, \ldots, 2p - \beta_g), \quad \text{if } p \geq q,$$

$$(p; p + q - \beta_1, \ldots, p + q - \beta_g), \quad \text{if } p < q.$$

To prove this proposition, we need some preparations. Let $S \subset \mathbb{Z}_{\geq 0}$ be an additive semigroup including 0 and S_0 a subset of S such that if $s = s' + s''$ with $s \in S_0$ and $s', s'' \in S$ then either $s' = 0$ or $s'' = 0$. Write $\mathcal{O}_S$ for the set of power series $\sum_r a_r t^r$ such that $a_r = 0$ for all $r \notin S$ and $\mathcal{O}_S^*$ for the subset satisfying the further condition $a_r \neq 0$ for all $r \in S_0$.

Lemma 3.4 (cf. Lemma 3.5.4 in [**29**]).

1. *Let* $(t\alpha(t))^m = t^m\gamma(t)$ *with* $m \in \mathbb{Z}$, $\alpha(t), \gamma(t) \in K[\![t]\!]$, $\alpha(0) \neq 0$. *Then* $\alpha \in \mathcal{O}_S$ *if and only if* $\gamma \in \mathcal{O}_S$, *and* $\alpha \in \mathcal{O}_S^*$ *if and only if* $\gamma \in \mathcal{O}_S^*$
2. *Let* $\alpha \in K[\![t]\!]$ *with* $\alpha(0) \neq 0$ *and let* $\beta \in K[\![t]\!]$ *be such that* $t = u\beta(u)$ *solves* $u = t\alpha(t)$. *Then* $\alpha \in \mathcal{O}_S$ *if and only if* $\beta \in \mathcal{O}_S$ *and* $\alpha \in \mathcal{O}_S^*$ *if and only if* $\beta \in \mathcal{O}_S^*$.

Proof. If we replace the condition $m \in \mathbb{Z}$ in (1) with $m \in \mathbb{N}$, then this is nothing but Lemma 3.4.5 in Wall's book [**29**]. Thus we show only the case $m = -1$ in (1). Although this follows from the same argument of the lemma in the Wall's book, we give a proof for the completeness of the paper. Write $\alpha(t) = \sum_{r=0}^{\infty} a_r t_r$ with $a_0 \neq 0$ and $\gamma(t) = \sum_{r=0}^{\infty} \gamma_r t^r$. Then

$$\sum_{r=0}^{\infty} \gamma_r t^r = \left(\sum_{r=0}^{\infty} a_r t_r\right)^{-1}.$$

We may assume that $a_0 = 1$. Then

$$\begin{aligned} \gamma_0 &= 1, \\ \gamma_1 + a_1\gamma_0 &= 0, \\ &\cdots \\ \gamma_k + a_1\gamma_{k-1} + \cdots + a_k\gamma_0 &= 0, \\ &\cdots. \end{aligned}$$

Thus γ_r are linear combinations of $a_{r_1} \cdots a_{r_m}$ with $r = r_1 + \cdots + r_m$. If $\gamma_r \neq 0$, then there exist $r_1, \ldots, r_m$ such that $r_1 + \cdots + r_m = r$ and $a_{r_1} \cdots a_{r_m} \neq 0$. Equivalently $r_1, \ldots, r_m \in S$. Since S is a semigroup, this means $r \in S$. Conversely, suppose $\gamma \in \mathcal{O}_S$ and that for each $r < k$ with $r \notin S$ we have $a_r = 0$. If $k \notin S$, $\gamma_k = na_k$ with a nonzero integer n. Thus $a_k = 0$ and it follows by induction on k that $\alpha \in \mathcal{O}_S$. If further $p \in S_0$, then p can not decompose by elements in S. Thus $\gamma_p = na_p$ with a nonzero integer n. Thus indeed $\gamma_p \neq 0$ if and only if $a_p \neq 0$. □

Proof of Proposition 3.3. We trace the argument of Theorem 3.5.5 in [**29**]. Set $x = t^q$ and $f(x^{\frac{1}{q}}) = t^{-p}\alpha(t)$ with $\alpha(t) \in K[\![t]\!]$, $\alpha(0) \neq 0$. Then the associated curve $C_{f,q}$ has a good parametrization $x = t^q$, $y = t^p\alpha(t)^{-1}$. Let

$$S = \{r \in \mathbb{Z} \mid \text{for some } q \geq 1,\ r \geq p - \beta_q \text{ and } e_q | r\},$$

and $S_0 = \{p - \beta_q \mid q \geq 1\}$. Then from the hypothesis, we have $\alpha \in \mathcal{O}_S^*$ which shows $\alpha^{-1} \in \mathcal{O}_S^*$ by Lemma 3.4. This shows the proposition in the case $p \geq q$. If $p < q$, we can put $y = t^p\alpha(t)^{-1} = (t\beta(t))^p$ with $\beta \in \mathcal{O}_S^*$ by Lemma 3.4. Set $u = t\beta(t)$, so that $y = u^p$. We can write $t = u\gamma(u)$ with $\gamma \in \mathcal{O}_S^*$. Thus $x = t^q = (u\gamma(u))^q = u^q(\delta(u))$ with $\delta(u) \in \mathcal{O}_S^*$ which completes the proof. □

3.2. Irregularity of connections and curve invariants

Let us see that the irregularity of connections relates some curve invariants, intersection numbers and Milnor numbers, of associated curve germs.

Theorem 3.5. *Let $E_{f,q} = (V,\nabla)$, $E_{g,q'} = (W,\nabla')$ be irreducible $K((x))$-connections. Set $-p/q = \mathrm{ord}(f)$, $-p'/q' = \mathrm{ord}(g)$. If $E_{f,q} \not\cong E_{g,q'}$, then*

$$I\left(C_{f,q}, C_{g,q'}\right) = pq' + p'q - \mathrm{Irr}(\mathrm{Hom}_{K((x))}(V,W)).$$

Here $\mathrm{Hom}_{K((x))}(V,W)$ can be naturally seen as a $K((x))$-connection through the actions of ∇ and ∇'. Namely the connection ∇'' on $\mathrm{Hom}_{K((x))}(V,W)$ is defined by

$$\nabla''(\phi)(v) = \nabla'(\phi(v)) - \phi(\nabla(v))$$

for $\phi \in \mathrm{Hom}_{K((x))}(V,W)$ and $v \in V$. Similarly we have

$$I\left(C_{f,q}, \frac{\partial}{\partial y} C_{f,q}\right) = 2p(q-1) - \mathrm{Irr}(\mathrm{End}_{K((x))}(V)).$$

Proof. The associated curve germ $C_{f,q}$ has a good parametrization, $x = t^q$, $y = \alpha(t) = \frac{1}{f(x^{\frac{1}{q}})}$. Thus

$$\begin{aligned}
I\left(C_{f,q}, C_{g,q'}\right) &= \mathrm{ord}_t C_{g,q'}(t^q, \alpha(t)) \\
&= \mathrm{ord}_t \prod_{i=1}^{q'} \left(\alpha(t) - \frac{1}{g_i(t^{\frac{q}{q'}})}\right) \\
&= q \cdot \mathrm{ord}_x \prod_{i=1}^{q'} \left(\frac{1}{f(x^{\frac{1}{q}})} - \frac{1}{g(x^{\frac{1}{q'}})}\right) \\
&= q \cdot \mathrm{ord}_x \prod_{i=1}^{q'} \left(\frac{g_i - f}{f g_i}\right) \\
&= pq' + p'q + q \cdot \mathrm{ord}_x \prod_{i=1}^{q'} (g_i - f).
\end{aligned}$$

Let us note that the intersection number does not depend on good parametrizations,

$x = t^q, y = 1/f_j$, $j = 0, \ldots, q-1$. Thus

$$\operatorname{ord}_x \prod_{i=1}^{q'} (g_i - f_j) = \operatorname{ord}_x \prod_{i=1}^{q'} (g_i - f_{j'})$$

for $1 \leq j, j' \leq q$. On the other hand, we have

$$\operatorname{Irr}(\operatorname{Hom}_{K((x))}(V, W)) = -\operatorname{ord}_x \prod_{i=1}^{q'} \prod_{j=1}^{q} (g_i - f_j).$$

Combining these equations, we have the required one.

Let us see the second assertion. Since $C_{f,q} = \prod_{i=1}^{q}(y - 1/f_i)$, we have $\frac{\partial}{\partial y} C_{f,q} = \sum_{i=1}^{q} \prod_{j \neq i} (y - 1/f_j)$. Thus we can show

$$I\left(C_{f,q}, \frac{\partial}{\partial y} C_{f,q}\right) = \operatorname{ord}_x \prod_{\substack{1 \leq i,j \leq q \\ i \neq j}} \left(\frac{1}{f_i} - \frac{1}{f_j}\right),$$

as above. Also recall

$$\operatorname{Irr}(\operatorname{End}_{K((x))}(V)) = -\operatorname{ord}_x \prod_{\substack{1 \leq i,j \leq q \\ i \neq j}} (f_i - f_j).$$

These equations show the required one as above. □

THEOREM 3.6. *Let $E_{f,q}$ be an irreducible $K((x))$-connection with* $\operatorname{ord}(f) = -p/q$. *Then the Milnor number μ of the associated curve $C_{f,q}$ is*

$$\mu = (2p-1)(q-1) - \operatorname{Irr}(\operatorname{End}_{K((x))}(V)).$$

PROOF. This follows from Lemma 1.11 and Proposition 3.5. □

We end this subsection with the following proposition which relates the irregularity and dual Puiseux characteristics.

PROPOSITION 3.7. *Let $E_{f,q}$ be an irreducible $K((x))$-connection with the dual Puiseux characteristic $(q, p; \beta_1, \ldots, \beta_g)$. Then we have*

$$\operatorname{Irr}(\operatorname{End}_{K((x))}(E_{f,q})) = \sum_{i=1}^{g} (e_{i-1} - e_i)\beta_i.$$

PROOF. If $p \geq q$, this follows from Lemma 1.10, Proposition 3.3 and Theorem 3.6. However Proposition 4.13 in [**29**] leads us to the following direct proof. Let us consider

$$\operatorname{ord}_x \prod_{i=1}^{q-1} (f - f_i).$$

Here $f_i(x^{\frac{1}{q}}) = f(\zeta_q^i x^{\frac{1}{q}})$. Since $i\frac{\beta_1}{q}$ is an integer if and only if i is divisible by $\frac{q}{e_1}$, we have $\operatorname{ord}_x(f - f_i) = -\beta_1/q$ for i such that $\frac{q}{e_1} \not| i$ and this happens $q - e_1$ times. Similarly, we can see that $\operatorname{ord}_x(f - f_i) = -\beta_j/q$ if and only if i is divisible by $\frac{q}{e_{j-1}}$ but not by $\frac{q}{e_j}$ and this happens $e_{j-1} - e_j$ times. Hence we have

$$\operatorname{ord}_x \prod_{i=1}^{q-1} (f - f_i) = -\frac{1}{q} \sum_{i=1}^{g} (e_{i-1} - e_i)\beta_i$$

which induces the required formula. □

3.3. Local Fourier transforms and birational transforms

In [**11**] and [**25**], Fang and Sabbah computed explicit structures of local Fourier transforms of $E_{f,q}$ as we saw in Theorem 2.4. Whereas Fang's computation is relatively direct algebraic calculation, Sabbah's is based on the blowing up technique of plane curve singularities. The proposition below may connect these two different approaches. Roughly to say, $\mathcal{F}^{(0,\infty)}$ and $\mathcal{F}^{(\infty,0)}$ can be seen as the blowing up of associated curves and $\mathcal{F}^{(\infty,\infty)}$ corresponds to the birational transform

$$\sigma_3\colon \begin{aligned} x &\mapsto x_1^{-1} y_1 \\ y &\mapsto \quad y_1. \end{aligned}$$

More precisely, let us consider an irreducible curve germ $C(x, y)$ with the good parametrization

$$\begin{aligned} &x = t^m \alpha(t), \quad \alpha(0) \neq 0, \\ &y = t^n \end{aligned}$$

and assume $m \leq n$. Then define an irreducible curve germ $\sigma_3^*(C(x, y))(x_1, y_1)$ so that the good parametrization of this curve is

$$\begin{aligned} x_1 &= t^{n-m} \alpha(t)^{-1}, \\ y_1 &= t^n. \end{aligned}$$

Proposition 3.8. *Let us take* $f \in K(\!(x^{\frac{1}{q}})\!)$ *so that the image is in* $R_q^o(x)\backslash\{0\}$ *and* $\operatorname{ord}(f) = -p/q$.

1. *Suppose that* $p < q$. *Let us take* $g \in K(\!(x_1^{\frac{1}{q-p}})\!)$ *so that*

$$\mathcal{F}^{(\infty,0)}(E_{f,q}) \cong E_{\dot{g},q-p},$$

as in Theorem 2.4 where $\dot{g} = g + \frac{p}{2(q-p)}$. *Then we have*

$$C_{g,q-p}(x_1, -y_1) = \sigma_2^*(C_{f,q}(x,y)).$$

2. *Let us take* $g \in K(\!(x_1^{\frac{1}{p+q}})\!)$ *so that*

$$\mathcal{F}^{(0,\infty)}(E_{f,q}) \cong E_{\dot{g},p+q},$$

as in Theorem 2.4 where $\dot{g} = g + \frac{p}{2(p+q)}$. *Then we have*

$$C_{f,q}(-x, y) = \sigma_2^*(C_{g,p+q}(x_1, y_1)).$$

3. *Suppose that* $p > q$. *Let us take* $g \in K(\!(x_1^{\frac{1}{p-q}})\!)$ *so that*

$$\mathcal{F}^{(\infty,\infty)}(E_{f,q}) \cong E_{\dot{g},p-q},$$

as in Theorem 2.4 where $\dot{g} = g + \frac{p}{2(p-q)}$. *Then we have*

$$C_{g,p-q}(x_1, -y_1) = \sigma_3^*(C_{f,q}(x,y)).$$

Proof. It may suffices to show (1), since the others are similar. The curve germs $C_{f,q}$ and $C_{\tilde{g},q-p}$ have good parametrizations $x = t^q$, $y = \alpha(t) = 1/f(x^{\frac{1}{q}})$ and $x_1 = u^{q-p}$, $y_1 = \beta(u) = 1/g(x_1^{\frac{1}{q-p}})$ respectively. By Theorem 2.4, we have

$$x_1 = xf(x^{\frac{1}{q}}), \qquad y_1 = \beta(u) = \frac{1}{g(x_1^{\frac{1}{q-p}})} = -\frac{1}{f(x^{\frac{1}{q}})},$$

that is,

$$x_1 y_1 = -x, \qquad y_1 = -\, y.$$

Since each irreducible curve germ is determined by a good parametrization, we are done. □

3.4. Resolution of ramified irregular singularities

In the previous section, we saw that local Fourier transforms could be regarded as the birational transforms of associated curve germs. As is well known, singularities of plane curve germs have a resolution via blowing up. We shall seek an analogy of the resolution of singularities for irreducible connections via local Fourier transforms.

First, let us see how the local Fourier transforms change dual Puiseux characteristics of connections.

Proposition 3.9. *Suppose that an irreducible $E_{f,q}$ has the dual Puiseux characteristic $(q, p; \beta_1, \dots, \beta_g)$.*

1. *If $p < q$, then the dual Puiseux characteristic of $\mathcal{F}^{(\infty,0)}(E_{f,q})$ is*

$$(q-p, \beta_1; \beta_2, \dots, \beta_g) \text{ if } (q-p)|\beta_1,$$
$$(q-p, \beta_1; \beta_1, \dots, \beta_g) \text{ otherwise.}$$

   Here we note that $p = \beta_1$ under the assumption $p < q$.

2. *The dual Puiseux characteristic of $\mathcal{F}^{(0,\infty)}(E_{f,q})$ is*

$$(p+q, \beta_1; \beta_1, \dots, \beta_g) \text{ if } p = \beta_1,$$
$$(p+q, p; p, \beta_1, \dots, \beta_g) \text{ otherwise.}$$

3. *If $p > q$, then the dual Puiseux characteristic of $\mathcal{F}^{(\infty,\infty)}(E_{f,q})$ is*

$$(p-q, \beta_1; \beta_2, \dots, \beta_g) \text{ if } p = \beta_1 \text{ and } (p-q)|\beta_1,$$
$$(p-q, p; \beta_1, \dots, \beta_g) \text{ otherwise.}$$

Proof. We use the same notation in the proof of Proposition 3.3. First we note that $E_{h,r}$ and $E_{h+\alpha,r}$ with $h \in R_r^o$ and $\alpha \in K$ have the same dual Puiseux characteristic. Thus it is enough to know the Puiseux characteristic of $C_{g,*}$ in Proposition 3.8. Let $x = t^q$, $y = t^p\alpha(t)$, $\alpha \in \mathcal{O}_S^*$, $\alpha(0) \neq 0$ be a good parametrization of $C_{f,q}$. As we see in the proof of Proposition 3.3, we have another good parametrization $x = u^q\delta(u)$, $y = u^p$, $\delta \in \mathcal{O}_S^*$, $\delta(0) \neq 0$. Then by Proposition 3.8, $C_{g,q-p}$ has a good parametrization $x_1 = -u^{q-p}\delta(u)$, $y_1 = -u^p$. Solving $x_1 = s^{q-p}$, we have another good parametrization $x_1 = s^{q-p}$, $y_1 = s^p\gamma(s)$, $\gamma \in \mathcal{O}_S^*$, $\gamma(0) \neq 0$. Thus we have (1). We can show (2) in the similar way as (1).

Let us see (3). We have a good parametrization $x = t^q$, $y = t^p\alpha(t)$, $\alpha \in \mathcal{O}_S^*$, $\alpha(0) \neq 0$ of $C_{f,q}$. By solving $u^p = t^p\alpha(t)$, we have another good parametrization $x = u^q\delta(u)$, $y = u^p$ as above. Then by Proposition 3.8, $C_{g,p-q}$ has a good parametrization $\xi_1 = -u^{q-p}\delta(u)$, $y_1 = u^p$. Here $\xi_1 = 1/x_1$. Lemma 3.4 allows us to find $\epsilon(u) \in \mathcal{O}_S^*$, $\epsilon(0) \neq 0$ such that $x_1 = u^{p-q}\epsilon(u)$. Finally solving $x_1 = s^{p-q}$, we have another good parametrization $x_1 = s^{p-q}$, $y_1 = s^p\gamma(s)$, $\gamma \in \mathcal{O}_S^*$, $\gamma(0) \neq 0$. Thus we have (3). □

We can obtain $E_{f+ax^{-n},q}$ from $E_{f,q}$ by the tensor product

$$E_{f,q} \otimes \left(\mathbb{C}((x)), \frac{d}{dx} + ax^{-n-1}\right) \cong E_{f+ax^{-n},q},$$

and call this process the *addition*. Suppose that $E_{f,q}$ is irreducible and has the dual Puiseux characteristic $(q,p;\beta_1,\dots,\beta_g)$ with $p\neq\beta_1$. Then applying the addition repeatedly, we can obtain a connection with the dual Puiseux characteristic $(q,\beta_1;\beta_1,\dots,\beta_g)$.

The following theorem determines a necessary and sufficient condition for an irreducible $E_{f,q}$ to have a resolution of ramified irregular singularity via local Fourier transforms.

THEOREM 3.10. *Suppose that an irreducible $E_{f,q}$ has the dual Puiseux characteristic $(q,p;\beta_1,\dots,\beta_g)$. Then we can reduce $E_{f,q}$ to a rank 1 connection by a finite iteration of local Fourier transforms and additions if and only if we have*

$$e_{i-1}\equiv\pm e_i\ (\operatorname{mod}\beta_i)$$

for all $i=1,\dots,g$. Here $e_0=q$.

PROOF. First we assume that $e_{i-1}\equiv\pm e_i\ (\operatorname{mod}\beta_i)$ for all $i=1,\dots,g$. Applying additions, we may suppose that $E_{f,q}$ has the dual Puiseux characteristic $(q,\beta_1;\beta_1,\dots,\beta_g)$. If $q=e_0\equiv e_1\ (\operatorname{mod}\beta_1)$, then $q\geq\beta_1$ and Proposition 3.9 shows that we can reduce the connection to one with the dual Puiseux characteristic $(e_1,\beta_1;\beta_2,\dots,\beta_g)$ by a finite iteration of $\mathcal{F}^{(\infty,0)}$. If $q=e_0\equiv -e_1\ (\operatorname{mod}\beta_1)$, then Proposition 3.9 shows that we can reduce the connection to one with the dual Puiseux characteristic $(\beta_1-e_1,\beta_1;\beta_1,\dots,\beta_g)$ by a finite iteration of $\mathcal{F}^{(\infty,0)}$. Applying $\mathcal{F}^{(\infty,\infty)}$ to this connection, we have one with $(e_1,\beta_1;\beta_2,\dots,\beta_g)$. Thus in both cases, we moreover apply the addition and obtain a connection with $(e_1,\beta_2;\beta_2,\dots,\beta_g)$. We can repeat this process to reduce the connection to a rank 1 connection with $(e_g=1,\beta_g;\ )$ by our hypothesis.

Conversely, we assume that $E_{f,q}$ can be reduced to a rank 1 connection by local Fourier transforms and additions. Namely $E_{f,q}$ is constructed from a rank 1 connection by the inversion of local Fourier transforms.

Step 1. Let us start from a rank 1 connection with the dual Puiseux characteristic $(1,p;\ )$, $p>1$. Then possible inverse transformations are $(\mathcal{F}^{(\infty,0)})^{-1}$ and $(\mathcal{F}^{(\infty,\infty)})^{-1}$.

(1-i) Let us apply $(\mathcal{F}^{(\infty,0,)})^{-1}$. Then we have the dual Puiseux characteristic $(1+p,p;p)$. After applying possible inverse local Fourier transforms, $(\mathcal{F}^{(\infty,0,)})^{-1}$ and $(\mathcal{F}^{(0,\infty)})^{-1}$, we obtain the dual Puiseux characteristic $(q,p;p)$ where $q\equiv 1\ (\operatorname{mod}p)$ or go back to $(1,p;\ )$.

(1-ii) Let us apply $(\mathcal{F}^{(\infty,\infty)})^{-1}$. Then the resulting dual Puiseux characteristic is $(p-1,p;p)$. After applying possible inverse local Fourier transforms, $(\mathcal{F}^{(\infty,\infty)})^{-1}$ and $(\mathcal{F}^{(\infty,0)})^{-1}$, we obtain the dual Puiseux characteristic $(q,p;p)$ where $q\equiv -1\ (\operatorname{mod}p)$ or go back to $(1,p;\ )$.

Step 2. Next let us start from the dual Puiseux characteristic $(q,p;p)$ with $q\equiv$

$\pm 1 \pmod p$ and apply an addition. Then we have the dual Puiseux characteristic $(q, p_1; p)$ with $p_1 > p$. Set $e_1 = \gcd(q, p_1)$. Now possible inverse transformations are $(\mathcal{F}^{(\infty,0)})^{-1}$ and $(\mathcal{F}^{(\infty,\infty)})^{-1}$.

(2-i) Applying $(\mathcal{F}^{(\infty,0)})^{-1}$, we obtain $(q+p_1, p_1; p_1, p)$. After applying possible inverse local Fourier transforms, $(\mathcal{F}^{(0,\infty)})^{-1}$ and $(\mathcal{F}^{(\infty,0,)})^{-1}$, we obtain the dual Puiseux characteristic $(q_1, p_1; p_1, p)$ with $q_1 \equiv e_1 \pmod{p_1}$ or go back to $(q, p_1; p)$.

(2-ii) Applying $(\mathcal{F}^{(\infty,\infty)})^{-1}$, we obtain $(p_1 - q, p_1; p_1, p)$. After applying possible inverse local Fourier transforms, $(\mathcal{F}^{(\infty,\infty)})^{-1}$ and $(\mathcal{F}^{(0,\infty)})^{-1}$, we obtain the dual Puiseux characteristic $(q_1, p_1; p_1, p)$ with $q_1 \equiv -e_1 \pmod{p_1}$ or go back to $(q, p_1; p)$.

Our possible transformations are the iteration of these process. Thus the obtained dual Puiseux characteristic $(p, q; \beta_1, \ldots, \beta_g)$ satisfies the required conditions. □

4. Sequences of total orders and Stokes structures

In this section we restrict the field K to the field of complex number field $\mathbb{C}$. We denote the ring of convergent power series, the field of meromorphic functions near 0 and the ring of convergent power series of x and y by $\mathbb{C}\{x\}$, $\mathbb{C}(\{x\})$ and $\mathbb{C}\{x, y\}$ respectively. Let us define k-th root $x^{\frac{1}{k}}$ of x so that it takes a real value when x is real and positive. Let us consider $f \in \mathbb{C}(\!(x^{\frac{1}{q}})\!)$ whose image is in $R_q^o(x)\backslash\{0\}$ and suppose that $E_{f,q}$ has the dual Puiseux characteristic $(q, p; \beta_1, \ldots, \beta_s)$. Then we define

$$\tilde{f}(x^{\frac{1}{q}}) = \sum_{i=1}^{g} a_{\beta_i} x^{-\frac{\beta_i}{q}}$$

and $\tilde{f}_i(x^{\frac{1}{q}}) = \tilde{f}(\zeta_q^i x^{\frac{1}{q}})$ for $i = 1, \ldots, q$, where we write $f(x^{\frac{1}{q}}) = a_p x^{-\frac{p}{q}} + a_{p-1} x^{-\frac{p-1}{q}} + \cdots$. If x moves in a small circle $S_\eta = \{z \in \mathbb{C} \mid |z| = \eta\}$, the order of sizes of $\mathrm{Re}(\tilde{f}_i(x^{\frac{1}{q}}))$ for $i = 0, \ldots, q-1$ change according to the argument of x. This is one of the reasons of the Stokes phenomenon. Thus to understand the Stokes phenomenon of the connections over $\mathbb{C}(\{x\})$ formally isomorphic to $E_{f,q}$, we study the closed curve

$$\mathrm{St} = \left\{(x, y) \,\middle|\, x \in S_\eta,\ y = \mathrm{Re}(\tilde{f}(x^{\frac{1}{q}}))\right\}$$

in this subsection. This curve can be seen as the projection of the closed curve

$$K = \left\{(x, y) \,\middle|\, x \in S_\eta,\ y = \frac{1}{\tilde{f}(x^{\frac{1}{q}})}\right\}$$

by $y \mapsto \mathrm{Re}(1/y)$. The closed curve K is obtained by restricting $x \in S_\eta$ in the associated curve germ $C_{\tilde{f},q}(x, y) \in \mathbb{C}\{x, y\}$ and it is well known that K can be seen as an iterated torus knot.

4.1. Braids and Plane curve germs

Let us recall the well known theorem by Brauner that irreducible plane curve germs describe iterated torus knots around the singular point . The detail can be found in standard references ([**10**] for instance). Let $C(x,y) \in \mathbb{C}\{x,y\}$ be an irreducible plane curve germ with the Puiseux characteristic $(m;\beta_1,\dots,\beta_g)$. Exchanging x and y if necessary, we may assume that f has a good parametrization $x=t^m$, $y=\sum_{i\geq n} a_i t^i \in \mathbb{C}\{t\}, (a_n\neq 0)$, with $n\geq m$. If we let x run around a sufficiently small circle ,then

$$K=\left\{(x,y)\,\middle|\, x\in S_\eta,\ y=\sum_{i\geq n} a_i x^{\frac{i}{m}}\right\}$$

describes a knot in a solid torus

$$S_\eta\times D_\delta=\left\{(\eta e^{\sqrt{-1}s},\epsilon e^{\sqrt{-1}t})\mid s,t\in\mathbb{R},\,0\leq\epsilon\leq\delta\right\}$$

with a suitable $\delta>0$.

Theorem 4.1 (K. Brauner [**9**]). *The above K is an iterated torus knot of order g and type $(m/e_1,\beta_1/e_1),(e_1/e_2,\beta_2/e_2),\dots,(e_{g-1}/e_g,\beta_g/e_g)$.*

Now let us recall the construction the iterated torus knot from the good parametrization. First we decompose $y(x)$ as $y(x)=\sum_{k=1}^g a_{\beta_k}x^{\frac{\beta_k}{m}}+r(x)$ where $r(x)$ is the term of small oscillations which may be ignored. Thus we focus only on $\tilde{y}(x)=\sum_{k=1}^g a_{\beta_k}x^{\frac{\beta_k}{m}}$. Let us first look at $\tilde{y}^{(1)}=a_{\beta_1}x^{\frac{\beta_1}{m}}$. Then

$$K_1=\{(x,\tilde{y}^{(1)}(x))\mid x\in S_\eta\}$$

is the torus knot of type $(m/e_1,\beta_1/e_1)$ which can be seen as the closed braid of the geometric braid B_1 with the m/e_1 strings

$$\tilde{y}^{(1)}_l(t)=a_{\beta_1}\eta^{\frac{\beta_1}{m}}e^{\sqrt{-1}\frac{\beta_1}{m}(t+l)}\quad(0\leq t\leq 2\pi),$$

for $l=1,\dots,m/e_1$. Here we note that there exists a permutation $\tau_1\in\mathfrak{S}_{m/e_1}$ such that

$$\tilde{y}^{(1)}_l(t+2\pi)=\tilde{y}^{(1)}_{\tau_1(l)}(t)$$

for $l=1,\dots,m/e_1$. Here $\mathfrak{S}_n$ denotes the symmetric group of n symbols.

Then next, $\tilde{y}^{(2)}=a_{\beta_1}x^{\frac{\beta_1}{m}}+a_{\beta_2}x^{\frac{\beta_2}{m}}$ improves the approximation and

$$K_2=\{(x,\tilde{y}^{(2)}(x))\mid x\in S_\eta\}$$

is the iterated torus knot of order 2 and type $(m/e_1, \beta_1/e_1), (e_1/e_2, \beta_2/e_2)$. Indeed, for each $l_1 = 1, \ldots, m/e_1$, one has e_1/e_2 points

$$\tilde{y}^{(2)}_{l_1,l_2}(t) = a_{\beta_1}\eta^{\frac{\beta_1}{m}} e^{\sqrt{-1}\frac{\beta_1}{m}(t+l_1)} + a_{\beta_2}\eta^{\frac{\beta_2}{m}} e^{\sqrt{-1}\frac{\beta_2}{m}(t+l_2)} \quad (l_2 = 1, \ldots, e_1/e_2)$$

in the circle of radius $|a_{\beta_2}|\eta^{\frac{\beta_2}{m}}$ around the point $\tilde{y}^{(1)}_{l_1}(t)$. Thus for each l_1, we have the set $\widehat{B}_{l_1}$ of the strings $\tilde{y}^{(2)}_{l_1,l_2}(t)$ for $l_2 = 1, \ldots, e_1/e_2$. As we noted above, we can identify $\widehat{B}_{l_1}$ and $\widehat{B}_{\tau_1(l_1)}$ by substituting $t + 2\pi$ for t. Thus it suffices to see $\widehat{B}_{l_1}$ for one $l_1 \in \{1, \ldots, n/e_1\}$. Then $\widehat{B}_{l_1}$ defines a geometric braid B_2 if t runs in the interval $[0, (m/e_1)2\pi]$ and we have the torus knot of type $(e_1/e_2, \beta_2/e_2)$ as the closed braid of B_2.

Then one can repeat this process to refine the approximation and obtain the iterated torus knot of the plane curve $C(x, y)$.

4.2. Representations of sequences of total orders and local moduli of differential equations

For a connection $(\widehat{V}, \widehat{\nabla})$ over $\mathbb{C}(\{x\})$, i.e., the pair of finite dimensional $\mathbb{C}(\{x\})$-vector space $\widehat{V}$ and the $\mathbb{C}$-linear connection $\widehat{\nabla}$, the *formalization* (V, ∇) is the connection over $\mathbb{C}((x))$ defined by $V = \mathbb{C}((x)) \otimes_{\mathbb{C}(\{x\})} \widehat{V}$ and $\nabla(f \otimes \hat{v}) = \frac{d}{dx} f \otimes \hat{v} + f \otimes \widehat{\nabla}(\hat{v})$ for $f \in \mathbb{C}((x))$ and $\hat{v} \in \widehat{V}$. Let us fix a connection (V_0, ∇_0) over $\mathbb{C}((x))$ and consider a $\mathbb{C}(\{x\})$-connection $(\widehat{V}, \widehat{\nabla})$ whose formalization is isomorphic to (V_0, ∇_0). Let us fix an isomorphism $\xi\colon (V, \nabla) \to (V_0, \nabla_0)$ and call $((\widehat{V}, \widehat{\nabla}), \xi)$ a *marked pair* formally isomorphic to (V_0, ∇_0). We say that marked pairs $((\widehat{V}, \widehat{\nabla}), \xi)$ and $((\widehat{V}', \widehat{\nabla}'), \xi')$ are isomorphic if there exists an isomorphism $\hat{u}\colon (\widehat{V}, \widehat{\nabla}) \to (\widehat{V}', \widehat{\nabla}')$ as $\mathbb{C}(\{x\})$-connections such that $\xi = \xi' \circ u$ where u is the isomorphism between the formalizations of them induced by $\hat{u}$. The isomorphism class of marked pairs formally isomorphic to (V_0, ∇_0) is denoted by $\mathfrak{M}((V_0, \nabla_0))$. This local moduli space $\mathfrak{M}((V_0, \nabla_0))$ is studied by many authors (see for instance [**4**] and its references) and it is known that there exists a one to one correspondence from a space of certain unipotent matrices, so called Stokes matrices, to $\mathfrak{M}((V_0, \nabla_0))$ (see Theorem 4.5 for example).

In this subsection we see first that the structure of the space of Stokes matrices, i.e., the local moduli space $\mathfrak{M}((V_0, \nabla_0))$ is determined by a sequence of total orders of a finite set. Next we focus on the moduli of $E_{f,q}$ and show a structure theorem of the sequence of total orders by using the iterated torus knot of the associated curve.

4.2.1 Representations of sequences of total orders

Let I be a finite set and $<_0, <_1, \ldots, <_h$ $(h \geq 1)$ a sequence of total orders of I. We shortly denote the pair of I and the sequence by

$$\mathcal{I} = (I, (<_i)_{i=0,\ldots,h}).$$

Let us define a representation of $\mathcal{I}$. For $\nu = 1, \ldots, h$, define subsets of $I \times I$ by

$$\rho_\nu = \{(j,k) \in I \times I \mid j \neq k, k <_{\nu-1} j, j <_\nu k\}.$$

Here we note that ρ_ν is *anti-symmetric*, i.e., $(j,k) \in \rho_\nu$ contradicts $(k,j) \in \rho_\nu$ and *transitive*, i.e., $(j,k) \in \rho_\nu$ and $(k,l) \in \rho_\nu$ implies $(j,l) \in \rho_\nu$. For each $k \in I$, take a finite dimensional $\mathbb{C}$-vector space V_k. Then *representations of* $\mathcal{I}$ are elements in

$$\mathrm{Rep}(\mathcal{I}, (V_k)_{k \in I}) = \bigoplus_{\nu=1}^{h} \bigoplus_{(j,k) \in \rho_\nu} \mathrm{Hom}_{\mathbb{C}}(V_k, V_j).$$

We call $(\dim_{\mathbb{C}}(V_k))_{k \in I} \in (\mathbb{Z}_{\geq 0})^I$ the *dimension vector* of $\mathrm{Rep}(\mathcal{I}, (V_k)_{k \in I})$. For a vector $\alpha = (\alpha_i) \in (\mathbb{Z}_{\geq 0})^I$, we write

$$\mathrm{Rep}(\mathcal{I}, \alpha) = \mathrm{Rep}(\mathcal{I}, (\mathbb{C}^{\alpha_k})_{k \in I}).$$

4.2.2 Sequence of total orders and that of permutations

Let us fix a sequence of total orders $\mathcal{I} = (I, (<_i)_{i=0,\ldots,h})$. For each $i = 0, \ldots, h$ let us arrange the elements in I,

$$t_1^{(i)} <_i t_2^{(i)} <_i \cdots <_i t_n^{(i)},$$

and define the bijection

$$\begin{array}{cccc} \phi_i \colon & I & \longrightarrow & \{1, \ldots, n\} \\ & t_k^{(i)} & \longmapsto & k \end{array}.$$

Here n is the cardinality $\# I$ of I. Then we have a sequence of permutations of $\{1, \ldots, n\}$,

$$r_\nu = \phi_\nu \circ \phi_{\nu-1}^{-1} \text{ for } \nu = 1, \ldots, h.$$

Conversely if we fix a bijection ϕ_0 from I to $\{1, \ldots, n\}$ and a sequence of permutations of $\{1, \ldots, n\}$,

$$r_1, \ldots, r_h,$$

then we can define a sequence of total orders as follows. Let us define bijections $\phi_\nu\colon I \to \{1,\dots,n\}$ by $\phi_\nu = r_\nu \circ \phi_{\nu-1}$ for $\nu = 1,\dots,h$. For each $i = 0,\dots,h$ define the total ordering $<_i$ of I as the pull back of the natural ordering of $\{1,\dots,n\}$ by ϕ_i. Thus we have the following.

PROPOSITION 4.2. *Let I be a finite set of the cardinality n. Then there exists a one to one correspondence between sequences of total orders of I and the pairs of a bijection $\phi_0\colon I \to \{1,\dots,n\}$ and a sequence of elements in $\mathfrak{S}_n$.*

The identity element $\mathrm{id} \in \mathfrak{S}_n$ may be included in the sequence of permutations $r_1,\dots,r_h$ corresponding to $(I,(<_i)_{i=0,\dots,h})$. It is equivalent to the existence of $i \in \{1,\dots,h\}$ such that $<_i$ and $<_{i+1}$ define the same order. Thus we may omit $\mathrm{id} \in \mathfrak{S}_n$ in the sequence of permutations and call the consequent sequence $r'_1,\dots,r'_{h'}$ without $\mathrm{id} \in \mathfrak{S}_n$ the *reduced sequence* of permutations.

DEFINITION 4.3. Two sequences of total orders $\mathcal{I}$ and $\mathcal{I}'$ are said to be *conjugate* if the corresponding reduced sequences of permutations are conjugate. Namely, let $r_1,\dots,r_h$ and $r'_1,\dots,r'_{h'}$ be reduced sequences of permutations corresponding to $\mathcal{I}$ and $\mathcal{I}'$ respectively. Then $h = h'$ and there exists $\omega \in \mathfrak{S}_n$ such that $r_\nu = \omega^{-1} r'_\nu \omega$ for all $\nu = 1,\dots,h$.

4.2.3 Local moduli space and representations of sequences of total orders

We shall construct a sequence of total orders from the Stokes structure of connections.

Let us consider a $\mathbb{C}(\!(x)\!)$-connection (V,∇) with a normalized matrix $A(x) \in M(n,\mathbb{C}[x^{-1}])$. Then it is known that there exists $F \in \mathrm{GL}(n,\mathbb{C}(\!(x^{\frac{1}{r}})\!))$ with $r \in \mathbb{Z}_{>0}$ such that

$$FA(x)F^{-1} + \left(\frac{d}{dx}F\right)F^{-1} = \begin{pmatrix} q_1 I_{m_1} & & & \\ & q_2 I_{m_2} & & \\ & & \ddots & \\ & & & q_s I_{m_s} \end{pmatrix} t^{-1} + \begin{pmatrix} L_1 & & & \\ & L_2 & & \\ & & \ddots & \\ & & & L_s \end{pmatrix} t^{-1}$$

where $t = x^{\frac{1}{r}}$, $q_i \in t^{-1}\mathbb{C}[t^{-1}]$ ($q_i \neq q_j$ if $i \neq j$) and $L_i \in M(m_i,\mathbb{C})$. For the finite set

$$Q_A = \{q_1, q_2, \dots, q_s\},$$

we define a sequence of total orders as follows. For $d \in \mathbb{R}$, we write

$$j <_d k \quad \text{if} \quad \mathrm{Re}(a_0 e^{-\sqrt{-1} l_0 d}) < 0$$

where $q_j - q_k = a_0 x^{-l_0} + a_1 x^{-l_1} + \cdots + a_t x^{-l_t}$ with $l_0 > l_1 > \cdots > l_t$, $a_0 \neq 0$ and say d is a *Stokes direction* if there exist two distinct integers $1 \leq j, k \leq s$ such that these are incomparable by $<_d$. Thus we note that if d is not a Stokes direction, $<_d$ defines a total order on Q_A.

Let $0 \leq d_1 < d_2 < \cdots < d_h < 2\pi$ be the collection of all Stokes directions in $[0, 2\pi)$, so called *basic Stokes directions* (see [**6**]). Let us choose $\varepsilon > 0$ so that $\tilde{d_i} = d_i + \varepsilon < d_{i+1}$ and for $i = 0, \ldots, h$, where d_0 is the maximum of Stokes directions $d < 0$ and we formally set $d_{h+1} = 2\pi$. Then we have the sequence of total orders

$$\mathcal{I}_A = (Q_A, (<_{\tilde{d}_i})_{i=0,\ldots,h}).$$

Remark 4.4. *In the above setting, we see only the basic Stokes directions d_i because there exists $\sigma \in \mathfrak{S}_s$ such that*

$$q_{\sigma(i)}(e^{2\pi\sqrt{-1}} x) = q_i(x)$$

for all $i = 1, \ldots, s$ and we have

$$j <_d k \text{ if and only if } \sigma(j) <_{d+2\pi} \sigma(k)$$

for $d \in \mathbb{R}$.

Let us associate the representations of $\mathcal{I}_A$ with the space of certain unipotent matrices, i.e., so called Stokes matrices. For each $\nu = 1, \ldots, h$, define

$$\mathrm{Sto}_{d_\nu}(A) = \left\{ (X_{i,j})_{1 \leq i,j \leq s} \in \bigoplus_{1 \leq i,j, \leq s} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{m_j}, \mathbb{C}^{m_i}) \;\middle|\; X_{i,j} = \begin{cases} \mathrm{id}_{\mathbb{C}^{m_i}} & \text{if } i = j \\ 0 & \text{if } (i,j) \notin \rho_\nu \end{cases} \right\}.$$

Then we have the isomorphism

$$\mathrm{Rep}(\mathcal{I}_A, (m_i)_{i=1,\ldots,s}) \cong \bigoplus_{\nu=1}^{h} \mathrm{Sto}_{d_\nu}(A)$$

as $\mathbb{C}$-vector spaces.

The following is the direct consequence of Theorem VII and its Remark 2 of [**6**] (see also [**4, 20**]).

THEOREM 4.5. *We have a one to one correspondence*

$$\mathrm{Rep}(\mathcal{I}_A, (m_i)_{i=1,\dots,s}) \cong \bigoplus_{\nu=1}^{h} \mathrm{Sto}_{d_\nu}(A) \cong \mathfrak{M}((V,\nabla)).$$

4.2.4 Sequences of total orders of irreducible connections and iterated torus knots of plane curves

Let us return to our irreducible connection $E_{f,q}$ with the dual Puiseux characteristic $(q,p;\beta_1,\dots,\beta_g)$. Then we set $Q_{E_{f,q}} = \{\tilde{f}_1,\dots,\tilde{f}_q\}$ and define the sequence of total orders $\mathcal{I}_{E_{f,q}} = (Q_{E_{f,q}}, (<_{\tilde{d}_i})_{i=0,\dots,h})$ as in the previous subsection. Recalling that

$$\tilde{f}_i(\zeta_q x^{\frac{1}{q}}) = \tilde{f}_{i+1}(x^{\frac{1}{q}})$$

for $i = 1,\dots,q$ where we set $\tilde{f}_{q+1} = \tilde{f}_1$, we see that the substitution $x^{\frac{1}{q}} \mapsto \zeta_q x^{\frac{1}{q}}$ defines the action of $\mathbb{Z}/q\mathbb{Z}$ on $Q_{E_{f,q}}$.

For the latter use, we introduce the product of sequences of total orders $\mathcal{I}_1 = (I_1, (<^{(1)}_i)_{i=0,\dots,h^{(1)}})$, $\mathcal{I}_2 = (I_2, (<^{(2)}_i)_{i=0,\dots,h^{(2)}})$ with $\#I_1 = \#I_2$. First suppose that $I_1 = I_2$ and $<^{(1)}_{h^{(1)}} = <^{(2)}_0$, then the product

$$(I, (\widetilde{<}_i)_{i=0,\dots,h^{(1)}+h^{(2)}}) = \mathcal{I}_1 * \mathcal{I}_2$$

is defined by

$$\widetilde{<}_i = \begin{cases} <^{(1)}_i & \text{if } 0 \le i \le h^{(1)}, \\ <^{(2)}_{i-h^{(1)}} & \text{if } h^{(1)}+1 \le i \le h^{(1)}+h^{(2)}. \end{cases}$$

For general cases, find the bijection $\phi\colon I_1 \to I_2$ such that

$$u <^{(1)}_{h^{(1)}} v \text{ if and only if } \phi(u) <^{(2)}_0 \phi(v)$$

in I_1 and define $\phi_*(\mathcal{I}_2) = (I_1, (<^{\phi}_i)_{i=0,\dots,h^{(2)}})$ so that

$$u <^{\phi}_k v \text{ if } \phi(u) <^{(2)}_k \phi(v)$$

in I_1. Then the product of $\mathcal{I}_1$ and $\mathcal{I}_2$ is defined by $\mathcal{I}_1 * \mathcal{I}_2 = \mathcal{I}_1 * \phi_*(\mathcal{I}_2)$.

For $k \in \mathbb{Z}_{>0}$ we write $\tilde{f}_i \sim_k \tilde{f}_j$ if $\deg_{x^{-\frac{1}{q}}}(\tilde{f}_i - \tilde{f}_j) < k$. Let us note that each $\sim_k$ preserves orders $<_d$ for $d \in \mathbb{R}$, i.e., if $\tilde{f}_{i_1} \sim_k \tilde{f}_{i_2}$, $\tilde{f}_{j_1} \sim_k \tilde{f}_{j_2}$, $\tilde{f}_{i_1} \not\sim_k \tilde{f}_{j_1}$ and $\tilde{f}_{i_1} <_d \tilde{f}_{j_1}$, then we have $\tilde{f}_{i_{\epsilon_1}} <_d \tilde{f}_{j_{\epsilon_2}}$ for all $\epsilon_1, \epsilon_2 \in \{1,2\}$. Thus we can consider

$$\mathcal{I}^{(k)} = \mathcal{I}_{E_{f,q}}/\sim_k = (Q_{E_{f,q}}/\sim_k, (<_i)_{i=0,\dots,h}).$$

Since $\mathcal{I}^{(k)}$ define the same sequences for $\beta_i \geq k > \beta_{i+1}$, it suffices to consider

$$\mathcal{I}^{(\beta_i)},\ i = 1, \ldots, g.$$

We write $I^{(\beta_i)} = Q_{E_{f,q}} / \sim_{\beta_i}$ for short. Let us note that $I^{(\beta_i)}$ has the cardinality q/e_i and the action of $\mathbb{Z}/(q/e_i)\mathbb{Z}$ induced from the $\mathbb{Z}/q\mathbb{Z}$ action on $Q_{E_{f,q}}$.

The natural projections

$$\mathcal{I}_{E_{f,q}} = \mathcal{I}^{(\beta_g)} \xrightarrow{\pi_g} \mathcal{I}^{(\beta_{g-1})} \xrightarrow{\pi_{g-1}} \cdots \xrightarrow{\pi_2} \mathcal{I}^{(\beta_1)},$$

give decompositions

$$\mathcal{I}^{(\beta_i)} = \bigsqcup_{a \in I^{(\beta_{i-1})}} \mathcal{I}_a^{(\beta_i)},$$

where $\mathcal{I}_a^{(\beta_i)} = (\pi_i^{-1}(a), (<_i)_{i=0,\ldots,h})$ for $a \in I^{(\beta_{i-1})}$ and $i = 2, \ldots, g$. This decomposition induces a decomposition of representations of $\mathcal{I}_{E_{f,q}}$ as follows. The decomposition below is well known as the decomposition of Stokes matrices (see Theorem 8 in [**22**] or Proposition I.5.5 in [**20**] for example).

Proposition 4.6. *We have a decomposition*

$$\mathrm{Rep}(\mathcal{I}_{E_{f,q}}, (1)_{i=1,\ldots,q}) \cong \mathrm{Rep}(\mathcal{I}^{(\beta_1)}, (e_1)_{i=1,\ldots,q/e_1}) \oplus \bigoplus_{j=2}^{g} \bigoplus_{a \in I^{(\beta_{j-1})}} \mathrm{Rep}(\mathcal{I}_a^{(\beta_j)}, (e_j)_{i=1,\ldots,e_{j-1}/e_j}).$$

Proof. This follows from the decomposition of $M(q, \mathbb{C})$ as below. For each $k = 1, \ldots, g$, define

$$M(q,\mathbb{C})^{(\beta_i)} = \left\{ (a_{i,j})_{1 \leq i,j \leq q} \in M(q,\mathbb{C}) \,\middle|\, a_{i,j} = 0 \text{ if } \deg_{x^{-\frac{1}{q}}}(\tilde{f}_i - \tilde{f}_j) \neq \beta_k \right\}.$$

Then we have a decomposition

$$M(q,\mathbb{C}) = \{\mathrm{diag}(a_1, \ldots, a_q) \mid a_i \in \mathbb{C}\} \oplus \bigoplus_{i=1}^{g} M(q,\mathbb{C})^{(\beta_i)}$$

as a $\mathbb{C}$-vector space. □

For each $i = 2, \ldots, g$ let us fix $o \in I^{(\beta_{i-1})}$ as the image of $\tilde{f}_1 \in Q_{E_{f,q}}$ and define a product of $\mathcal{I}_a^{(\beta_i)}$ for $a \in I^{(\beta_{i-1})}$ by

$$\widetilde{\mathcal{I}}^{(\beta_i)} = (\widetilde{I}^{(\beta_i)}, (\widetilde{<}_j)_{j=0,\ldots,h_i}) = \mathcal{I}_o^{(\beta_i)} * \mathcal{I}_{e(o)}^{(\beta_i)} * \mathcal{I}_{e^2(o)}^{(\beta_i)} * \cdots * \mathcal{I}_{e^{q/e_{i-1}-1}(o)}^{(\beta_i)},$$

and for $i = 1$ set $\widetilde{\mathcal{I}}^{(\beta_1)} = \mathcal{I}^{(\beta_1)}$. Here $e \in \mathbb{Z}/(q/e_{i-1})\mathbb{Z}$ is the image of $1 \in \mathbb{Z}$. Let us note that there exists the natural isomorphism

$$\mathrm{Rep}(\widetilde{\mathcal{I}}^{(\beta_j)}, (e_j)_{i=1,\dots,e_{j-1}/e_j}) \xrightarrow{\sim} \bigoplus_{a \in I^{(\beta_{j-1})}} \mathrm{Rep}(\mathcal{I}_a^{(\beta_j)}, (e_j)_{i=1,\dots,e_{j-1}/e_j})$$

as $\mathbb{C}$-vector spaces for each $j = 2, \dots, g$. Thus by Proposition 4.6 we have

$$\mathrm{Rep}(\mathcal{I}_{E_{f,q}}, (1)_{i=1,\dots,q}) \cong \bigoplus_{j=1}^{g} \mathrm{Rep}(\widetilde{\mathcal{I}}^{(\beta_j)}, (e_j)_{i=1,\dots,e_{j-1}/e_j}).$$

Namely the structure of $\mathrm{Rep}(\mathcal{I}_{E_{f,q}}, (1)_{i=1,\dots,q}) \cong \mathfrak{M}(E_{f,q})$ is determined by $\widetilde{\mathcal{I}}^{(\beta_j)}$, $j = 1, \dots, g$.

The following is the main theorem of this subsection which shows that the structure of $\mathcal{I}_{E_{f,q}}$ is determined by the dual Puiseux characteristic. This can be seen as an analogy of plane curve germs for which Puiseux characteristics are topological invariants of knot structures, namely, if two curve germs have the same Puiseux characteristic, then the knots of them are isotopic.

Theorem 4.7. *For each $i = 1, \dots, g$, the reduced sequence of permutations attached to $\widetilde{\mathcal{I}}^{(\beta_i)}$ is conjugate with*

$$(s_1 s_2 \cdots s_{e_{i-1}/e_i - 1})^{\beta_i/e_i}$$

where $s_j \in \mathfrak{S}_{e_{i-1}/e_i}$ are transpositions $(j, j+1)$.

Proof. Let us proceed as the argument in the subsection 4.1. We write $\tilde{f}(x^{\frac{1}{q}}) = \sum_{k=1}^{g} a_{\beta_k} x^{-\frac{\beta_k}{q}}$. Let us first look at $\tilde{f}^{(1)}(x) = a_{\beta_1} x^{-\frac{\beta_1}{q}}$. If x moves in S_η for a sufficiently small $\eta > 0$, then $\tilde{f}^{(1)}(x)$ moves along a small circle centered at ∞. The geometric braid B_1 with the q/e_1 strings

$$\tilde{f}_l^{(1)}(t) = a_{\beta_1} \eta^{-\frac{\beta_1}{q}} e^{-\sqrt{-1}\frac{\beta_1}{q}(t+l)} \quad (0 \leq t \leq 2\pi)$$

for $l = 1, \dots, q/e_1$ define the torus knot of type $(q/e_1, \beta_1/e_1)$ as the closed braid of B_1. As is well known, if we number the strings in B_1 suitably, we have the braid words

$$(\sigma_1 \sigma_2 \cdots \sigma_{q/e_1 - 1})^{\beta_1/e_1},$$

where σ_i are standard generators of the braid group $\mathcal{B}_{q/e_1}$ on q/e_1 strings. On the other hand, let us consider the finite set

$$\mathfrak{I}^{(\beta_1)} = \{\mathrm{Re}(\tilde{f}_1^{(1)}(t)), \dots, \mathrm{Re}(\tilde{f}_{q/e_1}^{(1)}(t))\}.$$

Here if t moves from 0 to 2π then $\mathfrak{I}^{(\beta_1)}$ defines a sequence of total orders which is nothing but $\mathcal{I}^{(\beta_1)}$ by a suitable identification $\mathfrak{I}^{(\beta_1)} \cong I^{(\beta_1)}$. Since this can be seen as the projection of B_1 by $\tilde{f}_l^{(1)}(t) \mapsto \mathrm{Re}(\tilde{f}_l^{(1)}(t))$, thus the sequence of total orders defines the sequence of permutations

$$(s_1 s_2 \cdots s_{q/e_1-1})^{\beta_1/e_1}$$

as required.

Next let us fix $j \in \{2, \dots, g\}$ and consider

$$\tilde{f}^{(j)}(x) = \sum_{k=1}^{j} a_{\beta_k} x^{-\frac{\beta_k}{q}}.$$

For $1 \leq k \leq j$ and $1 \leq l_k \leq e_{k-1}/e_k$ let us define

$$\tilde{f}^{(k)}_{l_1,\dots,l_k}(t) = \sum_{i=1}^{k} a_{\beta_i} \eta^{-\frac{\beta_i}{q}} e^{-\sqrt{-1}\frac{\beta_i}{q}(t+l_i)}.$$

Then as we see in the subsection 4.1, for a fixed $(l_1, \dots, l_{j-1})$ and $l_j = 1, \dots, e_{j-1}/e_j$, one has the e_{j-1}/e_j points $f^{(j)}_{l_1,\dots,l_j}(t)$ in the circle around the point $\tilde{f}^{(j-1)}_{l_1,\dots,l_{j-1}}(t)$. Moreover the strings

$$\tilde{f}^{(j)}(t)_{l_1,\dots,l_{j-1},l_j}(t) \quad (t \in [0, (q/e_{j-1})2\pi])$$

for $l_j = 1, \dots, e_{j-1}/e_j$ define a geometric braid B_j and we have a torus knot of type $(e_{j-1}/e_j, \beta_j/e_j)$ as the closed braid of B_j. Thus B_j defines the braid words

$$(\sigma_1 \sigma_2 \cdots \sigma_{e_{j-1}/e_j-1})^{\beta_j/e_j}.$$

By the same argument as above, if t moves form 0 to $(q/e_{j-1})2\pi$ then

$$\mathfrak{I}^{\beta_1}_{l_1,\dots,l_{j-1}} = \left\{ \mathrm{Re}(\tilde{f}^{(j)}_{l_1,\dots,l_{j-1},1}(t)), \dots, \mathrm{Re}(\tilde{f}^{(j)}_{l_1,\dots,l_{j-1},e_{j-1}/e_j}(t)) \right\}$$

defines a sequence of total orders which induces the sequence of permutations

$$(s_1 s_2 \cdots s_{e_{j-1}/e_j-1})^{\beta_j/e_j}.$$

Meanwhile this sequence of total orders can be identified with $\widetilde{\mathcal{I}}^{(\beta_j)}$ by a suitable identification $\mathfrak{I}^{(\beta_j)}_{l_1,\dots,l_{j-1}} \cong \widetilde{I}^{(\beta_j)}$. Thus we are done.

□

Thus if we fix a dual Puiseux characteristic $(q, p; \beta_1, \dots, \beta_g)$, then the conjugacy classes of $\widetilde{\mathcal{I}}^{(\beta_i)}$, $i = 1, \dots, g$, are determined.

Corollary 4.8. *Let $E_{f,q}$ and $E_{f',q}$ be irreducible $\mathbb{C}(\!(x)\!)$-connections with the same dual Puiseux characteristic $(q,p;\beta_1,\dots,\beta_g)$ and set $\mathcal{I}=\mathcal{I}_{E_{f,q}}$, $\mathcal{I}'=\mathcal{I}_{E_{f',q}}$. Then the sequences of total orders $\widetilde{\mathcal{I}}^{(\beta_i)}$ and $\widetilde{\mathcal{I}'}^{(\beta_i)}$ defined as above are conjugate for each $i=1,\dots,g$.*

References

[1] D. Arinkin, *Fourier transform and middle convolution for irregular $\mathscr{D}$-modules*, preprint, arXiv:0808.0699, (2009).

[2] D. Arinkin, *Rigid irregular connections on $\mathbb{P}^1$*, Compos. Math. **146** (2010), no. 5, 1323–1338.

[3] D. Babbitt and V. Varadarajan, *Formal reduction theory of meromorphic differential equations: a group theoretic view*, Pacific J. Math. **109** (1983), no. 1, 1–80.

[4] D. Babbitt and V. Varadarajan, *Local moduli for meromorphic differential equations*, Astérisque No. 169-170 (1989), 217 pp.

[5] W. Balser, W. Jurkat and D. Lutz, *A general theory of invariants for meromorphic differential equations. I. Formal invariants*, Funkcial. Ekvac. **22** (1979), no.2, 197–221.

[6] W. Balser, W. Jurkat and D. Lutz, *A general theory of invariants for meromorphic differential equations. II. Proper invariants*, Funkcial. Ekvac. **22** (1979), no. 3, 257–283.

[7] S. Bloch and H. Esnault, *Local Fourier transforms and rigidity for $\mathscr{D}$-modules*, Asian J. Math. **8** (2004), no. 4, 587–605.

[8] P. Boalch, *Geometry and braiding of Stokes data; fission and wild character varieties*, Ann. of Math. (2) **179** (2014), no. 1, 301–365.

[9] K. Brauner, *Zur Geometrie der Funktionen zweier komplexen Verändlichen III, IV*, Abh. Math. Sem. Hamburg **6** (1928), 8–54.

[10] E. Brieskorn and H. Knörrer, *Plane algebraic curves. Translated from the German original by John Stillwell*, Birkhäuser/Springer, Basel, 1986.

[11] J. Fang, *Calculation of local Fourier transforms for formal connections*, Sci. China Ser A **52** (2009), no. 10, 2195–2206.

[12] R. García López, *Microlocalization and stationary phase*, Asian J. Math. **8** (2004), no. 4, 747–768.

[13] A. Graham-Squire, *Calculation of local formal Fourier transforms*, Ark. Mat. **51** (2013), no. 1, 71–84.

[14] A. Hefez, *Irreducible plane curve singularities* in Real and complex singularities, 1–120, Lecture Notes in Pure and Appl. Math., 232, Dekker, 2003.

[15] M. Hien and C. Sabbah, *The local Laplace transform of an elementary irregular meromorphic connection*, preprint, arXiv:1405.5310 (2014).

[16] M. Hukuhara, *Sur les points singuliers des équation différentielles linéaires. III.*, Mem. Fac. Sci. Kyusyu Imp. Univ. A. **2** (1942), 125–137.

[17] H. Komatsu, *On the index of ordinary differential operators*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **18** (1971), 379–398.

[18] Laumon, *Transformation de Fourier, constantes d'équations fonctionnelles et conjecture de Weil*, Inst. Hautes Études Sci. Publ. Math. No. 65 (1987), 131–210.

[19] A. Levelt, *Jordan decomposition for a class of singular differential operators*, Ark. Math. **13** (1975), 1–27.

[20] M. Loday-Richaud, *Stokes phenomenon, multisummability and differential Galois groups*, Ann. Inst. Fourier (Grenoble) **44** (1994), no. 3, 849–906.

[21] B. Malgrange, *Remarques sur les équations différentielles á points siguliers irréguliers* in Lecurete Notes in Mathematics 712, Springer-Verlag 1979, 77–86.

[22] J. Martinet and J.-P. Ramis, *Elementary acceleration and multisummability. I*, Ann. Inst. H. Poincaré Phys. Théor. **54** (1991), no. 4, 331–401.

[23] T. Mochizuki, *Note on the Stokes structure of Fourier transform*, Acta. Math. Vietnam. **35** (2010), no. 1, 107–158.

[24] C. Roucairol, *Formal structure of direct image of holonomic $\mathscr{D}$-modules of exponential type*, Manuscripta Math. **124**, (2007), no.3, 299–318.
[25] C. Sabbah, *An explicit stationary phase formula for the local formal Fourier-Laplace transform*, in Singularities I, 309–330, Contemp. Math., 474, Amer. Math. Soc., Providence, RI, 2008.
[26] C. Sabbah, *Differential systems of pure Gaussian type*, preprint, arXiv:1311.4132., (2013).
[27] Y. Sibuya, *Linear differential equations in the complex domain; problems of analytic continuation* (in Japanese), Kinokuniya, Tokyo 1976, Translations of Mathematical Monographis (English translation), vol. 82, AMS, 1990.
[28] H. Turrittin, *Convergent solutions of linear homogeneous differential equations in the neighborhood of an irregular singular point*, Acta. Math. **93**, (1955), 27–66.
[29] C. Wall, *Singular points of plane curves*, London Mathematical Society Student Texts, 63, Cambridge University Press, Cambridge, 2004.

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